PDEs for a Metro Ride

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Chapter 1

Maximum principle and symmetry of solutions of elliptic equations

The purpose of this chapter is to provide a brief introduction to some of the results on the qualitative behavior and symmetry of non-negative solutions of semi-linear elliptic and parabolic PDE's. The choice of the material is absolutely subjective. Most of the proofs will be either skipped completely or presented in every excruciating detail with the hope that this will enable the reader to believe that the omitted ones are correct. We will avoid each and every regularity result to the greatest extent possible. Sometimes this will be impossible. A standard reference to the elliptic regularity theory is the classical book by Gilbarg and Trudinger [36]. A recent book with the lecture notes on the elliptic equations by Han and Lin [38] is a very enjoyable read.

1 The strong maximum principle

This material is absolutely standard and is taken from [38].

Let Ω be a bounded connected domain in $\mathbb{R}^n$. Consider the operator

$$Lu = a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

with $u \in C^2(\Omega) \cap C(\overline{\Omega})$[1]. The functions a_{ij}, b_i and c are always assumed to be continuous in $\overline{\Omega}$ while L is assumed to be uniformly elliptic:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n$$

with a positive constant $\lambda > 0$.

Let us recall the weak maximum principle.

Theorem 1.1 *(The Weak Maximum Principle) Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $u \geq 0$ and $Lu \geq 0$ in Ω with $c(x) \leq 0$ in Ω. Then u attains its maximum on $\partial\Omega$.*

[1] Repeated indices are always summed over, at least unless they are not.

Proof. The main observation is that if x_0 is an interior maximum then $\nabla u(x_0) = 0$ while the matrix $D^2u = \left(\dfrac{\partial^2 u}{\partial x_i \partial x_j}\right)$ is non-positive semi-definite. Hence, the left side of $Lu \geq 0$ may not be positive at an interior maximum, as $c \leq 0$ and $u \geq 0$. This would be a contradiction were it not for the possibility $Lu = 0$ that arises if $D^2u(x_0)$ is degenerate – the rest of the proof fights this.

Given $\varepsilon > 0$ define $w(x) = u(x) + \varepsilon e^{\alpha x_1}$ with α to be determined – this will ensure that $D^2w(x_0)$ is non-degenerate. Then we have

$$Lw = Lu + \varepsilon e^{\alpha x_1}\left(a_{11}\alpha^2 + b_1\alpha + c\right).$$

Recall that $a_{11} \geq \lambda > 0$ and $|b_1|, |c| \leq \text{const}$. Thus we may choose $\alpha > 0$ sufficiently large so that

$$a_{11}\alpha^2 + b_1\alpha + c > 0$$

and thus $Lw > 0$ in Ω. Therefore the function w attains its maximum in $\bar{\Omega}$ at the boundary $\partial\Omega$. Indeed, as before, if w attains its (non-negative) maximum at $x_0 \notin \partial\Omega$ then $\nabla w(x_0) = 0$ and the matrix $D_{ij} = \dfrac{\partial^2 w}{\partial x_i \partial x_j}$ is non-positive definite. Hence we would have

$$Lw(x_0) = a_{ij}(x_0)D^2_{ij}w(x_0) + c(x_0)w(x_0) \leq 0$$

which is a contradiction. Thus $x_0 \in \partial\Omega$ and we obtain

$$\sup_\Omega u \leq \sup_\Omega w \leq \sup_{\partial\Omega} w \leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} e^{\alpha x_1} \leq C\varepsilon + \sup_{\partial\Omega} u$$

with the constant C independent of ε. As $\varepsilon > 0$ is arbitrary we may let $\varepsilon \to 0$ to finish the proof. □

Corollary 1.2 *The Dirichlet problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \phi \text{ on } \partial\Omega \end{aligned}$$

with $f \in C(\Omega)$, $\phi \in C(\partial\Omega)$ has at most one solution if $c(x) \leq 0$.

Remark 1.3 We first note that the assumption of non-negativity of u is not needed if $c = 0$. However, non-positivity of $c(x)$ is essential: otherwise the Dirichlet problem may have a non-unique solution, as $u(x, y) = \sin x \sin y$ solves

$$\begin{aligned} \Delta u + 2u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

with $\Omega = [0, \pi] \times [0, \pi] \subset \mathbb{R}^2$. Second, the assumption that Ω is bounded is also essential both for uniqueness and for the maximum principle to hold: the function $u(x) = \log |x|$ solves

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

with $\Omega = \{|x| > 1\}$.

The Hopf lemma guarantees that the point on the boundary where the maximum is attained is not a critical point.

Theorem 1.4 *(The Hopf Lemma) Let B be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup x_0)$ satisfies $Lu \geq 0$ in B with $c(x) \leq 0$ in B. Assume in addition that $u(x) < u(x_0)$ for any $x \in B$ and $u(x_0) \geq 0$. Then we have*

$$\liminf_{t \to 0^+} \frac{u(x_0) - u(x_0 - tm)}{t} > 0$$

for each outward direction m: $m \cdot \nu(x_0) > 0$.

Remark 1.5 If the normal derivative exists at x_0 then $\dfrac{\partial u}{\partial \nu}(x_0) < 0$.

Proof. We may assume without loss of generality that B is centered at the origin and has radius r. We may also assume that $u \in C(\bar{B})$ and that $u(x) < u(x_0)$ for all $x \in \bar{B} \backslash \{x_0\}$ – otherwise we would simply consider a smaller ball $B_1 \subset B$ that is tangent to B at x_0.

Consider $w(x) = u(x) + \varepsilon h(x)$ with $h(x) = e^{-\alpha |x|^2} - e^{-\alpha r^2}$ and define the domain

$$\Sigma = B \cap B(x_0, r/2).$$

We observe first that $h > 0$ and $Lh > 0$ in Σ with an appropriate choice of α:

$$\begin{aligned} Lh &= e^{-\alpha|x|^2}\left[4\alpha^2 a_{ij}(x)x_i x_j - 2\alpha a_{ii}(x) - 2\alpha b_i(x)x_i + c(x)\right] - c(x)e^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\left[4\alpha^2 a_{ij}(x)x_i x_j - 2\alpha[a_{ii}(x) + b_i(x)x_i] + c(x)\right] \\ &\geq e^{-\alpha|x|^2}\left[4\alpha^2 \lambda |x|^2 - 2\alpha[a_{ii}(x) + b_i(x)x_i] + c(x)\right] \\ &\geq e^{-\alpha|x|^2}\left[4\alpha^2 \lambda \frac{r^2}{4} - 2\alpha[a_{ii}(x) + b_i(x)x_i] + c(x)\right] > 0 \end{aligned}$$

for all $x \in \Sigma$ for a sufficiently large $\alpha > 0$. Hence, we have $Lw > 0$ for all $\varepsilon > 0$ and thus w may not attain its non-negative maximum inside Σ. We now show that if $\varepsilon > 0$ is sufficiently small then w attains its non-negative maximum only at x_0. Indeed, we may find δ so that $u(x) < u(x_0) - \delta$ for $x \in \partial\Sigma \cap B$. Take ε so that $\varepsilon h(x) < \delta$ on $\partial\Sigma \cap B$, then $w(x) < u(x_0) = w(x_0)$ for all $x \in \partial\Sigma \cap B$. On the other hand, for $x \in \partial\Sigma \cap \partial B$ we have $h(x) = 0$ and $w(x) = u(x) < u(x_0) = w(x_0)$. We conclude that $w(x)$ attains its non-negative maximum in $\bar{\Sigma}$ at x_0 if ε is sufficiently small. This implies

$$\frac{w(x_0) - w(x_0 - tm)}{t} \geq 0$$

for any small $t > 0$. Letting $t \to 0$ we obtain

$$\liminf_{t \to 0^+} \frac{u(x_0) - u(x_0 - tm)}{t} \geq -\varepsilon \frac{\partial h}{\partial n}(x_0) = \varepsilon \alpha r e^{-\alpha r^2} > 0.$$

This finishes the proof. □

Theorem 1.6 *(The Strong maximum Principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. Then the non-negative maximum of u in Ω may be attained only on $\partial\Omega$ unless u is a constant.*

Proof. Let $M = \sup_{\bar{\Omega}} u(x)$ and define the set $\Sigma = \{x \in \Omega : \ u(x) = M\}$, where the maximum is attained. We need to show that either Σ is empty or $\Sigma = \Omega$. Assume that Σ is non-empty but $\Sigma \neq \Omega$. Choose $p \in \Omega \backslash \Sigma$ so that $d_0 = d(p, \Sigma) < d(p, \partial\Omega)$. Consider the ball $B_0 = B(p, d_0)$ and let $x_0 \in \partial B_0 \cap \partial\Sigma$. Then we have $Lu \geq 0$ in B_0 and $u(x) < u(x_0) = M$, $M \geq 0$ for all $x \in B_0$. The Hopf Lemma implies that $\frac{\partial u}{\partial n}(x_0) > 0$ where n is the normal to B_0 at x_0. However, x_0 is an internal maximum of u in Ω and hence $\nabla u(x_0) = 0$. This is a contradiction. □

Corollary 1.7 *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. If $u \leq 0$ on $\partial\Omega$ then either $u \equiv 0$ in Ω or $u < 0$ in Ω.*

Corollary 1.8 *Let Ω have an internal sphere property and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. Assume that u attains its maximum in $\bar{\Omega}$ at a point x_0. Then $x_0 \in \partial\Omega$ and*

$$\frac{\partial u}{\partial n}(x_0) > 0$$

unless $u = const$ in Ω.

The restriction $c(x) \leq 0$ may be eliminated if we know a priori that $u \leq 0$.

Corollary 1.9 *(Another version of the strong maximum principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $u \leq 0$ in Ω, with a bounded function $c(x)$. Then either $u \equiv 0$ in Ω or $u < 0$ in Ω.*

Proof. As $u \leq 0$ in Ω, the inequality $Lu \geq 0$ implies that, for any $M > 0$ we have

$$Lu - Mu \geq -Mu \geq 0.$$

However, if $M > \|c\|_{L^\infty(\Omega)}$ then the operator $L'u = Lu - Mu$ has the zero order coefficient $c'(x) = c(x) - M \leq 0$, hence we may apply Corollary 1.7 to conclude that either $u < 0$ in Ω or $u \equiv 0$ in Ω. □

Exercise 1.10 Let Ω be a bounded subset of $\mathbb{R}^n$ that has an internal sphere property and consider the boundary value problem

$$\begin{aligned} &Lu = f \text{ in } \Omega \\ &\frac{\partial u}{\partial n} + \alpha(x)u = \phi \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

with $f \in C(\bar{\Omega})$ and $\phi \in C(\partial\Omega)$. Assume in addition that $c(x) \leq 0$ and the "heat-loss" parameter $\alpha(x) \geq 0$. Show that (1.1) has at most one solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ provided that either c or α is not identically equal to zero. If both c and α are zero then solution (if it exists) is unique up to an additive constant.

A priori estimates

The most immediate application of the maximum principle is to obtain the very basic uniform a priori estimates on the solutions. We still assume that the matrix a_{ij} is uniformly elliptic in $\bar{\Omega}$: $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\lambda > 0$, and a_{ij}, b_i and c are continuous in $\bar{\Omega}$. We assume in addition that

$$\sup_{\Omega} |a_{ij}| + \sup_{\Omega} |b_i| \leq \Lambda.$$

The first result deals with the Dirichlet boundary conditions.

Theorem 1.11 *Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \phi \text{ on } \partial\Omega \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\phi \in C(\partial\Omega)$. There exists a constant $C(\lambda, \Lambda, \mathrm{diam}(\Omega))$ so that

$$|u(x)| \leq \max_{\partial\Omega} |\phi| + C \max_{\Omega} |f| \text{ for all } x \in \Omega \tag{1.2}$$

provided that $c(x) \leq 0$.

Proof. Let us denote $F = \max_{\Omega} |f|$ and $\Phi = \max_{\partial\Omega} |\phi|$ and assume that Ω lies inside a strip $\{0 < x_1 < d\}$. Define $w(x) = \Phi + \left(e^{\alpha d} - e^{\alpha x_1}\right) F$ with $\alpha > 0$ to be chosen so as to ensure

$$\begin{aligned} Lw &\leq -F \text{ in } \Omega \\ w &\geq \Phi \text{ on } \partial\Omega. \end{aligned} \tag{1.3}$$

We calculate $w \geq \Phi$ on $\partial\Omega$ and

$$-Lw = (a_{11}\alpha^2 + b_1\alpha)Fe^{\alpha x_1} - c\Phi - c\left(e^{\alpha d} - e^{\alpha x_1}\right) F \geq (a_{11}\alpha^2 + b_1\alpha)F \geq (\lambda\alpha^2 + b_1\alpha)F \geq F$$

when α is large enough. Hence w satisfies (1.3). The comparison principle implies that $-w \leq u \leq w$ in Ω and in particular

$$\sup_{\Omega} |u| \leq \Phi + \left(e^{\alpha d} - 1\right) F$$

so that (1.2) holds. □

A similar estimate holds with the "heat-loss" Robin boundary conditions.

Theorem 1.12 *Let f and ϕ be as before. Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{aligned} &Lu = f \text{ in } \Omega \\ &\frac{\partial u}{\partial n} + \alpha(x)u = \phi \text{ on } \partial\Omega \end{aligned} \tag{1.4}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$, with $\alpha(x) \geq \alpha_0 > 0$. Then there exists a constant $C = C(\lambda, \Lambda, \mathrm{diam}(\Omega), \alpha_0)$ so that

$$|u(x)| \leq C\left[\max_{\partial\Omega} |\phi| + \max_{\Omega} |f|\right] \text{ for all } x \in \Omega \tag{1.5}$$

provided that $c(x) \leq 0$.

Proof. *Case 1: $c(x) \le -c_0 < 0$.* We will show that

$$|u(x)| \le \frac{F}{c_0} + \frac{\Phi}{\alpha_0}. \tag{1.6}$$

We define

$$v^{\pm} = \frac{F}{c_0} + \frac{\Phi}{\alpha_0} \pm u.$$

Then we have

$$Lv^{\pm} = c(x)\left(\frac{F}{c_0} + \frac{\Phi}{\alpha_0}\right) \pm f \le -F \pm f \le 0 \text{ in } \Omega$$

and

$$\frac{\partial v^{\pm}}{\partial n} + \alpha(x)v^{\pm} = \alpha(x)\left(\frac{F}{c_0} + \frac{\Phi}{\alpha_0}\right) \pm \phi \ge \Phi \pm \phi \ge 0 \text{ on } \partial\Omega.$$

If $v^{\pm}$ attains a negative minimum in $\bar{\Omega}$ then it has to be on $\partial\Omega$ by the maximum principle. Then $\dfrac{\partial v^{\pm}}{\partial n}(x_0) \le 0$ and hence

$$\frac{\partial v^{\pm}}{\partial n}(x_0) + \alpha(x)v^{\pm}(x_0) \le \alpha(x)v^{\pm}(x_0) < 0$$

which is a contradiction. Hence $v^{\pm} > 0$ in $\bar{\Omega}$ and thus (1.6) holds.

Case 2: $c(x) \le 0$ for any $x \in \Omega$. We consider $w(x) = u(x)/z(x)$ and seek $z(x)$ positive to be determined so that Case 1 would apply to $w(x)$. A direct computation shows that w satisfies

$$a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + B_i\frac{\partial w}{\partial x_i} + \left(c + \frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i\frac{\partial z}{\partial x_i}\right]\right) w = \frac{f}{z} \text{ in } \Omega$$
$$\frac{\partial w}{\partial n} + \left(\alpha(x) + \frac{1}{z}\frac{\partial z}{\partial n}\right) w = \frac{\phi}{z} \text{ on } \partial\Omega.$$

Hence we need to choose a bounded function $z(x)$ so that

$$c + \frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i\frac{\partial z}{\partial x_i}\right] \le -c_0$$

and

$$\alpha(x) + \frac{1}{z}\frac{\partial z}{\partial n} \ge \beta_0$$

to reduce this case to Case 1. It suffices to require that

$$\frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i\frac{\partial z}{\partial x_i}\right] \le -c_0 < 0$$

and

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \le \frac{\alpha_0}{2}.$$

As before we assume that $\Omega \subset \{0 < x_1 < d\}$ and choose $z(x) = A + e^{\beta d} - e^{\beta x_1}$ with $A > 0$ and $\beta > 0$ to be determined. We calculate

$$-\frac{1}{z}\left[a_{ij}\frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i}\right] = \frac{(\beta^2 a_{11} + \beta b_1)e^{\beta x_1}}{A + e^{\beta d} - e^{\beta x_1}} \geq \frac{(\beta^2 a_{11} + \beta b_1)}{A + e^{\beta d}} \geq \frac{1}{A + e^{\beta d}} > 0$$

when β is sufficiently large. Having chosen such β we choose A so large that

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \leq \frac{\beta e^{\beta d}}{A} \leq \frac{\alpha_0}{2}.$$

Now the problem is reduced to Case 1 for the function w. □

Note that these results may not be extended to the Neumann problem ($\alpha(x) = 0$ in (1.4)) since solution is unique only up to addition of an arbitrary constant.

2 The isoperimeteric inequality and sliding

The simplest situation when the sliding idea arises is in a very simple proof of the isoperimetric inequality. We follow here the proof given by X. Cabré in [16][2]. The isoperimetric inequality says that among all domains of a given volume the ball has the smallest perimeter.

Theorem 2.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Then,*

$$\frac{|\partial \Omega|}{|\Omega|^{(n-1)/n}} \geq \frac{|\partial B_1|}{|B_1|^{(n-1)/n}}, \tag{2.1}$$

where B_1 is the open unit ball in $\mathbb{R}^n$, $|\Omega|$ denotes the measure of Ω and $|\partial\Omega|$ is the perimeter of Ω (the $(n-1)$-dimensional measure of the boundary of Ω). In addition, equality in (2.1) holds if and only if Ω is a ball.

The proof will use the area formula (see [26] for the proof) which generalizes the usual change of variables formula in multi-variable calculus (that corresponds to one-to-one mappings f).

Theorem 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with Jacobian Jf. Then for each $g \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} g(x) Jf(x) dx = \int_{\mathbb{R}^n} \left[\sum_{x \in f^{-1}\{y\}} g(x)\right] dy. \tag{2.2}$$

We will, in particular, need the following corollary.

Corollary 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with Jacobian Jf. Then for each measurable set $A \subset \mathbb{R}^n$ we have*

$$|f(A)| \leq \int_A Jf(x) dx. \tag{2.3}$$

[2]Readers with ordinary linguistic powers may consult [17].

Proof. We will use the area formula with $g(x) = \chi_A(x)$:

$$\int_A Jf(x)dx = \int_{\mathbb{R}^n} \chi_A(x)Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x\in f^{-1}\{y\}} \chi_A(x) \right] dy$$
$$= \int_{\mathbb{R}^n} [\#x \in A : \; f(x) = y]\, dy \geq \int_{\mathbb{R}^n} \chi_{f(A)}(y)dy = |f(A)|,$$

and we are done. $\square$

A more general form of this corollary is the following.

Corollary 2.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with Jacobian Jf. Then for each nonnegative function $p \in L^1(\mathbb{R}^n)$ and each measurable set A we have*

$$\int_{f(A)} p(y)dy \leq \int_A p(f(x))Jf(x)dx. \tag{2.4}$$

Proof. The proof is as in the previous corollary. We will use the area formula with $g(x) = p(f(x))\chi_A(x)$:

$$\int_A p(f(x))Jf(x)dx = \int_{\mathbb{R}^n} \chi_A(x)p(f(x))Jf(x)dx = \int_{\mathbb{R}^n} \left[\sum_{x\in f^{-1}\{y\}} \chi_A(x)p(f(x)) \right] dy$$
$$= \int_{\mathbb{R}^n} [\#x \in A : \; f(x) = y]\, p(y)dy \geq \int_{f(A)} p(y)dy,$$

and we are done. $\square$

Proof of Theorem 2.1. We now proceed with Cabré's proof of the isoperimetric inequality. Let $v(x)$ be the solution of the Neumann problem

$$\Delta v = k, \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 1 \text{ on } \partial\Omega. \tag{2.5}$$

Integrating the first equation above and using the boundary condition, we obtain

$$k|\Omega| = \int_\Omega \Delta v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = |\partial\Omega|.$$

Hence, solution exists only if

$$k = \frac{|\partial\Omega|}{|\Omega|}. \tag{2.6}$$

It is a classical result that with this particular value of k there exist infinitely many solutions that differ by addition of an arbitrary constant. We let v be any of them. As Ω is a smooth domain, v is also smooth.

Let Γ_v be the lower contact set of v, that is, the set of all $x \in \Omega$ such that the tangent hyperplane to the graph of v at x lies below v in all of $\bar{\Omega}$. More formally, we define

$$\Gamma_v = \{x \in \Omega : \; v(y) \geq v(x) + \nabla v(x) \cdot (y - x) \text{ for all } y \in \bar{\Omega}.\} \tag{2.7}$$

The main observation is that

$$B_1 \subset \nabla v(\Gamma_v). \tag{2.8}$$

Here B_1 is the open unit ball centered at the origin. The geometric reason for this is as follows: take any $p \in B_1$ and consider the graphs of the functions

$$r_c(y) = p \cdot y + c.$$

There exists $M > 0$ so that if $c < -M$ then $r_c(y) < v(y) - 100$ for all $y \in \bar{\Omega}$, and $r_c(y) > v(y) + 100$ for all $y \in \bar{\Omega}$ if $c > M$. Let

$$\alpha = \sup\{c \in \mathbb{R} : \ r_c(y) < v(y) \text{ for all } y \in \bar{\Omega}\},$$

then it is easy to see that there exists $y_0 \in \bar{\Omega}$ such that $r_\alpha(y_0) = v(y_0)$ and

$$r_\alpha(y) \le v(y) \text{ for all } y \in \bar{\Omega}. \tag{2.9}$$

Furthermore, as $|\partial r_c/\partial \nu| = |p \cdot \nu| \le |p| < 1$ and $\partial v/\partial \nu = 1$ for $y \in \partial\Omega$, it is impossible that $y_0 \in \partial\Omega$. Thus, y_0 is an interior point of Ω, $\nabla v(y_0) = p$, the graph of $r_\alpha(y)$ is the tangent plane to v at y_0, and then (2.9) implies that $y_0 \in \Gamma_v$.

The rest is simple: we deduce from (2.8) and Corollary 2.3 that

$$|B_1| \le |\nabla v(\Gamma_v)| = \int_{\nabla v(\Gamma_v)} dp \le \int_{\Gamma_v} \det[D^2 v(x)] dx. \tag{2.10}$$

We used in the last inequality above the fact that $\det[D^2 v]$ is non-negative for $x \in \Gamma_v$ – this follows immediately from the definition of the set Γ_v, and, moreover, all eigenvalues of $D^2 v$ are nonnegative on this set. It remains to notice that by the classical arithmetic mean-geometric mean inequality applied to these eigenvalues we have

$$\det[D^2 v(x)] \le \left(\frac{\mathrm{Tr}[D^2 v]}{n}\right)^n = \left(\frac{\Delta v}{n}\right)^n \text{ for } x \in \Gamma_v. \tag{2.11}$$

Using this inequality in (2.10) and recalling that v solves (2.5) we deduce that

$$|B_1| \le \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Gamma_v| \le \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Omega|.$$

However, for the unit ball we have $|\partial B_1| = n|B_1|$, and the isoperimetric inequality (2.1) follows.

In order to see the converse, we observe that it follows from the above that for the equality to hold in (2.1) we must have equality in (2.11), which implies that $D^2 v(x)$ is a multiple of the identity matrix at each $x \in \Omega$ so that

$$v(x) = \frac{k}{2n}\left[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2\right],$$

with some $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. The boundary condition $\partial v/\partial \nu = 1$ implies then that Ω is a ball. □

Exercise 2.5 *Give a non-geometric proof of the inclusion (2.8) using the Legendre transform.*

We note that the sliding idea used in this proof will soon reappear in the proof of the ABP maximum principle below.

3 The ABP estimate and the maximum principle for small domains

3.1 The maximum principle for narrow domains

The usual maximum principle in the form "$Lu \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ implies either $u \equiv 0$ or $u < 0$ in Ω" can be interpreted physically as follows. If u is the temperature distribution then the boundary condition $u \leq 0$ means that "the boundary is cold" while the term $c(x)u$ can be viewed as a heat source if $c(x) \geq 0$ or as a heat sink if $c(x) \leq 0$. The condition $c(x) \leq 0$ means that the boundary is cold and there are no heat sources – therefore, the temperature is cold everywhere, and we get $u \leq 0$. On the other hand, if the domain is such that each point inside Ω is "close to the boundary" then the effect of the cold boundary can dominate over a heat source, and even if $c(x) \geq 0$ at some $x \in \Omega$, the maximum principle still holds.

Mathematically, the first step in that direction is the maximum principle for narrow domains.

Theorem 3.1 *(The Maximum Principle for Narrow Domains) Let* $\mathbf{e}$ *be a unit vector. There exists* $d_0 > 0$ *that depends on the coercivity constant* λ*, and the* L^∞*-norms* $\|b_i\|_\infty$ *and* $\|c^+\|_\infty$ *so that if* $|(y-x)\cdot\mathbf{e}| < d_0$ *for all* $(x,y) \in \Omega$ *then maximum principle holds for the operator* L*. That is, if* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies* $Lu \geq 0$ *and* $u \leq 0$ *on* $\partial\Omega$ *then either* $u \equiv 0$ *or* $u < 0$ *in* Ω*.*

The main point is that in a narrow domain we need not assume $c \leq 0$.

Proof. Assume that $\mathbf{e}$ is the unit vector in the direction x_1 so that $\bar{\Omega} \subset \{0 < x_1 < d\}$, and that $\|b_i\|_\infty, \|c^+\|_\infty \leq N$. Consider the function $w = e^{\alpha d} - e^{\alpha x_1} > 0$ in $\bar{\Omega}$, as we did in the proof of the a priori estimates in Theorem 1.11 We compute

$$Lw = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -\lambda\alpha^2 + N\alpha + Ne^{\alpha d} < 0$$

if we first choose α sufficiently large and then d sufficiently small. We now set $v = u/w$, then v satisfies

$$\frac{Lu}{w} = a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + B_i\frac{\partial v}{\partial x_i} + \left(\frac{Lw}{w}\right)v \geq 0 \tag{3.1}$$

with

$$B_i = b_i + \frac{2}{w}a_{ij}\frac{\partial w}{\partial x_j}.$$

We may now apply the usual maximum principle to the operator in the middle of (3.1) acting on v, as $\tilde{c} = (Lw/w) < 0$ and conclude that $v = u/w$ is non-positive inside Ω, and hence so is u. $\Box$

3.2 The ABP Maximum Principle

The simple a priori estimates we have obtained in Theorem 1.11 for the problem

$$Lu = f,$$

with the Dirichlet boundary conditions do not take into account whether the right side f is large on a small set or not – after all, if, say, $\|f\|_\infty = 1$ but f is supported on a very small set, we would expect u to be close to the solution of $Lu = 0$, not $Lu = 1$, and that is not seen in Theorem 1.11!

When the operator L is not in the divergence form, and not much is known about the regularity of the coefficients a_{ij}, a useful tool is the ABP (Alexandrov, Bakelman and Pucci) maximum principle. It allows to estimate the supremum of the solution of equations in the non-divergence form in terms of the L^n-norm of the right hand side. This is useful in various applications. First, it is the standard starting point in fully nonlinear equations: let us very briefly explain how it is used. Consider, as the simplest example, an equation of the form

$$F(D^2u) = f. \tag{3.2}$$

Here $F(M)$ is a real-valued function of a matrix argument. Differentiating (3.2) with respect to x_j we obtain an equation for the partial derivatives $u_j = \partial u/\partial x_j$ of the form

$$b_{mk}\frac{\partial^2 u_j}{\partial x_m \partial x_k} = \frac{\partial f}{\partial x_j}, \tag{3.3}$$

with

$$b_{mk}(x) = \frac{\partial F}{\partial M_{mk}}(D^2u(x)).$$

Note that unless we already know something a priori about the solution $u(x)$, we can not say anything about the regularity of the coefficients $b_{mk}(x)$. It is reasonable to assume that F is uniformly elliptic, that is, that the matrix b_{mk} is bounded and uniformly positive definite for all u:

$$\lambda I \le \left\{\frac{\partial F(M)}{\partial M_{mk}}\right\} \le \Lambda I,$$

for all $M \in \mathrm{Mat}_{n\times n}$. Therefore, all a priori information for (3.3) we have is that the coefficients b_{mk} are measurable and uniformly elliptic, and this equation is in the non-divergence form. The ABP maximum principle allows us to bound the derivatives u_j which gives more regularity on the coefficients b_{mk} and one may bootstrap this argument to develop regularity theory. This problem is very interesting but rather technical, and we will not consider it here. Another immediate application of the ABP maximum principle that we will consider in some detail is to the maximum principle for domains of small volume and in the sliding method.

Let us now state the ABP estimate. As before, we let Ω be a bounded domain in $\mathbb{R}^n$ and consider an elliptic operator

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

with bounded measurable coefficients a_{ij}, b_i and c[3]. We assume that L is uniformly elliptic: there exist $\lambda > 0$ and $\Lambda > 0$ so that

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \tag{3.4}$$

[3]When we talk about bounds for equations with bounded measurable coefficients our true concern is not the lack of regularity of a_{ij}, b_i or c (they may lie in $C^\infty(\Omega)$) but rather that any bound on the solution $u(x)$ that we obtain should not involve anything but L^∞ or L^p norms of the coefficients in the equation. That is, the constants in our estimates can not depend on, say, Hölder norms of the coefficients or on their derivatives.

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$, and that

$$\left(\sum_{i=1}^{n} |b_i(x)|^2 \right)^{1/2} \leq \bar{b}, \quad |c(x)| \leq \bar{c}, \tag{3.5}$$

for all $x \in \Omega$.

Given a function $u \in C^2(\Omega)$ we define its upper contact set Γ as

$$\Gamma = \{ y \in \Omega : \; u(x) \leq u(y) + \nabla u(y) \cdot (x - y) \text{ for any } x \in \Omega \},$$

that is, the set of all points $x \in \Omega$ so that the tangent hyperplane of v at x lies above the graph of v in all of $\bar{\Omega}$. The Hessian matrix $D^2 u(x)$ is non-positive on Γ. This notion may be extended to continuous functions u as

$$\Gamma = \{ y \in \Omega : \; \exists p(y) \text{ so that } u(x) \leq u(y) + p(y) \cdot (x - y) \text{ for any } x \in \Omega \}.$$

Here $z(x) = u(y) + p \cdot (x - y)$ is the supporting hyper-plane at y. Clearly, the function u is concave if and only if its upper contact set is all of Ω, and if $u \in C^1(\Omega)$ then $p(y) = \nabla u(y)$.

Theorem 3.2 *(The ABP Maximum Principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq f$ in Ω and assume that $f \in L^n(\Omega)$ and $c \leq 0$ in Ω. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \mathrm{diam}(\Omega) \left\| f^- \right\|_{L^n(\Gamma)}, \tag{3.6}$$

where Γ is the upper contact set of $u^+(x)$, and the constant C depends only on $\|b\|_{L^n(\Gamma)}$, dimension n and the constant λ.

The precise form of the constant can be found in [38]. Note that the upper contact set of u^+ lies in the intersection of the upper contact set of u and the set $\{u > 0\}$.

3.3 The maximum principle for small domains

An important consequence of the ABP maximum principle is a maximum principle for a domain with a small volume [5]. Despite a simple proof and beautiful applications it has been observed only fairly recently, at least in the West where it was discovered in the 1990's by Varadhan[4]. Consider

$$Lu = a_{ij} D_{ij} u + b_i D_i u + cu$$

where a_{ij} is point-wise positive definite with

$$|b_i| + |c| \leq \Lambda, \quad \mathrm{det} a_{ij} \geq \lambda$$

for some positive constants λ and Λ.

Theorem 3.3 *(The Maximum Principle for Domains of a Small Volume) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ in Ω and assume that $u \leq 0$ on $\partial\Omega$. Then there exists a positive constant $\delta = \delta(n, \lambda, \Lambda, diam(\Omega))$ so that if $|\Omega| \leq \delta$ then $u \leq 0$ in Ω.*

[4] It was first noted by Bakelman in USSR.

Proof. If $c \le 0$ then $u \le 0$ by the usual maximum principle (or by the ABP maximum principle). In general write $c = c^+ - c^-$, then

$$a_{ij} D_{ij} u + b_i D_i u - c^- u \ge -c^+ u.$$

Then the ABP maximum principle implies that

$$\sup_\Omega u \le C \|c^+ u^+\|_{L^n(\Omega)} \le C \mathrm{diam}(\Omega) \|c^+\|_\infty |\Omega|^{1/n} \sup_\Omega u \le \frac{1}{2} \sup_\Omega u$$

with the constant $C = C(n, \lambda, \Lambda)$, when $|\Omega|$ is small. Hence we have $u \le 0$ in Ω. $\square$

3.4 Proof of the ABP maximum principle

Note that we may assume without loss of generality that

$$u \le 0 \text{ on } \partial\Omega. \tag{3.7}$$

Indeed, if that is not the case, we define

$$u' = u - \sup_{\partial\Omega} u^+.$$

The function u' satisfies

$$Lu' = Lu - c(x) \sup_{\partial\Omega} u^+ \ge f,$$

as $c \le 0$. The upper contact sets of u and u' are also the same. Therefore, we will assume (3.7) throughout the proof.

The case $b = 0$ and $c = 0$

First, we consider the special case $b = 0$ and $c = 0$. The idea is as in the proof of the isoperimetric inequality. If $M := \sup_\Omega u \le 0$ then there is nothing to prove, hence we assume that $M > 0$. The maximum is achieved at an interior point $x_0 \in \Omega$, $M = u(x_0)$, as $u(x) \le 0$ on $\partial\Omega$. Consider the function $v = -u^+$, then $v \le 0$ in Ω, $v \equiv 0$ on $\partial\Omega$ and

$$-M = \inf_\Omega v = v(x_0).$$

We proceed as in the proof of the isoperimetric inequality. Let Γ be the upper contact set of u^+, that is, the lower contact set of v. As $v \le 0$ in Ω, we have $v < 0$ on Γ, or, equivalently $u > 0$ on Γ. Let $A(x) := [a_{ij}(x)]$, then

$$\mathrm{Tr}(A(x) D^2 v) = -a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} = -Lu \le -f(x), \text{ for } x \in \Gamma. \tag{3.8}$$

The analog of the inclusion (2.8) that we will now prove is

$$B(0; M/d) \subset \nabla v(\Gamma), \tag{3.9}$$

with $d = \mathrm{diam}(\Omega)$ and $B(0, M/d)$ the open ball centered at the origin of radius M/d. One way to see that is by sliding: let $p \in B(0; M/d)$ and consider the hyperplane that is the graph of

$$z_k(x) = p \cdot x - k.$$

Clearly, $z_k(x) < v(x)$ for k sufficiently large. As we decrease k, let $\bar{k}$ be the first value when the graphs of $v(x)$ and $z_{\bar{k}}(x)$ touch at a point x_1. Then we have $v(x) \geq z_{\bar{k}}(x)$ for all $x \in \Omega$. If x_1 is on the boundary $\partial\Omega$ then $v(x_1) = z_{\bar{k}}(x_1) = 0$, and we have

$$p \cdot (x_0 - x_1) = z_k(x_0) - z_k(x_1) \leq v(x_0) - 0 = -M,$$

whence $|p| \geq M/d$, which is a contradiction. Therefore, x_1 is an interior point, $x_1 \in \Gamma$, and $p = \nabla v(x_1)$.

Mimicking the proof of the isoperimetric inequality we use the area formula:

$$c_n \left(\frac{M}{d}\right)^n = |B(0; M/d)| \leq |\nabla v(\Gamma)| \leq \int_\Gamma \det(D^2 v(x))dx. \tag{3.10}$$

We now need a slightly more general version of (2.11): given any two $n \times n$ non-negative semi-definite matrices A and B we have

$$\det(AB^*) \leq \left(\frac{1}{n}\mathrm{Tr}(AB^*)\right)^n,$$

which is just geometric mean-arithmetic mean inequality for the eigenvalues of the matrix AB^*. The latter are all positive, as $AB^*w = \lambda w$ implies that $(B^*w, w) = \lambda(A^{-1}w, w)$ so that $\lambda = (B^*w, w)/(A^{-1}w, w) \geq 0$. Then (3.8) implies, for $x \in \Gamma$:

$$\det(D^2 v(x)) = \frac{1}{\det(A(x))}\det(A(x)D^2 v(x)) \leq \lambda^{-n}\det(A(x)D^2 v(x)) \tag{3.11}$$

$$\leq \lambda^{-n}\left(\frac{\mathrm{Tr}(A(x)D^2 v(x))}{n}\right)^n \leq \frac{1}{(n\lambda)^n}(-f(x))^n.$$

Integrating (3.11) and using (3.10) we get

$$M^n \leq \frac{(\mathrm{diam}(\Omega))^n}{c_n(n\lambda^n)} \int_\Gamma |f^-(x)|^n dx, \tag{3.12}$$

which is the ABP estimate.

The proof in the general case

We now drop the assumption that $b = 0$ and $c = 0$. As before, we assume that $M = \sup_\Omega u(x) > 0$ as otherwise there is nothing to prove, set $v = -u^+$ and let Γ be the lower contact set of v, which is also the upper contact set of u^+. Recall that $v < 0$ on Γ, hence it is smooth on this set and locally satisfies

$$Lv \leq -f \text{ on } \Gamma,$$

and, moreover, the matrix D^2v is non-negative on Γ.

In the general case we will need a slightly more general version of (3.10). Let g be a non-negative locally integrable function. We claim that

$$\int_{B(0;M/d)} g(z)dz \le \int_\Gamma g(\nabla v)\det[D^2 v]dx, \tag{3.13}$$

where $d = \operatorname{diam}(\Omega)$. In order to show that (3.13) holds we use use the version (2.4) of the area formula from Corollary 2.4, applied to ∇v in the set Γ:

$$\int\limits_{\nabla v(\Gamma)} g(z)dz \le \int\limits_\Gamma g(\nabla v)|\det(D^2 v)|dx, \tag{3.14}$$

where $|\det(D^2v)|$ is the Jacobian of the map $\nabla v : \Gamma \to \mathbb{R}^n$. All that remains is to recall that we have already shown that $B(0; M/d) \subset \nabla v(\Gamma)$ (see (3.9)), from which (3.13) follows immediately.

Note that while (3.13) seems to be an integral inequality, it is usually used with a given function $g(u)$. Thus the left side may be evaluated explicitly in terms of M – this gives an estimate of $\sup_\Omega u$ in terms of the right side that is an integral quantity and will be estimated in terms of the forcing f from the equation.

As in the previous step (see (3.11)), using the fact that $D^2v = -D^2u$ is non-negative on Γ, we obtain another form of (3.13):

$$\int_{B(0;M/d)} g(x)dx \le \int_\Gamma g(\nabla u)\left(-\frac{a_{ij}D_{ij}u}{n\lambda}\right)^n. \tag{3.15}$$

Note that neglecting f and c we have, roughly "$(-a_{ij}D_{ij}u)^n \le |b|^n|\nabla u|^n$" which suggests to take $g(p) = 1/p^n$ in (3.15). This function, however, is not integrable at the origin. Hence we take

$$g(p) = (|p|^n + \mu^n)^{-1} \tag{3.16}$$

with $\mu > 0$ to be determined.

First, we correct the above rough estimate in Γ, using the Hölder inequality, to

$$\begin{aligned}
0 &\le -a_{ij}D_{ij}^2u \le b_iD_iu + cu - f \le b_iD_iu - f \le |b|\cdot|\nabla u| + \frac{f^-}{\mu}\cdot\mu \\
&\le \left(|b|^n + \frac{(f^-)^n}{\mu^n}\right)^{1/n}\left(|\nabla u|^{n/(n-1)} + \mu^{n/(n-1)}\right)^{(n-1)/n} \\
&\le \left(|b|^n + \frac{(f^-)^n}{\mu^n}\right)^{1/n}\left(|\nabla u|^n + \mu^n\right)^{1/n}(1+1)^{(n-2)/n},
\end{aligned}$$

so that

$$\left(-a_{ij}D_{ij}^2u\right)^n \le \left(|b|^n + \frac{(f^-)^n}{\mu^n}\right)\left(|\nabla u|^n + \mu^n\right)2^{n-2}. \tag{3.17}$$

We now use (3.17) in (3.15) with g as in(3.16) to get

$$\int\limits_{|p|\le M/d} \frac{dp}{|p|^n + \mu^n} \le \frac{2^{n-2}}{\lambda^n n^n}\int\limits_\Gamma [|b|^n + \mu^{-n}(f^-)^n]dx.$$

The integral on the left side is easily computed to be

$$\int\limits_{|p|\le M/d} \frac{dp}{|p|^n+\mu^n} = c_n \int_0^{M/d} \frac{r^{n-1}dr}{r^n+\mu^n} = \frac{c_n}{n}\log\left(\frac{M^n}{d^n\mu^n}+1\right).$$

Thus we get, with the constant K_n that depends only on dimension n:

$$M \le \mu d\left(\exp\left\{\frac{K_n}{\lambda^n}\left[\|b\|^n_{L^n(\Gamma^+\cap\Omega^+)} + \frac{1}{\mu^n}\left\|f^-\right\|^n_{L^n(\Gamma^+\cap\Omega^+)}\right]\right\}-1\right)^{1/n}.$$

Now, if $f^-=0$ we let $\mu\to 0^+$ and get $M=0$, that is, $\sup_\Omega u\le 0$. If $f^-\ne 0$ then we choose

$$\mu = \left\|f^-\right\|_{L^n(\Gamma)}$$

and get

$$\sup_\Omega u \le C\left\|f^-\right\|_{L^n(\Gamma)}$$

with the constant C that depends only on dimension n, the ellipticity constant λ and the norm $\|b\|_{L^n(\Gamma)}$. □

4 The moving plane method

The following result on the radial symmetry of non-negative solutions was first proved by Gidas, Ni and Nirenberg. It is one of the first examples of a general phenomenon that positive solutions of elliptic equations tend to be monotonic in one form or other: another example we will treat in detail below is a result of Berestycki, Caffarelli and Nirenberg on monotonicity of solutions in unbounded domains, monotonicity of solutions is also a recurring theme in the theory of travelling waves in reaction-diffusion equations. It has been even shown that solutions of initial value semi-linear diffusion problems become monotonic in a finite time [53]. We present a simpler proof of the Gidas-Ni-Nirenberg theorem from [13]. The proof uses the moving plane method combined with the maximum principle for domains of a small volume.

Theorem 4.1 *Let $u\in C(\bar B_1)\cap C^2(B_1)$ be a positive solution of*

$$\begin{aligned}&\Delta u + f(u) = 0 \quad \text{in } B_1\\ &u=0 \ \text{on } \partial B_1\end{aligned}\tag{4.1}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\dfrac{\partial u}{\partial r}(x)<0$ for $x\ne 0$.

The proof is based on the following lemma.

Lemma 4.2 *Let Ω be a bounded domain that is convex in the x_1-direction and symmetric with respect to the plane $\{x_1=0\}$. Let $u\in C(\bar\Omega)\cap C^2(\Omega)$ be a positive solution of*

$$\begin{aligned}&\Delta u + f(u) = 0 \quad \text{in } \Omega\\ &u=0 \ \text{on } \partial\Omega\end{aligned}\tag{4.2}$$

with the function f that is locally Lipschitz in $\mathbb{R}$. Then u is symmetric with respect to x_1 and $\dfrac{\partial u}{\partial x_1}(x)<0$ for any $x\in\Omega$ with $x_1>0$.

Proof of Theorem 4.1. Theorem 4.1 follows immediately from Lemma 4.2. Indeed, first, Lemma 4.2 implies that $u(x)$ is decreasing in any given radial direction. Second, it follows from the same lemma that $u(x)$ is symmetric with respect to reflection with respect to any hyper-plane plane passing through the origin – this trivially implies that u is radially symmetric. □

Proof of Lemma 4.2. We write $x = (x_1, y) \in \Omega$ with $y \in \mathbb{R}^{n-1}$. We will prove that

$$u(x_1, y) < u(x_1^*, y) \text{ for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1. \tag{4.3}$$

This implies monotonicity for $x_1 > 0$. Next, letting $x_1^* \to -x_1$ we get $u(x_1, y) \le u(-x_1, y)$ for any $x_1 > 0$. Changing direction we get the reflection symmetry: $u(x_1, y) = u(-x_1, y)$.

Given any $\lambda \in (0, a)$, with $a = \sup_\Omega x_1$, define

$$\begin{aligned}
&\Sigma_\lambda = \{x \in \Omega : \ x_1 > \lambda\} \\
&T_\lambda = \{x_1 = \lambda\} \\
&\Sigma'_\lambda = \text{the reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda \\
&x_\lambda = (2\lambda - x_1, x_2, \dots, x_n), \text{the reflection of } x = (x_1, x_2, \dots, x_n) \text{ with respect to } T_\lambda.
\end{aligned}$$

Note that T_λ is our "moving plane". We define

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

The mean value theorem implies that w_λ satisfies

$$\Delta w_\lambda = f(u(x_\lambda)) - f(u(x)) = \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} w_\lambda = -c(x, \lambda) w_\lambda$$

in Σ_λ. This is a recurring trick that we use very often: the difference of two solutions of a semi-linear equation satisfies a "linear" equation with an unknown function c. However, we know a priori that the function c is bounded:

$$|c(x)| \le \mathrm{Lip}(f), \text{ for all } x \in \Omega. \tag{4.4}$$

The boundary $\partial\Sigma_\lambda$ consists of a piece of $\partial\Omega$, where $w_\lambda = -u(x_\lambda) < 0$ and of T_λ, where $w_\lambda = 0$. Summarizing, we have

$$\begin{aligned}
&\Delta w_\lambda + c(x, \lambda) w_\lambda = 0 \text{ in } \Sigma_\lambda \\
&w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 \text{ on } \partial\Sigma_\lambda,
\end{aligned} \tag{4.5}$$

with a bounded function $c(x, \lambda)$. We will show that

$$w_\lambda < 0 \text{ inside } \Sigma_\lambda \text{ for all } \lambda \in (0, a). \tag{4.6}$$

This implies in particular that w_λ assumes its maximum (equal to zero) over $\bar{\Sigma}_\lambda$ along T_λ. The Hopf lemma implies that

$$\left.\frac{\partial w_\lambda}{\partial x_1}\right|_{x_1 = \lambda} = 2 \left.\frac{\partial u}{\partial x_1}\right|_{x_1 = \lambda} < 0.$$

Given that λ is arbitrary it remains only to show that $w_\lambda < 0$ inside Σ_λ to establish monotonicity of u in x_1 for $x_1 > 0$. Another consequence of (4.6) is that

$$u(x_1, x') < u(2\lambda - x_1, x') \text{ for all } \lambda \text{ such that } x \in \Sigma_\lambda,$$

that is, for $\lambda \in (0, x_1)$, which is the same as (4.3).

In order to show that $w_\lambda < 0$ one would like to apply the maximum principle to (4.5). However, a priori the function $c(x, \lambda)$ does not have a sign so the usual maximum principle may not be used. Still, when λ is close to a, the maximum principle for narrow domains, as well as the maximum principle for a domain of small volume imply that $w_\lambda < 0$ inside Σ_λ. This is because $w_\lambda \leq 0$ on $\partial\Sigma_\lambda$, and $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$, and, in addition, $|\Sigma_\lambda| \leq \delta_c$. Here, δ_c is the volume so that the maximum principle for domains of small volume holds for the operator

$$Lu = \Delta u + c(x)u,$$

and domains D of volume $|D| \leq \delta_c$. Note that δ_c depends only on $\|c\|_{L^\infty(D)}$ that is controlled in our case by (4.4) and $\mathrm{Diam}(D)$, and we have $\mathrm{diam}(\Sigma_\lambda) \leq \mathrm{diam}(\Omega)$ for all λ, so, indeed, we may apply the maximum principle for domains of small volume to Σ_λ when λ is sufficiently close to a.

Let us now decrease λ and let (λ_0, a) be the largest interval of values so that $w_\lambda < 0$ inside Σ_λ for all $\lambda \in (\lambda_0, a)$. We want to show that $\lambda_0 = 0$. If $\lambda_0 > 0$ then by continuity $w_{\lambda_0} \leq 0$ in Σ_{λ_0}. Moreover, w_{λ_0} is not identically equal to zero on $\partial\Sigma_{\lambda_0}$. The strong maximum principle implies that

$$w_{\lambda_0} < 0 \text{ in } \Sigma_{\lambda_0}. \tag{4.7}$$

We will show that then

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \tag{4.8}$$

for sufficiently small $\varepsilon < \varepsilon_0$.

Here is the key idea and the reason why the maximum principle for domains of small volume is useful: choose a closed subset K of Σ_{λ_0} so that $|\Sigma_{\lambda_0} \backslash K| < \delta/2$ with $\delta > 0$ to be determined. Inequality (4.7) implies that there exists $\eta > 0$ so that $w_{\lambda_0} \leq -\eta < 0$ for any $x \in K$. By continuity we have $w_{\lambda_0-\varepsilon} < 0$ for any $x \in K$. Therefore, $w_{\lambda_0-\varepsilon} \leq 0$ both on $\partial\Sigma_{\lambda_0-\varepsilon}$ and on ∂K and thus on $\partial(\Sigma_{\lambda_0-\varepsilon} \backslash K)$. However, when ε is sufficiently small we have $|\Sigma_{\lambda_0-\varepsilon} \backslash K| < \delta$. Choose δ (once again, solely determined by $\|c\|_{L^\infty(\Omega)}$), so small that we may apply the maximum principle for domains of small volume to $w_{\lambda_0-\varepsilon}$ in $\Sigma_{\lambda_0-\varepsilon} \backslash K$. Then, we obtain $w_{\lambda_0-\varepsilon} \leq 0$ in $\Sigma_{\lambda_0-\varepsilon} \backslash K$. The strong maximum principle implies that $w_{\lambda_0-\varepsilon} < 0$ in $\Sigma_{\lambda_0-\varepsilon} \backslash K$. Putting two and two together we see that (4.8) holds. This, however, contradicts the choice of λ_0. □

5 The sliding method

The sliding method differs from the moving plane method in that one compares translations of a function rather than its reflections. We will illustrate it in this and the following sections on examples taken from [13] and [15]. The following result is the simplest application of the method.

Theorem 5.1 *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ which is convex in the x_1-direction. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \text{ in } \Omega \\ &u = \eta \text{ on } \partial\Omega \end{aligned} \tag{5.1}$$

with a Lipschitz continuous function f. Assume that for any three points $x' = (x_1', y)$, $x = (x_1, y)$ and $x'' = (x_1'', y)$ lying on a segment parallel to the x_1-axis, $x_1' < x_1 < x_1''$ with $x', x'' \in \partial\Omega$, the following hold:

$$\eta(x') < u(x) < \eta(x'') \text{ if } x \in \Omega \tag{5.2}$$

and

$$\eta(x') \leq \eta(x) \leq \eta(x'') \text{ if } x \in \partial\Omega. \tag{5.3}$$

Then u is monotone in x_1 in Ω:

$$u(x_1 + \tau) > u(x_1, y) \text{ for } (x_1, y), (x_1 + \tau, y) \in \Omega \text{ and } \tau > 0.$$

Furthermore, if f is differentiable, then $\dfrac{\partial u}{\partial x_1} > 0$ in Ω. Finally, u is the unique solution of (5.1) in $C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (5.2).

Assumption (5.2) is usually checked in applications from the maximum principle and is not as unverifiable and restrictive in practice as it might seem at a first glance. For instance, one may look at (5.1) in a rectangle $[-a, a]_x \times [0, 1]_y$ with the Dirichlet data $\eta = 0$ and $\eta = 1$ prescribed at $x = -a$ and $x = a$, respectively, while the zero Neumann data is prescribed along the lines $y = 0$ and $y = 1$. The function f is assumed to vanish at $u = 0$ and $u = 1$: $f(0) = f(1) = 0$, $f(s) \leq 0$ with $u \notin [0, 1]$. Then Theorem 5.1 still applies with these boundary conditions, and the maximum principle implies that $0 \leq u \leq 1$ so that (5.2) holds.

Proof. For $\tau \geq 0$ we let $u^\tau(x_1, y) = u(x_1 + \tau, y)$ be a shift of u. The function u^τ is defined on the set $\Omega^\tau = \Omega - \tau \mathbf{e}_1$ obtained from Ω by sliding it to the left a distance τ parallel to the x_1-axis. The monotonicity of u may be restated as

$$u^\tau > u \text{ in } D^\tau = \Omega^\tau \cap \Omega \text{ for any } \tau > 0, \tag{5.4}$$

and this is what we will prove. The strategy will be exactly the same as in the moving plane method: we first establish (5.4) using the maximum principle for domains of a small volume for τ close to the largest value τ_0 – that is, those that have been slid almost all the way to the left. Then we will start decreasing τ, sliding the domain Ω^τ to the right, and will show that you may decrease it all the way to $\tau = 0$ keeping (5.4) enforced.

Set $w^\tau(x) = u^\tau(x) - u(x) = u(x_1 + \tau, y) - u(x_1, y)$ – the function w^τ is defined in D^τ. Since u^τ satisfies the same equation as u, we have from the mean value theorem

$$\begin{aligned} &\Delta w^\tau + c^\tau(x) w^\tau = 0 \text{ in } D^\tau \\ &w^\tau \geq 0 \text{ on } \partial D^\tau \end{aligned} \tag{5.5}$$

where

$$c^\tau(x) = \frac{f(u^\tau(x)) - f(u(x))}{u^\tau(x) - u(x)}$$

is a uniformly bounded function:

$$|c^\tau(x)| \le \mathrm{Lip}(f). \tag{5.6}$$

The inequality on the boundary ∂D^τ in (5.5) follows from assumptions (5.2) and (5.3). Let $\tau_0 = \sup\{\tau > 0 : \ D^\tau \neq \emptyset\}$ be the largest shift of Ω to the left that we can make so that Ω and Ω^τ still have a non-zero intersection. The volume $|D^\tau|$ is small when τ is close to τ_0. As in the moving plane method, since the function $c^\tau(x)$ is uniformly bounded by (5.6), we may apply the maximum principle for small domains to w^τ in D^τ for τ close to τ_0, and conclude that $w^\tau > 0$ for such τ.

Then we start sliding Ω^τ back to the right, that is, we decrease τ from τ_0 to a critical position τ_1: let (τ_1, τ_0) be a maximal interval with $\tau_1 \ge 0$ so that $w^\tau \ge 0$ in D^τ for all $\tau \in (\tau_1, \tau_0]$. We want to show that $\tau_1 = 0$ and argue by contradiction assuming that $\tau_1 > 0$.

Continuity implies that $w^{\tau_1} \ge 0$ in D^{τ_1}. Furthermore, (5.2) implies that $w^{\tau_1}(x) > 0$ for all $x \in \Omega \cap \partial D^{\tau_1}$. The strong maximum principle then implies that $w^{\tau_1} > 0$ in D^{τ_1}.

Now we use the same idea as in the proof of Lemma 4.2: choose $\delta > 0$ so that the maximum principle holds for any solution of (5.5) in a domain of volume less than δ. Carve out of D^{τ_1} a closed set $K \subset D^{\tau_1}$ so that

$$|D^{\tau_1} \backslash K| < \delta/2.$$

We know that $w^{\tau_1} > 0$ on K, hence for ε small $w^{\tau_1-\varepsilon}$ is also positive on K. Moreover, for $\varepsilon > 0$ small, $|D^{\tau_1-\varepsilon} \backslash K| < \delta$. Furthermore, since

$$\partial(D^{\tau_1-\varepsilon} \backslash K) \subset \partial D^{\tau_1-\varepsilon} \cup K,$$

we see that

$$w^{\tau_1-\varepsilon} \ge 0 \text{ on } \partial(D^{\tau_1-\varepsilon} \backslash K).$$

Thus, $w^{\tau_1-\varepsilon}$ satisfies

$$\begin{aligned} &\Delta w^{\tau_1-\varepsilon} + c^{\tau_1-\varepsilon}(x) w^{\tau_1-\varepsilon} = 0 \text{ in } D^{\tau_1-\varepsilon} \backslash K \\ &w^{\tau_1-\varepsilon} \ge 0 \text{ on } \partial(D^{\tau_1-\varepsilon} \backslash K). \end{aligned} \tag{5.7}$$

The maximum principle for domains of small volume implies that $w^{\tau_1-\varepsilon} \ge 0$ on $D^{\tau_1-\varepsilon} \backslash K$. Hence $w^{\tau_1-\varepsilon} \ge 0$ in all of $D^{\tau_1-\varepsilon}$, and, as $w^{\tau_1-\varepsilon} \not\equiv 0$ on $\partial D^{\tau_1-\varepsilon}$, it is positive in $D^{\tau_1-\varepsilon}$. However, this contradicts the choice of τ_1. Therefore, $\tau_1 = 0$ and the function u is monotone in the x_1-variable.

Moreover, if f is differentiable, the derivative $u_1 = \dfrac{\partial u}{\partial x_1}$ satisfies an equation

$$\Delta u_1 + f'(u) u_1 = 0 \text{ in } \Omega.$$

As we already know $u_1 \ge 0$, and $u_1 \not\equiv 0$ identically, we conclude that $\dfrac{\partial u}{\partial x_1} > 0$ in Ω.

Finally, to show that such solution u is unique, we suppose that v is another solution. We argue exactly as before but with $w^\tau = u^\tau - v$. The same proof shows that $u^\tau \ge v$ for all $\tau \ge 0$. In particular, $u \ge v$. Interchanging the role of u and v we conclude that $u = v$. □

Another beautiful application of the sliding method allows to extend lower bounds obtained in one part of a domain to a different part by moving a sub-solution around the domain and observing that it may never touch a solution.

Lemma 5.2 *Let u be a positive function in an open connected set D satisfying*

$$\Delta u + f(u) \leq 0 \text{ in } D$$

with a Lipschitz function f. Let B be a ball with its closure $\bar{B} \subset D$, and suppose z is a function in $\bar{B}$ satisfying

$$\begin{aligned} &z \leq u \text{ in } B \\ &\Delta z + f(z) \geq 0, \text{ wherever } z > 0 \text{ in } B \\ &z \leq 0 \text{ on } \partial B. \end{aligned}$$

Then for any continuous one-parameter family of Euclidean motions (rotations and translations) $A(t)$, $0 \leq t \leq T$, so that $A(0) = \mathrm{Id}$ and $A(t)\bar{B} \subset D$ for all t, we have

$$z_t(x) = z(A(t)^{-1}x) < u(x) \text{ in } B_t = A(t)B. \tag{5.8}$$

Proof. The rotational invariance of the Laplace operator implies that the function z_t satisfies

$$\begin{aligned} &\Delta z_t + f(z_t) \geq 0, \text{ wherever } z_t > 0 \text{ in } B_t \\ &z_t \leq 0 \text{ on } \partial B_t. \end{aligned}$$

Thus the difference $w_t = z_t - u$ satisfies $\Delta w_t + c_t(x)w_t \geq 0$ wherever $z_t > 0$ in B_t with c_t bounded in B_t, where, as always,

$$c_t(x) = \begin{cases} \dfrac{f(z_t(x)) - f(u(x))}{z_t(x) - u(x)}, \text{ if } z_t(x) \neq u(x) \\ 0, \text{ otherwise.} \end{cases}$$

In addition, $w_t < 0$ on ∂B_t.

We now argue by contradiction. Suppose that there is a first t so that the graph of z_t touches the graph of u at a point x_0. Then, for that t, still $w_t \leq 0$ in B_t, $w_t(x_0) = 0$. As $u > 0$ in D, and $z_t \leq 0$ on ∂B_t, the point x_0 has to be inside B_t and thus $w_t \equiv 0$ in the whole component G of the set of points in B_t where $z_t > 0$ that contains x_0. Consequently, this is still true for all $\tilde{x} \in \partial G$ and thus ∂G lies inside B_t. But then $z_t(\tilde{x}) = u(\tilde{x}) > 0$ on ∂G, which contradicts the fact that $z_t = 0$ on ∂G. Hence the graph of z_t may not touch that of u and (5.8) follows. □

Lemma 5.2 is often used to "slide around" a sub-solution that is positive somewhere to show that solution itself is uniformly positive.

6 Monotonicity in Unbounded Domains

We now consider the monotonicity properties of bounded solutions of

$$\Delta u + f(u) = 0 \text{ in } \Omega \tag{6.1}$$

when the domain Ω is not bounded so that monotonicity may not be "forced" on the solution as in (5.2)-(5.3). We will consider two examples, the first one deals with the whole space and the version we present is fairly simple – the main result is that solution depends only on one variable, the second is more difficult – it addresses domains bounded by a graph of a function and shows monotonicity in any direction that does not touch the graph.

6.1 Monotonicity in $\mathbb{R}^n$

Our first example taken from the paper [9] by Berstycki, Hamel and Monneau deals with the whole space. We consider solutions of

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^n \tag{6.2}$$

which satisfy $|u| \le 1$ together with the asymptotic conditions

$$u(x', x_n) \to \pm 1 \text{ as } x_n \to \pm\infty \text{ uniformly in } x' = (x_1, \dots, x_{n-1}). \tag{6.3}$$

The given function f is Lipschitz-continuous on $[-1, 1]$. We assume that there exists $\delta > 0$ so that

$$f \text{ is non-increasing on } [-1, -1+\delta] \text{ and on } [1-\delta, 1]; \text{ and } f(\pm 1) = 0. \tag{6.4}$$

The prototypical example is $f(u) = u - u^3$. We will show that any solution of (6.2) with the asymptotic conditions (6.3) is actually one-dimensional.

Theorem 6.1 *Let u be any solution of (6.2)-(6.3) such that $|u| \le 1$. Then $u(x', x_n) = u_0(x_n)$ where u_0 is a solution of*

$$u_0'' + f(u_0) = 0 \text{ in } \mathbb{R},\ u_0(\pm\infty) = \pm 1.$$

Moreover, u is increasing with respect to x_n. Finally, such solution is unique up to a translation.

Without the uniformity assumption in (6.3) this is known as "the weak form" of the De Giorgi conjecture, and was resolved by Savin [55] who showed that all solutions are one-dimensional in $n \le 8$, and del Pino, Kowalczyk and Wei [23] who showed that non-planar solutions exist $n \ge 9$. The additional assumption of uniform convergence at infinity made in this section makes this question much easier. The full De Giorgi conjecture is that any solution of (6.4) in dimension $n \le 8$ with $f(u) = u - u^3$ such that $-1 \le u \le 1$ is one-dimensional. It is still open in this generality, to the best of our knowledge.

First, we state a version of the maximum principle for unbounded domains.

Lemma 6.2 *Let D be an open connected set in $\mathbb{R}^n$, possibly unbounded. Assume that $\bar{D}$ is disjoint from the closure of an infinite open cone Σ. Suppose there is a function $z \in C(\bar{D})$ that is bounded from above and satisfies for some continuous function $c(x) \le 0$*

$$\begin{aligned} &\Delta z + c(x) z \ge 0 \text{ in } D \\ &z \le 0 \text{ on } \partial D. \end{aligned} \tag{6.5}$$

Then $z \le 0$.

Proof. If the function $z(x)$ would, in addition, vanish at infinity:

$$\limsup_{|x| \to +\infty} z(x) = 0, \tag{6.6}$$

then the proof would be easy. Indeed, if (6.6) holds then we can find a sequence $R_n \to +\infty$ so that

$$\sup_{\Omega\cap\{|x|=R_n\}} z(x) \le \frac{1}{n}. \tag{6.7}$$

The usual maximum principle in the domain $D_n = D \cap B(0; R_n)$ imples that $z(x) \le 1/n$ in D_n. Letting $n \to \infty$ gives

$$z(x) \le 0 \text{ in } \Omega.$$

Our next task is to reduce the case of a bounded function z to (6.7). To do this we will construct a harmonic function $g(x)$ in D such that

$$|g(x)| \to +\infty \text{ as } |x| \to +\infty. \tag{6.8}$$

Since g is harmonic, the ratio $\sigma = z/g$ will satisfy the following equation in D:

$$\Delta\sigma + \frac{2}{g}\nabla g \cdot \nabla\sigma + c\sigma \ge 0.$$

This is similar to (6.5) but now σ does satisfy the asymptotic condition

$$\limsup_{x\in D, |x|\to\infty} \sigma(x) \le 0,$$

uniformly in $x \in D$. Moreover, $\sigma \le 0$ on ∂D. Hence one may apply the usual maximum principle to $\sigma(x)$, and conclude that $\sigma(x) \le 0$, which, in turn, implies that $z(x) \le 0$ in D.

In order to construct such harmonic function $g(x)$ in D, the idea is to decrease the cone Σ to a cone $\tilde{\Sigma}$ and to consider the principal eigenfunction ψ (with the corresponding eigenvalue $\mu > 0$) of the spherical Laplace-Beltrami operator in the region $G = S^{n-1}\backslash\tilde{\Sigma}$ with $\psi = 0$ on ∂G:

$$\begin{aligned} &\Delta_S\psi + \mu\psi = 0, \quad \psi > 0 \text{ in } G,\ \mu, \\ &\psi = 0 \text{ on } \partial G. \end{aligned}$$

Then, going to the polar coordinates $x = r\xi$, $r > 0$, $\xi \in S^{n-1}$, we set $g(x) = r^\alpha\psi(\xi)$, $\xi \in G$, defined on D, with $\alpha(n + \alpha - 2) = \mu$. With this choice of α the function g is harmonic:

$$\Delta g = \frac{\partial^2 g}{\partial r^2} + \frac{n-1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\Delta_S g = [\alpha(\alpha-1) + \alpha(n-1) - \mu]r^{\alpha-2}\Psi = 0.$$

Moreover, as $\mu > 0$ (the operator $(-\Delta_S)$ is positive), we have $\alpha > 0$, thus (6.8) also holds, and the proof is complete. □

This lemma will be most important in the proof of the Berestycki-Caffarelli-Nirenberg result later on. For now we will need the following corollary that we will use for half-spaces.

Corollary 6.3 *Let f be a Lipschitz continuous function, non-increasing on $[-1, -1+\delta]$ and on $[1-\delta, 1]$ for some $\delta > 0$. Assume that u_1 and u_2 satisfy*

$$\Delta u_i + f(u_i) = 0 \textit{ in } \Omega$$

and are such that $|u_i| \le 1$. Assume furthermore that $u_2 \ge u_1$ on $\partial\Omega$ and that either $u_2 \ge 1-\delta$ or $u_1 \le -1+\delta$ in Ω. If $\Omega \subset \mathbb{R}^n$ is an open connected set so that $\mathbb{R}^n\backslash\bar{\Omega}$ contains an open infinite cone then $u_2 \ge u_1$ in Ω.

Proof. Assume, for instance, that $u_2 \geq 1-\delta$ and set $w = u_1 - u_2$. Then

$$\Delta w + c(x,z)w = 0 \text{ in } \Omega$$

with

$$c(x) = \frac{f(u_1) - f(u_2)}{u_1 - u_2} \leq 0 \text{ where } w \geq 0.$$

Hence if the set $G = \{w > 0\}$ is not empty, we may apply the maximum principle to w in G (note that $w = 0$ on ∂G), and conclude that $w \leq 0$ in G giving a contradiction. □

Proof of Theorem 6.1. We are going to prove that u is increasing in any direction $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_n > 0$. This is sufficient as this means that

$$\frac{1}{\nu_n}\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} \alpha_j \frac{\partial u}{\partial x_j} > 0$$

for any choice of $\alpha_j = \nu_j/\nu_n$. It follows that all $\partial u/\partial x_j = 0$, $j = 1, \ldots, n-1$, so that u depends only on x_n and, moreover, $\partial u/\partial x_n > 0$.

Define $u^t(x) = u(x + t\nu)$, the goal is to show that $u^t(x) \geq u(x)$ for all $t \geq 0$. We start the sliding method with a very large t. Observe that there exists a real $a > 0$ so that

$$u(x', x_n) \geq 1 - \delta \text{ for all } x_n \geq a,$$

and

$$u(x', x_n) \leq -1 + \delta \text{ for all } x_n \leq -a.$$

Take $t \geq 2a/\nu_n$, then the functions u and u^t are such that

$$\begin{aligned} u^t(x', x_n) &\geq 1 - \delta && \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \geq -a \\ u(x', x_n) &\leq -1 + \delta && \text{for all } x' \in \mathbb{R}^{n-1} \text{ and for all } x_n \leq -a \\ u^t(x', -a) &\geq u(x', -a) && \text{for all } x' \in \mathbb{R}^{n-1}. \end{aligned} \tag{6.9}$$

Hence we may apply Corollary 6.3 separately in $\Omega_1 = \mathbb{R}^{n-1} \times (-\infty, -a)$ and $\Omega_2 = \mathbb{R}^{n-1} \times (-a, +\infty)$. In both cases we conclude that $u^t \geq u$ and thus

$$u^t \geq u \text{ in all of } \mathbb{R}^n \text{ for } t \geq 2a/nu_n.$$

Following the sliding method, we start to decrease t, and let

$$\tau = \inf\{t > 0,\ u^t \geq u \text{ in } \mathbb{R}^n\}.$$

By continuity, we still have $u^\tau \geq u$ in $\mathbb{R}^n$. We need to show that $\tau = 0$, and argue by contradiction. Assume that $\tau > 0$ and consider two cases.

Case 1. Suppose that

$$\inf_{D_a}(u^\tau - u) > 0, \quad D_a = \mathbb{R}^{n-1} \times [-a, a]. \tag{6.10}$$

The function u is globally Lipschitz continuous – this follows from the standard elliptic estimates [36]. This implies that there exists $\eta_0 > 0$ so that for all $\tau - \eta_0 < t < \tau$ we still have

$$u^t(x', x_n) > u(x', x_n) \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and for all } -a \leq x_n \leq a. \tag{6.11}$$

As $u(x',x_n)\geq 1-\delta$ for all $x_n\geq a$, it follows that

$$u^t(x',x_n)\geq 1-\delta \text{ for all } x_n\geq a \text{ and } t>0. \tag{6.12}$$

We may now apply Corollary 6.3 in the half-spaces $\{x_n>a\}$ and $\{x_n<-a\}$ to conclude that

$$u^{\tau-\eta}(x)>u(x)$$

everywhere in $\mathbb{R}^n$ for all $\eta\in[0,\eta_0]$. This contradicts the choice of τ. Hence the case (6.10) is impossible.

Case 2. Suppose that

$$\inf_{D_a}(u^\tau-u)=0,\quad D_a=\mathbb{R}^{n-1}\times[-a,a]. \tag{6.13}$$

We will use the usual trick of moving "the interesting part" of the domain to the origin and passing to the limit. There exists a sequence $\xi_k\in D_a$ so that

$$u^\tau(\xi_k)-u(\xi_k)\to 0 \text{ as } k\to\infty.$$

Let us translate: set

$$u_k(x)=u(x+\xi_k).$$

Then the standard elliptic estimates imply that $u_k(x)$ converge along a subsequence to a function $u_\infty(x)$, uniformly on compact sets. We have

$$u_\infty^\tau(0)=u_\infty(0),$$

and

$$u_\infty^\tau(x)\geq u_\infty(x),\quad \text{for all } x\in\mathbb{R}^n,$$

because $u_k^\tau\geq u_k$ for all k. The strong maximum principle implies that $u_\infty^\tau=u_\infty$, that is,

$$u_\infty(x+\tau\nu)=u_\infty(x),$$

that is, the function u_∞ is periodic in the ν-direction. However, as all $\xi_k\in D_a$, their n-th components are uniformly bounded $|(\xi_k)_n|\leq a$. It follows that the function u_∞ must satisfy the boundary conditions (6.3). This is a contradiction. Hence, this case is also impossible, and thus $\tau=0$. □

6.2 Monotonicity in general unbounded domains

We now consider the monotonicity properties of bounded solutions of

$$\begin{aligned}\Delta u+f(u)&=0 \text{ in } \Omega\\ u&=0 \text{ on } \partial\Omega,\end{aligned} \tag{6.14}$$

when the domain Ω is not bounded so that monotonicity may not be "forced" on the solution as in (5.2)-(5.3). We assume that u is bounded: $0<u\leq M<\infty$ in Ω and the domain Ω is defined by

$$\Omega=\{x\in\mathbb{R}^n:\ x_n>\phi(x_1,\ldots,x_{n-1})\}. \tag{6.15}$$

For simplicity, we assume that $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth, globally Lipschitz function, an interested reader may consult [15] for the additional slightly technical arguments required if we only assume that ϕ is a globally Lipschitz continuous function. A typical example would be a half-space Ω – the main result we are going to prove says that u has to be a monotonic function of the single variable x_n in this case.

We will assume that f is Lipschitz continuous on $\mathbb{R}^+$, $f(s) > 0$ on $(0,1)$ and $f(s) \le 0$ for $s \ge 1$. Furthermore, we assume that f satisfies

$$f(s) \ge \delta_0 s \text{ on } [0, s_0] \text{ for some } s_0 > 0, \tag{6.16}$$

and

$$\text{there exists } s_1 \text{ so that } f \text{ is non-increasing on } (s_1, 1). \tag{6.17}$$

The prototypical example is[5] $f(s) = s(1-s)$. The main result says that such u is unique, monotonic in x_n and tends to one as distance to the boundary tends to infinity.

Theorem 6.4 *The function u has the following properties:*
(i) it is monotonic with respect to x_n:

$$\frac{\partial u}{\partial x_n} > 0 \text{ in } \Omega,$$

(ii) $0 < u < 1$ *in* Ω
(iii) $u(x) \to 1$ *as* $\mathrm{dist}(x, \partial\Omega) \to \infty$, *uniformly in* Ω.
(iv) u is the unique bounded solution of (6.14) that is positive inside Ω.
(v) Let κ be the Lipschitz constant of the graph function ϕ, then given any collection of constants a_j, $j = 1, \ldots, n-1$ so that $\sum_{j=1}^{n-1} a_j^2 < \frac{1}{\kappa^2}$, we have

$$\frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} a_j \frac{\partial u}{\partial x_j} > 0 \text{ in } \Omega. \tag{6.18}$$

Part (v) implies that u is increasing in any direction ξ such that there exists an orthonormal change of variables $x \to z$ with the z_n-axis in the direction of ξ and $\partial\Omega = \{z_n = \tilde{\phi}(z')\}$ with a smooth function $\tilde{\phi}$.

When $\phi = 0$, that is, when Ω is a half-space, the constants α_j in (6.18) may be arbitrary which immediately implies that

$$\frac{\partial u}{\partial x_j} = 0 \text{ for all } j = 1, \ldots, n-1,$$

so that u has to be a function of x_n only in this case.

[5] Such nonlinearities arise naturally in reaction-diffusion modeling and are known as nonlinearities of the Fisher-KPP (for Kolmogorov, Petrovskii, Piskunov) type. Apart from the original papers [34, 43] which are both masterpieces, good recent introductions to reaction-diffusion problems are the books [10, 59], and the review [58], where many more references can be found.

Let us explain heuristically why the limit in (iii) holds. Under our assumptions on the function f, the ODE

$$\dot{u} = f(u)$$

has two steady states: $u = 0$ is unstable,while $u = 1$ is stable. Solutions of the elliptic problem (6.14) can be thought of as steady solutions of the parabolic problem

$$v_t = \Delta v + f(v) \text{ in } \Omega \tag{6.19}$$
$$v = 0 \text{ on } \partial\Omega,$$
$$v(0, x) = v_0(x). \tag{6.20}$$

The parabolic problem inherits from the ODE the stability of the steady state $v = 1$. The boundary condition $v = 0$ on $\partial\Omega$ prevents v from being close to one near the boundary but far away from the boundary its effect is weak, hence solutions tend to one as both distance from the boundary and time tend to infinity. This, in turn, is reflected in the behavior of the solutions of the elliptic problem as $|x| \to +\infty$.

Outline of the proof

The proof of Theorem 6.4 is fairly long and we prove each part separately. The general flow is as follows. First one uses the maximum principle of Lemma 6.2 to show that $0 < u < 1$, so that (ii) holds. Second, we show that $f(u) \to 0$ as $\text{dist}(x, \partial\Omega)) \to \infty$ – roughly speaking, because otherwise u would satisfy

$$\Delta u < -\varepsilon_0, \tag{6.21}$$

at infinity, with some $\varepsilon_0 > 0$ which is impossible as $0 < u < 1$.

It is easy to conclude from $f(u) \to 0$ that $u \to 1$. In the third step, uniqueness is proved by the sliding method. Finally, monotonicity is established by constructing a solution that is positive and monotonic. Uniqueness implies that the original solution coincides with that one and hence is itself monotonic. Such solution is constructed first on bounded domains and then we pass to the limit of the full domain. The tricky part is to make sure that the limit is positive – this is done by ensuring that solution we construct stays above u.

Proof of (ii) in Theorem 6.4

Let us assume that $u > 1$ somewhere and let D be a connected component of the set where $u > 1$. The set D lies outside an open cone since the function ϕ that defines the boundary $\partial\Omega$ is Lipschitz. Consider the function $z = u - 1$ in D. It satisfes

$$\Delta z = -f(u) \geq 0 \text{ in } D,$$

as $f(u) \leq 0$ in D. Furthermore, z vanishes on ∂D and thus Lemma 6.2 implies that $z \leq 0$ in D which is a contradiction. Thus, we have $u \leq 1$. Suppose $u(x_0) = 1$ for some $x_0 \in \Omega$, then $z = u - 1$ satisfies $z \leq 0$ in Ω, $z(x_0) = 0$ and

$$\Delta z + c(x)z = 0 \text{ in } \Omega,$$

with

$$c(x)=\begin{cases}\dfrac{f(u(x))}{u(x)-1}, \text{ if } u(x)<1\\ \quad 0, \text{ if } u(x)=1.\end{cases}$$

The function $c(x)$ is bounded and hence the strong maximum principle implies that $z \equiv 0$ in Ω which contradicts the fact that $z=-1$ on $\partial\Omega$. $\Box$

Proof of (iii) in Theorem 6.4

The proof that $u(x) \to 1$ as $\text{dist}(x, \partial\Omega) \to \infty$ is in two steps. First, we show that u is bounded away from zero at a fixed distance away from the boundary: $u(x) \geq \varepsilon_1$ if $\text{dist}(x, \partial\Omega) > R_0$. Second, we show that $f(u(x)) \to 0$ as $\text{dist}(x, \partial\Omega) \to \infty$. This implies that $u \to 1$, as u is bounded away from zero in this region, and $u=0$ and $u=1$ are the only zeros of $f(u)$ in the interval $0 \leq u \leq 1$.
Step 1: u is strictly positive away from the boundary.

Lemma 6.5 *There exist $\varepsilon_1 > 0$ and $R_0 > 0$ so that*

$$u(x) > \varepsilon_1 \textit{ if } \text{dist}(x, \partial\Omega) > R_0. \tag{6.22}$$

Proof. Let R_0 be so large that the principle eigenvalue λ_1 of the Dirichlet Laplacian in a ball $B(0; R_0)$ of radius R_0 is smaller than the constant δ_0 in (6.16). Let ϕ_1 be the corresponding positive eigenfunction with $\max \phi_1 = 1$:

$$\begin{aligned}&-\Delta\phi_1 = \lambda_1\phi_1 \text{ in } B(0; R_0),\\ &\phi_1 = 0 \text{ on } \partial B(0; R_0),\\ &\phi_1 > 0 \text{ in } B(0; R_0),\\ &\max_{x\in B(0;R_0)} \phi_1(x) = 1.\end{aligned} \tag{6.23}$$

Then the function $z = \varepsilon\phi_1$ is a sub-solution of our equation for $0 < \varepsilon \leq s_0$, that is,

$$\begin{aligned}&\Delta z + f(z) \geq 0 \text{ in } B(0; R_0)\\ &z = 0 \text{ on } \partial B(0; R_0).\end{aligned} \tag{6.24}$$

Let us choose a large enough so that $\overline{B(a; R_0)} \subset \Omega$ and set

$$\varepsilon_0 = \min_{\bar{B}(a;R_0)} u.$$

Clearly, we have $\varepsilon_0 > 0$. We set $\varepsilon_1 = \min(\varepsilon_0, s_0)$, then, as $\phi_1 \leq 1$ we have

$$\varepsilon_1\phi_1(x-a) \leq u(x) \text{ in } B(a; R_0).$$

Lemma 5.2 (we slide the ball $B(a; R_0)$ around the domain) implies now that

$$\varepsilon_1\phi_1(x-y) \leq u(x) \text{ in } B(y; R_0)$$

for all $y \in \Omega$ with $\text{dist}(y, \partial\Omega) > R_0$. In particular, $u(y) > \varepsilon_1$ for such y. $\Box$

We see from the proof that R_0 depends only on the function $f(s)$. The constant ε_1 in the above proof depends on the function u. However, we will next show that $u(x) \to 1$ as $\text{dist}(x, \partial\Omega) \to +\infty$. Therefore, we may choose the center a in the proof of Lemma 6.5 sufficiently far from the boundary so that

$$\min_{x \in B(a,R_0)} > s_0.$$

This will allow us to set $\varepsilon_1 = s_0$, hence both R_0 and ε_1 in the statement of Lemma 6.5 depend only on the function $f(s)$.

Step 2: the nonlinearity vanishes at infinity. As we have explained above, in order to complete the proof of part (iii) of Theorem 6.4 we show that $f(u(x)) \to 0$ as $\text{dist}(x, \partial\Omega) \to \infty$. Once again, the heuristic reason is that the function u can not have a uniformly negative Laplacian in too big a region without violating the condition $0 < u < 1$. Here is how that is formalized. Let $v(x)$ be the solution of

$$\begin{aligned} -\Delta v &= 1 \text{ in } B(0;1) \\ v &= 0 \text{ on } \partial B(0;1). \end{aligned}$$

It is given explicitly by

$$v(x) = \frac{1 - |x|^2}{2n},$$

with

$$\max_{B(0;1)} v = v(0) = 1/(2n).$$

Let also $|y| \geq R_0$ with R_0 as in the previous lemma, and set

$$\gamma(y) = \min\{f(s) : \ s \in [\varepsilon_1, u(y)]\}.$$

Here ε_1 is also as in (6.22). We claim that

$$\gamma(y) \leq \frac{2n}{[\text{dist}(y, \partial\Omega) - R_0]^2}. \tag{6.25}$$

This estimate immediately implies that $f(u(x)) \to 0$ as $\text{dist}(x, \partial\Omega) \to +\infty$.

In order to prove(6.25) we argue by contradiction: suppose that (6.25) fails, that is,

$$\gamma(y_0) > \frac{2n}{[\text{dist}(y_0, \partial\Omega) - R_0]^2}$$

for some y_0 with $\text{dist}(y_0, \partial\Omega) > R_0$. Fix $R < \text{dist}(y_0, \partial\Omega) - R_0$ so that

$$\frac{\gamma(y_0)}{2n} > \frac{1}{R^2}. \tag{6.26}$$

The function u cannot have a local minimum at y_0 since

$$\Delta u(y_0) = -f(y_0) < 0.$$

Thus, we may find y_1 close to y_0 so that $u(y_1) < u(y_0)$ and $\mathrm{dist}(y_1, \partial\Omega) > R_0 + R$. Lemma 6.5 implies that $u \geq \varepsilon_1$ in $B := B(y_1; R)$. Let

$$z(y) = \gamma(y_0) R^2 v \left(\frac{y - y_1}{R} \right),$$

then $\max_B z = \gamma(y_0) R^2/(2n)$ and z satisfies

$$\begin{aligned} -\Delta z &= \gamma(y_0) \text{ in } B \\ z &= 0 \text{ on } \partial B. \end{aligned} \tag{6.27}$$

Now we do "ballooning" (as opposed to "sliding") of z: let $z^\tau(x) = \tau z(x)$. Then for $\tau > 0$ small we have

$$z^\tau(x) < \varepsilon_1 \leq u(x) \text{ in } B.$$

As we increase τ, there is the first value τ_0 so that the graph of z^{τ_0} touches the graph of $u(y)$ at some point x_0. Since $z = 0$ on ∂B, x_0 has to be inside B. Also, as $z^{\tau_0}(y_1) \leq u(y_1)$, we have

$$u(x_0) = \tau_0 z(x_0) \leq \tau_0 z(y_1) = \frac{\tau_0 \gamma(y_0) R^2}{2n} \leq u(y_1) < u(y_0) < 1 \tag{6.28}$$

Hence, by the choice of R (see (6.26)) we have

$$\tau_0 < \frac{2n}{\gamma(y_0) R^2} < 1.$$

It follows that

$$w := \tau_0 z - u \leq 0 \text{ in } B, \; w(x_0) = \tau_0 z(x_0) - u(x_0) = 0. \tag{6.29}$$

Note that according to (6.28), $u(x_0) < u(y_0)$ and so in an neighborhood N of x_0 we still have $u(x) < u(y_0)$. The definition of $\gamma(y_0)$ implies that

$$\Delta u(x) \leq -\gamma(y_0) \text{ for } x \in N,$$

and thus

$$\Delta w(x) \geq -\tau_0 \gamma(y_0) + \gamma(y_0) > 0 \text{ for } x \in N,$$

as $\tau_0 < 1$. This contradicts the fact that w has a local maximum at x_0.

Therefore $\gamma(y)$ satisfies $\gamma(y) \to 0$ as $\mathrm{dist}(y, \partial\Omega) \to \infty$ and thus $f(u(y)) \to 0$ in this limit, which implies $u(y) \to 1$, since $u(y) \geq \varepsilon_1$ in this region. Moreover, the above proof shows that the rate at which $u(x) \to 1$ as $\mathrm{dist}(x, \partial\Omega) \to \infty$ depends only on function $f(s)$, that is, for any $\varepsilon > 0$ there exists L_ε so that for any positive solution $u(x)$ of (6.14) we have

$$u(x) > 1 - \varepsilon, \text{ if } \mathrm{dist}(x, \partial\Omega) > L_\varepsilon.$$

Proof of (iv) in Theorem 6.4

We now show uniqueness of a positive bounded solution of (6.14). Naturally, we will do this by sliding. In order to start sliding we will need the following estimate in strips.

Lemma 6.6 *For any $h > 0$ the solution is bounded away from 1 in the strip*

$$\Omega_h = \{x \in \Omega : \phi(x') < x_n < \phi(x') + h\}.$$

Proof. This follows immediately from the regularity of the solution $u(x)$ up to the boundary that follows from the standard elliptic estimates [36]. It is instructive also to see the argument is by contradiction and shifting. Assume there exists a sequence $\xi_j \in \Omega_h$ so that $u(\xi_j) \to 1$. We may shift all ξ_j to the origin, with Ω shifted to a domain $\Omega_j = \{x_n > \phi_j(x')\}$. The functions $\phi_j(x')$ are all translations of $\phi(x')$ (up to an additive constant) and thus all have the same Lipschitz constant. Thus along a subsequence they converge to a function $\hat{\phi}(x)$. The shifted domains converge to a domain $\hat{\Omega} = \{x_n > \hat{\phi}(x')\}$, with 0 an interior point of $\hat{\Omega}$, while the shifted solutions converge along a subsequence (this also follows from the standard elliptic estimates) to a solution of

$$\begin{aligned}
&\Delta \hat{u} + f(\hat{u}) = 0, \quad \text{in } \hat{\Omega} \\
&0 \le \hat{u} \le 1, \quad \text{in } \hat{\Omega}, \\
&\hat{u} = 0 \text{ on } \partial\Omega,
\end{aligned}$$

such that $\hat{u}(0) = 1$. This is impossible according to part (ii) of the present theorem that we have already proved. □

Let now u and w be a pair of positive bounded solutions of (6.14). Note that $d(x) := x_n - \phi(x') \to \infty$ implies $\text{dist}(x, \partial\Omega) \to \infty$, as the function ϕ is Lipschitz. Hence, part (iii) of Theorem 6.4 that we have already proved implies that both $u(x)$ and $w(x)$ tend to one uniformly as $d(x) \to \infty$. Hence there exists $A > 0$ so that

$$u(x), w(x) \ge s_1 \text{ if } d(x) \ge A, \tag{6.30}$$

with s_1 as in (6.17): $f(s)$ is non-increasing on $(s_1, 1)$. We set $\Omega^\varepsilon = \{x \in \Omega : \ d(x) > A\}$ and $\Omega_A = \{x \in \Omega : \ d(x) < A\}$. A key point is that once we show $u \ge w$ in Ω_A then this inequality propagates to the whole Ω. More generally, we have the following lemma.

Lemma 6.7 *Suppose that for some $\tau \ge 0$ the inequality*

$$u^\tau(x) = u(x + \tau \mathbf{e}_n) \ge w(x) \tag{6.31}$$

holds in $\bar{\Omega}_A$. Then (6.31) holds in all of Ω.

Proof. The proof is very similar to that of Corollary 6.3. Assume that (6.31) holds. The function $z = w - u^\tau$ satisfies an equation of the form

$$\Delta z + c(x) z = 0 \text{ in } \Omega^\varepsilon.$$

Both $w, u^\tau \ge s_1$ in Ω^ε, thus

$$c(x) = \frac{f(w(x)) - f(u^\tau(x))}{w(x) - u^\tau(x)} \le 0,$$

wherever $z(x) \geq 0$. Moreover, by assumption $z \leq 0$ on $\partial\Omega^\varepsilon$ – this is all we need from (6.31). We may now apply Lemma 6.2, the maximum principle for unbounded domains, to Ω^ε and conclude that $z \leq 0$ in Ω^ε, that is (6.31) holds in all of Ω. □

Let us now show that $u(x) \geq w(x)$ in Ω_A. We do that by the sliding method. Note that

$$u^\tau(x) = u(x + \tau\mathbf{e}_n) > w(x) \text{ in } \Omega_A \text{ for large } \tau > 0,$$

because $u^\tau(x) \to 1$ as $\tau \to +\infty$ (according to the already proved part (iii) of the present theorem) while Lemma 6.6 implies that $w(x)$ is bounded away from 1 in Ω_A. As has been our common practice, we let

$$T = \inf\{\tau > 0 : \ u^\tau(x) \geq w(x) \text{ in } \Omega_A\}.$$

By continuity,

$$u^T(x) \geq w(x) \text{ in } \Omega_A. \tag{6.32}$$

We have to prove that $T = 0$. Suppose that $T > 0$, then there is a sequence of points $x_j \in \Omega_A$ and a sequence $\tau_j < T$, $\tau_j \to T$, so that

$$u(x_j + \tau_j\mathbf{e}_n) < w(x_j). \tag{6.33}$$

Once again, we shift the points x_j to the origin. The domain Ω is moved to Ω_j, and, as before, along a subsequence, Ω_j converge to a domain $\hat{\Omega} = \{x : \ x_n > \hat{\phi}(x')\}$ with a Lipschitz function $\hat{\phi}$. The shifts of u and w converge to positive solutions $\hat{u}$ and $\hat{w}$ of (6.14) in $\hat{\Omega}$ – this also follows from the standard elliptic regularity estimates. As follows from (6.32), we have

$$\hat{u}(x + T\mathbf{e}_n) \geq \hat{w}(x) \text{ in } \hat{\Omega}_A. \tag{6.34}$$

Lemma 6.7 implies that this inequality holds in the whole domain $\hat{\Omega}$. But passing to the limit in (6.33) we obtain that at the origin

$$0 < \hat{u}(T\mathbf{e}_n) \leq \hat{w}(0).$$

This implies that

$$\hat{u}^T(0) = \hat{w}(0), \tag{6.35}$$

and in particular 0 is an interior point of $\hat{\Omega}$, as on the boundary of $\hat{\Omega}$ we have $\hat{w} = 0$ while $\hat{u}^T > 0$ on $\partial\hat{\Omega}$ since $T > 0$. We have now reached a contradiction: the function $\hat{z} = \hat{w} - \hat{u}^T$ satisfies an elliptic equation

$$\Delta\hat{z} + \hat{c}(x)\hat{z} = 0,$$

inequality (6.34) means that $\hat{z} \leq 0$, while (6.35) implies that $\hat{z}(0) = 0$ and $0 \in \hat{\Omega}$. It follows that $\hat{z} \equiv 0$. However, at the boundary $\partial\hat{\Omega}$, $\hat{w} = 0$ while $\hat{u}^T > 0$, hence $z < 0$ on $\partial\hat{\Omega}$, which is a contradiction. Thus, $T = 0$ and $u(x) \geq w(x)$ for all $x \in \Omega_A$, hence in all of Ω. Similarly, we can show that $w(x) \geq u(x)$ for all $x \in \Omega$, and uniqueness follows.

Proof of (i) in Theorem 6.4

One would like to use Theorem 5.1 to show monotonicity. The problem is that the domain Ω is unbounded. This should be remedied by the fact that $u \to 1$ as $\text{dist}(x, \partial\Omega) \to \infty$ – hence, one may think of infinity as another part of the boundary where the value $u = 1$ is prescribed that guarantees that condition (5.2) still "holds" with the boundary condition "$\eta(x') = 0$ and $\eta(x'') = 1$". In order to make this precise we will consider a family of approximating domains

$$\Omega_h = \{\phi(x') < x_n < \phi(x') + h\} \tag{6.36}$$

and consider a sequence $h_n \to \infty$. We will construct a monotonic solution w_h on Ω_h and let $h \to \infty$. The sequence w_{h_n} will converge to a limit function w along a subsequence $h_n \to \infty$. The function w will be a monotonic solution of (6.14) and uniqueness (the already proved part (iv) of the Theorem 6.4) will finish the proof. Moreover, in order to make sure that $w \neq 0$ identically we will construct w_h so that $w_h \geq u$ in Ω_h.

Let us construct the monotonic solution w. This is done in two steps. First, we consider the cylinder

$$\Omega_{h,R} = \{x \in \Omega_h : \ |x'| < R\}$$

with Ω_h as in (6.36). The standard Hölder regularity estimates up to the boundary (recall that the boundary of Ω is smooth) imply that there exist $M > 0$ and $\alpha > 0$ so that

$$|u(x) - u(y)| \leq M|x - y|^\alpha \text{ for } x, y \in \Omega. \tag{6.37}$$

Using the constants α and M as above, we define

$$\sigma(t) = \begin{cases} Mt^\alpha, & \text{for } 0 \leq t \leq M^{-1/\alpha}, \\ 1, & \text{for } t \geq M^{-1/\alpha}. \end{cases}$$

We consider $h > h_0 = 1 + M^{-1/\alpha}$ and define a continuous function σ_R on $\partial\Omega_{h,R}$:

$$\sigma_R(t) = \begin{cases} 0, & \text{for } x \in \partial\Omega, \\ 1, & \text{for } x \text{ s.t. } x_n = \phi(x') + h, \\ \sigma(x_n - \phi(x')), & \text{otherwise on } \partial\Omega_{h,R}. \end{cases}$$

Note that $\sigma_R \geq u$ on $\partial\Omega_h$ by (6.37). Let w the solution of

$$\begin{aligned} &\Delta w_{h,R} + f(w_{h,R}) = 0 \text{ in } \Omega_{h,R} \\ &w = \sigma_R \text{ on } \partial\Omega_{h,R}. \end{aligned} \tag{6.38}$$

Existence of a solution to (6.38) follows from the fact that it has a sub-solution $\underline{w} = u$ and a super-solution $\overline{w} = 1$. Indeed, start with $w_0 = \underline{w}$ and solve

$$\begin{aligned} &\Delta w_{j+1} - kw_{j+1} = -f(w_j) - kw_j \text{ in } \Omega_{h,R} \\ &w_{j+1} = \sigma_R \text{ on } \partial\Omega_{h,R}. \end{aligned}$$

Here k is the Lipschitz constant of f. First, we have

$$\Delta w_1 - kw_1 = -f(w_0) - kw_0 = -f(u) - ku = \Delta u - ku$$

and hence

$$\Delta(w_1 - u) - k(w_1 - u) = 0 \text{ in } \Omega_{h,R},$$

while $w_1 \geq u$ on $\partial\Omega_{h,R}$. Hence $w_1 \geq u \geq w_0$. The induction argument shows that

$$w_0 \leq w_1 \leq \cdots \leq \overline{w}, \tag{6.39}$$

because

$$\Delta(w_{j+1} - w_j) - k(w_{j+1} - w_j) = -k(w_j - w_{j-1}) - [f(w_j) - f(w_{j-1})] \leq 0$$

by the induction hypothesis. The last inequality in (6.39) also follows from induction applied to

$$\Delta(w_{j+1} - \bar{w}) - k(w_{j+1} - \bar{w}) = -k(w_j - \bar{w}) - (f(w_j) - f(\bar{w})) \geq 0.$$

Hence, w_j converge to a limit $w_{h,R}$ as $j \to +\infty$ – elliptic regularity implies that $w_{h,R}$ is a solution of (6.38).

Theorem 5.1 implies that $w_{h,R}$ is monotonic in x_n. The maximum principle implies that $w_{h,R} \geq u$.

We now pass to the limit $R \to \infty$. The standard elliptic estimates as before imply that $w_{h,R}$ converges along a subsequence $R_n \to \infty$ to a function $w_h \geq u$ that satisfies

$$\begin{aligned} &\Delta w_h + f(w_h) = 0 \text{ in } \Omega_h \\ &w_h = 0 \text{ on } \partial\Omega \\ &w_h = 1 \text{ on } \{x_n = \phi(x') + h\}. \end{aligned}$$

Finally we let $h \to \infty$, and by the same argument conclude that, along a subsequence, w_{h_n} converges to a monotonic solution of

$$\begin{aligned} &\Delta w + f(w) = 0 \text{ in } \Omega_h \\ &w = 0 \text{ on } \partial\Omega \end{aligned}$$

with $w \geq u$. Hence uniqueness of a positive bounded solution implies that u has to coincide with w and we are done. □

Proof of (v) of Theorem 6.4. This one is a trivial consequence of part (i): all it says is that u is monotonic in any direction ξ such that there exists an orthonormal basis with $\mathbf{e}_n$ along ξ so that the boundary $\partial\Omega$ may be represented as $z = \phi(z')$ in the new variables. □

Chapter 2

Heat kernel bounds

In this chapter we will mostly consider the Cauchy problem for the parabolic equations of the form

$$\begin{aligned}\phi_t &= \nabla \cdot (a(x)\nabla\phi), \\ \phi(0,x) &= \phi_0(x),\end{aligned} \tag{0.1}$$

in the whole space $x \in \mathbb{R}^n$, $t > 0$. The diffusion matrix $a(x)$ is assumed to be bounded and uniformly elliptic. We are interested in estimates for the solutions of (0.1) that would exhibit both temporal and spatial decay, as in the heat equation. Let us recall that solutions of the heat equation

$$\psi_t = \Delta\psi, \tag{0.2}$$

with the initial data $\psi(0,x) = \psi_0(x)$ are given by

$$\psi(t,x) = \int_{\mathbb{R}^n} G_0(t,x,y)\psi_0(y)dy. \tag{0.3}$$

The Green's function for the heat equation is given explicitly by

$$G_0(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-(x-y)^2/(4t)}. \tag{0.4}$$

Similarly, solutions of the inhomogeneous advection-diffusion equation (0.1) can be expressed in terms of the Green's function for this problem as

$$\phi(t,x,y) = \int_{\mathbb{R}^n} G(t,x,y)\phi_0(y)dy. \tag{0.5}$$

However, the Green's function is no longer given explicitly and the best we can hope for are interesting bounds for $G(t,x,y)$. We will show that, in some sense, solution of (0.1) behaves "almost exactly" as a solution of the heat equation. More precisely, there exists a constant $C > 0$ so that

$$\frac{1}{Ct^{n/2}} e^{-Cx^2/t} \leq G(t,x,y) \leq \frac{C}{t^{n/2}} e^{-x^2/(Ct)}. \tag{0.6}$$

This result is originally due to Nash [46]. We will follow here a more recent proof by Fabes and Stroock [27][1].

[1]The whole material of this chapter is taken from this paper by Fabes and Stroock.

Following Fabes and Stroock we will also explain that the heat kernel estimates (0.6) imply "everything you ever wanted to know about parabolic equations": such as the Hölder regularity of solutions, and Harnack inequality. There is a physical reason for that: thermodynamics tells us that solutions of heat equations tend to equilibrate. The bounds in (0.6) are simply a quantification of that. We will see that the Gaussian bounds imply that the oscillation $\mathrm{Osc}_{D_R}\phi$ of any solution ϕ over a set $D_R = \{|x - x_0| < R, |t - t_0| < R^2\}$ goes down to zero exponentially as $R \to 0$:

$$\mathrm{Osc}_{D_R} < \gamma \mathrm{Osc}_{D_{2R}},$$

with a constant $\gamma < 1$. This implies the Hölder bound on ϕ.

Divergence and non-divergence forms: intuition or integration?

We will consider in this section only equations in the divergence form, possibly with an incompressible drift:

$$\phi_t + u \cdot \nabla\phi = \nabla \cdot (a(x)\nabla\phi). \tag{0.7}$$

The practical reason to consider equations in the divergence form is that they are much more amenable to integration by parts than their counterpart in the non-divergence form

$$\phi_t + u \cdot \nabla\phi = a_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}. \tag{0.8}$$

On the other hand, solutions of (0.8) have a nice probabilistic interpretation. Consider the stochastic differential equation[2]

$$dX_t = -u(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x. \tag{0.9}$$

Here $W_t = (W_t^1, \dots, W_t^n)$ is the n-dimensional Brownian motion, that is, every component W_t^j, $j = 1, \dots, n$ is a standard Brownian motion, and B_t^j and B_t^k are independent for $k \neq j$. The matrix $\sigma(x)$ is symmetric and satisfies $\sigma^2(x) = 2a(x)$. This SDE is related to the PDE (0.8) in a way very similar to the connection between first order hyperbolic equations and ODE's. Let $\phi(t,x)$ be the solution of (0.8) with the initial data $\phi(0,x) = \phi_0(x)$. Then it is given "explicitly" by

$$\phi(t,x) = \mathbb{E}_x(\phi_0(X(t)). \tag{0.10}$$

Here X_t is the solution of the stochastic differential equation (0.9), and the subscript x in $\mathbb{E}_x$ refers to the fact that $X(t)$ starts at the point x at time $t = 0$.

This probabilistic interpretation provides a very good intuition for how solutions of the equations in the non-divergence form should behave. Much of this intuition applies also to solutions of equations in the divergence form (though the probabilistic interpretation has to be modified to take into account the additional drift coming from $\nabla a(x)$), and we will often appeal to it even when we consider equations in the divergence form.

[2]It would be far too ambitious for us to review the general theory of diffusions here, the reader may either think of solutions of stochastic differential equations as "randomized" solutions of classical ODEs, or consult [6, 7, 51] that all consider the connections between PDEs and diffusions.

1 The Nash inequality

Let us first review "why" solutions of the heat equation spread and decay. The integral of the solutions of the heat equation

$$\begin{aligned} &\psi_t = \Delta\psi, \\ &\psi(0,x) = \psi_0(x) \end{aligned} \tag{1.1}$$

with a rapidly decaying initial data $\psi_0(x)$ is conserved:

$$\int_{\mathbb{R}^n} \psi(t,x)dx = \int_{\mathbb{R}^n} \psi_0(x)dx := M_0. \tag{1.2}$$

They also preserve positivity: if $\psi_0 \geq 0$ then $\psi(t,x) > 0$ for all $t > 0$. Moreover, multiplying (1.1) by $|x|^2$ and integrating gives

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2\psi(t,x)dx = \int_{\mathbb{R}^n} |x|^2\Delta\psi(t,x)dx = 2n\int_{\mathbb{R}^n} \psi(t,x)dx = 2nM_0, \tag{1.3}$$

so that

$$M_2(t) := \int_{\mathbb{R}^n} |x|^2\psi(t,x)dx = 2nM_0t + \int_{\mathbb{R}^n} |x|^2\psi_0(x)dx. \tag{1.4}$$

As the second moment $M_2(t)$ grows linearly in time, while the total mass stays constant, solutions of the heat equation have to spread keeping their mass fixed. We learn from (1.4) two things: first, as an upper bound

$$\int_{\mathbb{R}^n} |x|^2\psi(t,x)dx \leq 2nM_0t + \int_{\mathbb{R}^n} |x|^2\psi_0(x)dx, \tag{1.5}$$

it tells you that "the mass outside of the ball $B(0,R)$ is small for any $R \gg t^{-1/2}$". On the other hand, as a lower bound

$$\int_{\mathbb{R}^n} |x|^2\psi(t,x)dx \geq 2nM_0t, \tag{1.6}$$

it tells you that there has to be some mass at distance $O(t^{1/2})$ from the origin – the mass can not be concentrated in a ball $B(0,R)$ of radius $R \ll t^{1/2}$. Hence, solutions have to spread over the ball of radius $O(t^{1/2})$. On the other hand, if it does so in a mass preserving way, the mass balance tells us that its maximum should roughly satisfy

$$I_0 \sim \psi_{max}(t)(t^{1/2})^n, \tag{1.7}$$

hence the maximum should decay as $\psi_{max} \sim t^{-n/2}$. The very last step (1.7), is, of course, just a rough ballpark estimate but a combination of the above bounds lies at the heart of the rigorous proof.

In order to estimate the decay for the heat equation in a more careful way, let us multiply (1.1) by ψ and integrate:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\psi(t,x)|^2dx = -\int_{\mathbb{R}^n} |\nabla\psi(t,x)|^2dx. \tag{1.8}$$

We now need a relation between the dissipation

$$D = \int_{\mathbb{R}^n} |\nabla\psi(t,x)|^2 dx, \tag{1.9}$$

the conserved mass

$$M_0 = \int_{\mathbb{R}^n} \psi(t,x) dx, \tag{1.10}$$

and the L^2-norm of ψ itself. It is given by the Nash inequality.

Theorem 1.1 *(The Nash inequality) There exists a constant $C > 0$ so that for any function $\phi(x) \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ we have*

$$\|\nabla\phi\|_{L^2}^2 \geq \frac{C\|\phi\|_{L^2}^{2+4/n}}{\|\phi\|_{L^1}^{4/n}}. \tag{1.11}$$

Proof. Using the Fourier transform

$$\hat{\phi}(\xi) = \int e^{-2\pi i\xi\cdot x}\phi(x)dx,$$

we have, for any $R > 0$:

$$\begin{aligned}\int |\phi(x)|^2 dx = \int |\hat{\phi}(\xi)|^2 d\xi &= \int_{|\xi|\leq R} |\hat{\phi}(\xi)|^2 d\xi + \int_{|\xi|\geq R} |\hat{\phi}(\xi)|^2 d\xi \\ &\leq C_n R^n \|\hat{\phi}\|_{L^\infty}^2 + \frac{1}{R^2}\int_{|\xi|\geq R} |\xi|^2 |\hat{\phi}(\xi)|^2 d\xi.\end{aligned} \tag{1.12}$$

We may now estimate the two terms in the right side as

$$\|\hat{\phi}\|_{L^\infty} \leq \|\phi\|_{L^1},$$

and

$$\int_{|\xi|\geq R} |\xi|^2|\hat{\phi}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |\xi|^2|\hat{\phi}(\xi)|^2 d\xi = \frac{1}{4\pi^2}\int |\nabla\phi(x)|^2 dx.$$

Going back to (1.12) we conclude that, for any $R > 0$ we have

$$\int |\phi(x)|^2 dx \leq C\left[R^n\|\phi\|_{L^1}^2 + R^{-2}\|\nabla\phi\|_{L^2}^2\right]. \tag{1.13}$$

Choosing

$$R = \left(\frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L^1}^2}\right)^{1/(n+2)}$$

leads to

$$\int |\phi(x)|^2 dx \leq C\|\nabla\phi\|_{L^2}^{2n/(n+2)}\|\phi\|_{L^1}^{4/(n+2)},$$

which is the same as (1.11). □

2 The temporal decay

Divergence form equations

Let us now explain how the Nash inequality can be used to obtain the temporal decay of solutions of the parabolic problem

$$\begin{aligned} &\phi_t = \nabla \cdot (a(x)\nabla\phi), \\ &\phi(0,x) = \phi_0(x), \end{aligned} \tag{2.1}$$

in the whole space $x \in \mathbb{R}^n$, $t > 0$. We assume that the matrix $a(x)$ is bounded and uniformly elliptic: for any $\xi \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$ we have

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \tag{2.2}$$

with some $\lambda, \Lambda > 0$. The next theorem shows that solutions of (2.1) obey the same decay bounds as solutions of the heat equation with constant coefficients.

Theorem 2.1 *Let the diffusion matrix $a(x)$ satisfy (2.2) and assume that the initial data $\psi_0(x)$ is sufficiently rapidly decaying. There exists a constant C that depends only on the dimension n so that the function $\psi(t,x)$ satisfies*

$$|\phi(t,x)| \le \frac{C}{(\lambda t)^{n/2}}\|\phi_0\|_{L^1}, \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$.

The proof of this theorem proceeds as follows. First, we will show, using the Nash inequality that $\phi(t,x)$ satisfies an $L^1 \to L^2$ decay estimate:

$$\|\phi(t)\|_{L^2} \le \frac{C}{(\lambda t)^{n/4}}\|\phi_0\|_{L^1}. \tag{2.4}$$

This means that the solution operator S_t that maps the initial data ϕ_0 to the solution of (2.1) at time t is a bounded operator from L^1 to L^2 for each $t > 0$ with its norm bounded as

$$\|S_t\|_{L^1 \to L^2} \le \frac{C}{(\lambda t)^{n/4}}. \tag{2.5}$$

Therefore, the adjoint operator S_t^* maps L^2 to L^∞ with the bound

$$\|S_t^*\|_{L^2 \to L^\infty} \le \frac{C}{(\lambda t)^{n/4}}. \tag{2.6}$$

We claim that the operator S_t is self-adjoint. Indeed, let $\phi(t,x)$ and $\psi(t,x)$ be the solutions of (2.1) with the initial data $\phi(0,x) = f(x)$ and $\psi(0,x) = g(x)$. We need to show that

$$\int f(x)\psi(t,x)dx = \int g(x)\phi(t,x)dx. \tag{2.7}$$

In order to see that, set

$$B(s) = \int \phi(s,x)\psi(t-s,x)dx,$$

then

$$\begin{aligned}\frac{dB}{ds} &= \int [\nabla\cdot(a(x)\nabla\phi(s,x))\psi(t-s,x) - \phi(s,x)\nabla\cdot(a(x)\nabla\psi(t-s,x))dx\\ &= \int [(a(x)\nabla\phi(s,x))\cdot\nabla\psi(t-s,x) - \nabla\phi(s,x)\cdot(a(x)\nabla\psi(t-s,x))dx = 0.\end{aligned}$$

It follows that

$$B(s) = B(0) \text{ for all } 0 \le s \le t.$$

Setting $s = t$ gives (2.7). Hence, the solution operator S_t is, indeed, self-adjoint and (2.6) means nothing but

$$\|S_t\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/4}}. \tag{2.8}$$

The next observation is that the operators S_t form a semi-group so that

$$S_t = S_{t/2}\circ S_{t/2}. \tag{2.9}$$

As $S_{t/2}$ maps L^1 to L^2 and, as we have just shown, it also maps L^2 to L^∞, we know from (2.9) that S_t maps L^1 to L^∞ with the norm bounded as

$$\|S_t\|_{L^1\to L^\infty} \le \|S_{t/2}\|_{L^1\to L^2}\|S_{t/2}\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/4}}\frac{C}{(\lambda t)^{n/4}} = \frac{C'}{(\lambda t)^{n/2}}. \tag{2.10}$$

This exactly means that estimate (2.3) holds. Therefore, it only remains to prove the $L^1 \to L^2$ estimate (2.4).

In order to show that (2.4) holds we multiply (2.1) by ϕ and integrate:

$$\frac{1}{2}\frac{d}{dt}\int |\phi(t,x)|^2dx = -\int (a(x)\nabla\phi(t,x)\cdot\nabla\phi(t,x))dx \le -\lambda\int |\nabla\phi(t,x)|^2dx. \tag{2.11}$$

We also integrate (2.1) in space to get

$$\int \phi(t,x)dx = \int \phi_0(x)dx := M_0. \tag{2.12}$$

We may assume that $\phi_0(x) \ge 0$, otherwise we decompose $\phi = \phi_1 - \phi_2$. Here ϕ_1 and ϕ_2 are solutions ϕ_1 and ϕ_2 of (2.1) with the initial data $\phi^+(x)$ and $\phi^-(x)$, respectively. The bound we prove for solutions with non-negative initial data will apply both to ϕ_1 and ϕ_2, hence to their difference ϕ.

If $\phi_0 \ge 0$, then (2.12) means that $\|\phi(t)\|_{L^1} = M_0$ for all $t > 0$. The Nash inequality implies then that

$$\int |\nabla\phi(t,x)|^2dx \ge \frac{C}{M_0^{4/n}}\left(M(t)\right)^{1+2/n}. \tag{2.13}$$

Here we have set

$$M(t) = \int |\phi(t,x)|^2dx.$$

We may now rewrite the inequality (2.11) as

$$\frac{dM}{dt} \le -\frac{C\lambda}{I_0^{4/n}} (M(t))^{1+2/n}. \tag{2.14}$$

Integrating this ODE in time gives

$$\frac{1}{M(0)^{2/n}} - \frac{1}{M(t)^{2/n}} \le -\frac{C\lambda t}{M_0^{4/n}}. \tag{2.15}$$

It follows that

$$M(t) \le \frac{CM_0^2}{(\lambda t)^{n/2}}. \tag{2.16}$$

This is exactly (2.4), hence the proof of Theorem 2.1 is complete. □

Equations with an incompressible drift

It turns out that the previous argument can be easily generalized to the Cauchy problem for parabolic equations with an incompressible drift, yielding decay estimates that are uniform in the drift. Consider the initial value problem

$$\begin{aligned} &\phi_t + u \cdot \nabla\phi = \nabla \cdot (a(x)\nabla\phi), \\ &\phi(0, x) = \phi_0(x), \end{aligned} \tag{2.17}$$

with a uniformly elliptic matrix $a(x)$ satisfying (2.2), and a divergence-free flow $u(x)$:

$$\nabla \cdot u(x) = 0 \text{ in } \mathbb{R}^n. \tag{2.18}$$

The divergence-free condition (2.18) means that the fluid is incompressible, that is, the solution map of an ODE

$$\dot{X}(t; x) = u(X), \quad X(0; x) = x \tag{2.19}$$

is measure-preserving. In other words, given any measurable set A and any $t > 0$ we have the following property: the Lebesgue measure of A equals to the Lebesgue measure of the set

$$A(t) = \{y \in \mathbb{R}^n : \ y = X(t; x) \text{ for some } x \in A\}. \tag{2.20}$$

In other words, the set A is not compressed, hence the term "incompressible". This property plays an enormously important role in the theory of fluids.

Theorem 2.2 *Solutions of (2.17) satisfy the estimate*

$$|\phi(t, x)| \le \frac{C}{(\lambda t)^{n/2}} \|\phi_0\|_{L^1}, \tag{2.21}$$

with a constant $C > 0$ that depends only dimension n and does not depend on the flow u, provided that the incompressibility constraint (2.18) holds.

The assumption that the flow $u(x)$ is divergence-free is very important and the conclusion of this theorem is false without this condition. The reason why estimate (2.21) holds with a constant that is uniform in all incompressible flows can be seen from the probabilistic interpretation of the solutions of (2.17) in the special case $a(x)$ is the identity matrix. Let X_t be the solution of a stochastic differential equation

$$dX_t = -u(X_t)dt + \sqrt{2}dW_t, \quad X_0 = x. \tag{2.22}$$

Here $W_t = (W_t^1, \ldots, W_t^n)$ is the n-dimensional Brownian motion. As we have mentioned, solution of the parabolic equation (2.17) can be written as[3]

$$\phi(t,x) = \mathbb{E}(\phi_0(X(t)). \tag{2.23}$$

Let us consider for simplicity the case when $\phi_0(x)$ is the characteristic function of a set $A \subset \mathbb{R}^n$, then

$$\phi(t,x) = \mathbb{P}(X(t) \in A), \tag{2.24}$$

and (2.21) says that

$$\mathbb{P}(X(t) \in A) \leq \frac{C}{(\lambda t)^{n/2}}|A|. \tag{2.25}$$

If the flow $u(x)$ is divergence free then the solution map $S_t : x \to y(t)$ of the ODE without diffusion,

$$\dot{y} = -u(y), \quad y(0) = x, \tag{2.26}$$

is measure preserving and thus "mixing things around". "Therefore", it is unable to keep the particle in any given set in the presence of a diffusion, and that is reflected in estimate (2.25) – the probability to visit a given set A tends to zero as $t \to +\infty$ uniformly in the flow u.

Proof. The proof follows that of Theorem 2.1 with one modification. We multiply (2.17) by ϕ and integrate. As u is divergence-free, the term involving the drift vanishes:

$$\int (u \cdot \nabla\phi)\phi dx = \frac{1}{2}\int u \cdot \nabla(\phi^2)dx = -\frac{1}{2}\int \phi(\nabla \cdot u)dx = 0. \tag{2.27}$$

This cancellation means that we still have the identity (2.9):

$$\frac{1}{2}\frac{d}{dt}\int |\phi(t,x)|^2 dx = -\int (a(x)\nabla\phi(t,x) \cdot \nabla\phi(t,x))dx \leq -\lambda \int |\nabla\phi(t,x)|^2 dx. \tag{2.28}$$

Moreover, as

$$\int (u \cdot \nabla\phi)dx = -\int \phi(\nabla \cdot u)dx = 0, \tag{2.29}$$

the integral of ϕ is still preserved:

$$\int \phi(t,x)dx = \int \phi_0(x)dx. \tag{2.30}$$

[3] As $a(x)$ is the identity matrix, there is no difference between the divergence and non-divergence forms of the equation, hence we may use the probabilistic interpretation directly.

Therefore, using the Nash inequality we may proceed as in the proof of Theorem 2.1 to obtain

$$\|\phi(t)\|_{L^2} \leq \frac{C}{(\lambda t)^{n/4}} \|\phi_0\|_{L^1}, \tag{2.31}$$

with a constant $C > 0$ that depends only on dimension n and not on the flow u. Once again, that means that the solution operator S_t for (2.17) satisfies the bound

$$\|S_t\|_{L^1 \to L^2} \leq \frac{C}{(\lambda t)^{n/4}}, \tag{2.32}$$

and its adjoint satisfies

$$\|S_t^*\|_{L^2 \to L^\infty} \leq \frac{C}{(\lambda t)^{n/4}}, \tag{2.33}$$

However, S_t is not self-adjoint when $u \not\equiv 0$. Rather, the adjoint operator S_t^* is the solution operator for the Cauchy problem

$$\begin{aligned} &\psi_t - u \cdot \nabla \psi = \nabla \cdot (a(x) \nabla \psi), \\ &\psi(0, x) = \psi_0(x). \end{aligned} \tag{2.34}$$

To verify this, set

$$B(s) = \int \phi(s, x) \psi(t - s, x) dx,$$

then

$$\begin{aligned} &\frac{dB}{ds} = \int [\nabla \cdot (a(x) \nabla \phi(s, x)) - u(x) \cdot \nabla \phi(s, x)] \psi(t - s, x) dx \\ &- \int \phi(s, x) [\nabla \cdot (a(x) \nabla \psi(t - s, x)) + u(x) \cdot \nabla \psi(t - s, x)] dx \\ &= \int [(a(x) \nabla \phi(s, x) \cdot \nabla \psi(t - s, x)) - \psi(t - s, x)(u(x) \cdot \nabla \phi(s, x))] dx \\ &- \int [(a(x) \nabla \psi(t - s, x) \cdot \nabla \phi(s, x)) - \psi(t - s, x)(u(x) \cdot \nabla \phi(s, x))] dx \\ &+ \int \psi(t - s, x) \phi(s, x) \nabla \cdot u(x) dx = 0, \end{aligned}$$

since $\nabla \cdot u = 0$. Therefore, $B(0) = B(t)$, that is,

$$\int \phi(0, x) \psi(t, x) dx = \int \phi(t, x) \psi(0, x),$$

which means exactly that S_t^* is the solution operator for (2.34). However, (2.34) has the same form as our original problem (2.17), with the flow u replaced by $(-u)$, which is also incompressible. Hence, from what we have already proved we know that

$$\|S_t^*\|_{L^1 \to L^2} \leq \frac{C}{(\lambda t)^{n/4}}. \tag{2.35}$$

This, in turn, implies that

$$\|S_t\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/4}}. \tag{2.36}$$

The rest is as in the proof of Theorem 2.1: the semigroup property implies that $S_t = S_{t/2}\circ S_{t/2}$ whence

$$\|S_t\|_{L^1\to L^\infty} \le \|S_{t/2}\|_{L^1\to L^2}\|S_t\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/2}}.$$

Therefore, (2.21) holds. □

3 Elliptic problems with an incompressible drift

Another application of the Nash inequality is to elliptic problems with an incompressible drift.

Theorem 3.1 *Let the flow $u(x)$ be divergence-free and let $\phi(x)$ be the solution of the elliptic problem*

$$\begin{aligned} &-\nabla\cdot(a(x)\nabla\phi) + u\cdot\nabla\phi = f(x) \ \text{in } \Omega, \\ &\phi = 0 \ \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

with $f(x)\in L^p(\Omega)$, $p > n/2$. There exists a constant $C(\Omega,n,p) > 0$ which depends on p, the ellipticity constant λ of the matrix a, and the domain Ω but not on the flow $u(x)$, so that

$$\|\phi\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{3.2}$$

The spirit of this theorem is very close to that of Theorem 2.2. Estimate (3.2) always holds for any flow u, whether divergence free or not – this is a standard elliptic regularity bound [36] – but with a constant C that depends on u in an uncontrolled way. The point is that the same constant in (3.2) works for all divergence free flows.

Exercise 3.2 Construct a flow u in the unit ball $B = \{|x|\le 1\}\subset\mathbb{R}^n$ which is not divergence-free, so that for the functions $\phi_A(x)$ which satisfy

$$\begin{aligned} &-\Delta\phi_A + u\cdot\nabla\phi = 1 \text{ in } B, \\ &\phi = 0 \text{ on } \partial B, \end{aligned} \tag{3.3}$$

we have

$$\lim_{A\to+\infty}\phi_A(x) = +\infty. \tag{3.4}$$

The reason for estimate (3.1) can be seen, once again, from the probabilistic interpretation of the solutions of (3.1). Let X_t be the solution of the SDE

$$dX_t = -u(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{3.5}$$

with a matrix σ such that $\sigma(x)\sigma^T(x) = 2a(x)$, and let τ be the first exit time from the domain Ω for the process X_t. Then solution of the boundary value problem (3.1) can be written as

$$\phi(x) = \mathbb{E}_x\left(\int_0^\tau f(X(s))ds\right). \tag{3.6}$$

Let us assume, once again, that $f(x) = \chi_A(x)$ is the characteristic function of a set A. Then (3.6) takes the form:

$$\phi(x) = \mathbb{E}_x\left(T_A\right), \tag{3.7}$$

where T_A is the total time the process X_t spends in the set A before exiting from Ω, and (3.2) says that

$$\mathbb{E}_x\left(T_A\right) \le C_p|A|^{1/p}, \tag{3.8}$$

for any $p > n/2$. This means that a combination of an incompressible flow and a diffusion can not keep a particle in any given set for too long time. If the flow is not divergence free then this is clearly not true – if a very strong flow points radially toward a given point then the particle will take a very long time to escape a small ball centered at that point.

One may wonder if we have some sort of a uniform lower bound for ϕ also: whether we can say, for instance, that if a ball $B(x_0, r)$ is contained strictly inside Ω, then solutions of, say,

$$\begin{aligned} &-\Delta\phi + u\cdot\nabla\phi = 1 \text{ in } \Omega, \\ &\phi = 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.9}$$

obey a uniform lower bound: $\phi(x) \ge C$ on $B(x_0, r)$ with the constant $C > 0$ that does not depend on u as long as u is divergence-free. The probabilistic interpretation for solutions of (3.9) is simple:

$$\phi(x) = \mathbb{E}_x(\tau), \tag{3.10}$$

where τ is the time the process X_t, solution of

$$dX_t = -u(X_t)dt + \sqrt{2}dW_t, \quad X_0 = x, \tag{3.11}$$

spends inside Ω before it exits this domain. It turns out that there is no lower bound on $\phi(x)$ that would be uniform in u – the reason is, roughly, that if u is very fast and very mixing then the particle will exit Ω very quickly with a very high probability – see [11, 18] for various results of this kind.

Proof of Theorem 3.1

We may assume that $f(x) \ge 0$ without loss of generality – if not, we decompose $f = f^+ - f^-$ and $\phi = \phi^+ - \phi^-$, where ϕ^+ and ϕ^- are solutions of (3.1) with f replaced by f^+ and f^-, respectively. We write $\phi(x)$, the solution of (3.1), as

$$\phi(x) = \int_0^\infty \psi(t, x)dt. \tag{3.12}$$

The function $\psi(t, x)$ satisfies the parabolic initial value problem

$$\begin{aligned} &\psi_t - \nabla\cdot(a(x)\nabla\psi) + u\cdot\nabla\psi = 0 \text{ in } \Omega, \\ &\psi(t, x) = 0 \text{ on } \partial\Omega, \\ &\psi(0, x) = f(x) \text{ in } \Omega. \end{aligned} \tag{3.13}$$

We will now show that there exists a pair of constants $C > 0$ and $\alpha > 0$ so that for any incompressible flow u and any solution of (3.13) with initial data $f(x)$ we have a uniform bound

$$|\psi(t,x)| \leq \frac{Ce^{-\alpha t}}{t^r}\|f\|_{L^1}, \tag{3.14}$$

with any $r > n/2$. The proof is close to that of Theorem 2.2, with a slight modification, we present the details for the convenience of the reader. First, multiplying (3.13) by ψ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 = -\int_\Omega (a(x)\nabla\psi\cdot\nabla\psi)dx \leq -\lambda\|\nabla\psi\|_{L^2}^2. \tag{3.15}$$

Using the Poincaré inequality

$$\|\psi\|_2 \leq C_p\|\nabla\psi\|_{L^2}, \tag{3.16}$$

for all functions $\psi \in H_0^1(\Omega)$, we conclude that there exists a constant $\alpha > 0$ so that

$$\|\psi(t_2)\|_2 \leq e^{-\alpha(t_2-t_1)}\|\psi(t_1)\|_2 \tag{3.17}$$

for any pair of times $t_2 \geq t_1 \geq 0$.

In order to estimate the dissipation term in (3.15) we will use the following Nash-type inequality in Ω.

Lemma 3.3 *For all $0 < s < 4/n$ there exists a constant C that depends on Ω and s so that for all smooth functions ϕ such that $\phi = 0$ on $\partial\Omega$, we have*

$$\|\nabla\phi\|_{L^2}^2 \geq C\frac{\|\phi\|_{L^2}^{s+2}}{\|\phi\|_{L^1}^s}. \tag{3.18}$$

Proof. The Poincaré inequality implies that we have

$$\|\phi\|_{L^q} \leq C_q\|\nabla\phi\|_{L^2}, \text{ for all } 1 < q < \frac{2n}{n-2}.$$

Next, using the Hölder inequality, with $1/\alpha + 1/\beta = 1$ we obtain:

$$\|\phi\|_{L^2}^2 = \int|\phi|^2 \leq \left(\int|\phi|\right)^{1/\alpha}\left(\int|\phi|^{(2-1/\alpha)\beta}\right)^{1/\beta} \leq C\|\phi\|_{L^1}^{1/\alpha}\|\nabla\phi\|_{L^2}^{2-1/\alpha},$$

provided that

$$\left(2-\frac{1}{\alpha}\right)\beta = \left(2-\frac{1}{\alpha}\right)\frac{\alpha}{\alpha-1} = \frac{2\alpha-1}{\alpha-1} < \frac{2n}{n-2},$$

or, equivalently:

$$\alpha > (n+2)/4. \tag{3.19}$$

Therefore, we have

$$\|\nabla\phi\|_{L^2}^2 \geq C\frac{\|\phi\|_{L^2}^{4\alpha/(2\alpha-1)}}{\|\phi\|_{L^1}^{2/(2\alpha-1)}} = C\frac{\|\phi\|_{L^2}^{s+2}}{\|\phi\|_{L^1}^s},$$

with $s = 2/(2\alpha - 1)$, that is, for $s < 4/n$. $\square$

We continue the proof of Theorem 3.1. Using Lemma 3.3 we may rewrite (3.15) as

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 \leq -C\frac{\|\psi\|_{L^2}^{s+2}}{\|\psi\|_{L^1}^s}. \tag{3.20}$$

In order to estimate the L^1-norm above we integrate (3.13) over Ω:

$$\frac{d}{dt}\int_\Omega \psi dx = \int_{\partial\Omega}(a(x)\nabla\psi\cdot\nu)dy, \tag{3.21}$$

as

$$\int_\Omega (u\cdot\nabla\psi)dx = 0$$

because u is divergence-free. Here ν is the outward normal to Ω. The parabolic maximum principle implies that $\psi(t,x) > 0$ for $x \in \Omega$ and $t > 0$, hence $\nabla\psi \cdot v < 0$ for any vector v such that $v \cdot \nu > 0$. The matrix $a(x)$ is positive-definite, hence the vector $v = a(x)\nu$ satisfies this condition for all $x \in \partial\Omega$, thus

$$(a(x)\nabla\psi\cdot\nu) < 0 \text{ on } \partial\Omega.$$

We conclude from (3.21) that

$$\|\psi(t)\|_{L^1} = \int_\Omega \psi(t,x)dx \leq \int_\Omega f(x)dx. \tag{3.22}$$

Using this inequality in (3.20) gives

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 \leq -C\frac{\|\psi\|_{L^2}^{s+2}}{\|f\|_{L^1}^s}, \tag{3.23}$$

hence $M(t) = \|\psi(t)\|_{L^2}$ satisfies

$$\frac{1}{M^{s+1}(t)}\frac{dM}{dt} \leq -\frac{C}{\|f\|_{L^1}^s}.$$

Integrating in time we obtain

$$\frac{1}{M^s(0)} - \frac{1}{M^s(t)} \leq -\frac{Ct}{\|f\|_{L^1}^s}.$$

Setting $q = 1/s$, we conclude that

$$\|\psi(t)\|_{L^2} \leq \frac{C}{t^q}\|f\|_{L^1}, \quad \text{for any } q > n/4. \tag{3.24}$$

Consider now the solution operator $\mathcal{P}_t:\ \psi_0 \to \psi(t)$. We have shown in (3.17) that

$$\|\mathcal{P}_t\|_{L^2\to L^2} \leq Ce^{-\alpha t}, \tag{3.25}$$

while (3.24) says that

$$\|\mathcal{P}_t\|_{L^1\to L^2} \le \frac{C}{t^q}, \quad \text{for any } q > n/4. \tag{3.26}$$

Once again, using the semi-group property we can write $\mathcal{P}_t = \mathcal{P}_{t/2} \circ \mathcal{P}_{t/2}$, and deduce that

$$\|\mathcal{P}_t\|_{L^1\to L^2} \le \|\mathcal{P}_{t/2}\|_{L^1\to L^2}\|\mathcal{P}_{t/2}\|_{L^2\to L^2} \le \frac{Ce^{-\alpha t/2}}{t^q}. \tag{3.27}$$

As we have already discussed, the adjoint operator $\mathcal{P}_t^*$ is simply the solution operator corresponding to the (also incompressible) flow $(-u)$. Therefore, we have the dual bound

$$\|\mathcal{P}_t^*\|_{L^1\to L^2} \le \frac{Ce^{-\alpha t/2}}{t^{1/s}},$$

which in turn implies that

$$\|\mathcal{P}_t\|_{L^2\to L^\infty} \le \frac{Ce^{-\alpha t/2}}{t^{1/s}}.$$

Putting these bounds together we obtain

$$\|\psi(t)\|_\infty = \|\mathcal{P}_t f\|_\infty = \|\mathcal{P}_{t/2} \circ \mathcal{P}_{t/2} f\|_\infty \le \|\mathcal{P}_{t/2}\|_{L^2\to L^\infty}\|\mathcal{P}_{t/2}\|_{L^1\to L^2}\|f\|_1 \le \frac{C_q e^{-\alpha t/2}}{t^{2q}}\|f\|_1,$$

which is (3.14).

The maximum principle also implies that we have a trivial bound

$$\|\psi\|_{L^\infty} \le \|f\|_{L^\infty}. \tag{3.28}$$

Interpolating between these two bounds[4] we get the estimate

$$\|\psi(t)\|_{L^\infty} \le \frac{C_\varepsilon e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}}\|f\|_{L^p}, \tag{3.29}$$

for any $\varepsilon > 0$. Now, (3.12) implies that

$$\|\phi\|_{L^\infty} \le C_\varepsilon \|f\|_{L^p} \int_0^\infty \frac{e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}} dt. \tag{3.30}$$

Note that for any $p > n/2$ we may choose $\varepsilon > 0$ sufficiently small so that the time integral in (3.30) is finite. It follows that

$$\|\phi\|_{L^\infty} \le C\|f\|_{L^p},$$

and the constant $C > 0$ is independent of the incompressible flow u. This finishes the proof of Theorem 3.1. □

[4]We use here the Riesz-Thorin interpolation theorem [40]. The corollary that we need says that if an operator A is a bounded linear operator from L^1 to L^∞ and also from L^∞ to L^∞ with $\|A\|_{L^\infty\to L^\infty} \le 1$, then A is also a bounded operator from L^p to L^∞ for any $p \in (1,\infty)$, with the norm bounded by $\|A\|_{L^p\to L^\infty} \le \|A\|_{L^1\to L^\infty}^{1/p}$.

4 The Gaussian bounds

The upper bounds in Theorems 2.1 and 2.2 are sharp – they have the correct decay in time as $t \to +\infty$ – but have no information about the spatial decay of solutions. Now, we will turn to the proof of Gaussian bounds on solutions of the parabolic equations in the divergence form. The matrix $a(x)$ is assumed to satisfy the usual uniform ellipticity condition:

$$\lambda|\xi|^2 \le (a(x)\xi \cdot \xi) \le \Lambda|\xi|^2, \tag{4.1}$$

for all $t > 0$, $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. As promised, we will follow the proof of Fabes and Stroock [27]. Consider the Cauchy problem:

$$\begin{aligned} &\phi_t = \nabla \cdot (a(x)\nabla\phi), \quad t > 0, \quad x \in \mathbb{R}^n, \\ &\phi(0,x) = g(x). \end{aligned} \tag{4.2}$$

Solution of (4.2) can be written in terms of Green's function $\Gamma(t,x,y)$ for (4.2) as

$$\phi(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x,y)g(y)dy. \tag{4.3}$$

Recall that $\Gamma(t,x,y)$ is the solution (in sense of distributions, if you wish) of the initial value problem

$$\begin{aligned} &\frac{\partial\Gamma(t,x,y)}{\partial t} = \nabla_x \cdot (a(x)\nabla\Gamma(t,x,y)), \quad t > 0, \quad x \in \mathbb{R}^n \\ &\Gamma(0,x,y) = \delta(x-y), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.4}$$

We will prove the following theorem.

Theorem 4.1 *There exists a constant $C > 0$ that depends only on the ellipticity constants λ and Λ of the matrix $a(t,x)$, and dimension n so that*

$$\frac{1}{Ct^{n/2}}e^{-C|x-y|^2/t} \le \Gamma(t,x,y) \le \frac{C}{t^{n/2}}e^{-|x-y|^2/(Ct)}, \tag{4.5}$$

for all $0 \le s < t$.

This result generalizes essentially verbatim to equations (4.1) with a diffusivity matrix $a(t,x)$ that depends both on t and x as long as the ellipticity condition (4.1) still holds for all $t \ge 0$ and all $x \in \mathbb{R}^n$. Then solution of the Cauchy problem starting at $t = s$:

$$\begin{aligned} &\phi_t = \nabla \cdot (a(t,x)\nabla\phi), \quad t > s, \quad x \in \mathbb{R}^n, \\ &\phi(s,x) = g(x) \end{aligned} \tag{4.6}$$

is expressed via Greens' function (which depends now both on t and s) as

$$\phi(t,x) = \int_{\mathbb{R}^n} \Gamma(t,s,x,y)g(y)dy. \tag{4.7}$$

Then one can show that

$$\frac{1}{C(t-s)^{n/2}}e^{-C|x-y|^2/(t-s)} \le \Gamma(t,s,x,y) \le \frac{C}{(t-s)^{n/2}}e^{-|x-y|^2/[C(t-s)]}. \tag{4.8}$$

The proof is nearly identical to what we will present except for somewhat more cumbersome notation so we will stick to the case when $a(x)$ is time-independent.

4.1 The proof of the upper bound

We first prove the upper bound.

Theorem 4.2 *There exists a constant $C > 0$ that depends on the dimension n so that*

$$\Gamma(t,x,y) \leq \frac{C}{(\lambda t)^{n/2}} e^{-|x-y|^2/8\Lambda t)}, \tag{4.9}$$

for all $t > 0$.

The constant 8 in the exponent in (4.9) is, of course, not optimal, we will point out the place in the proof where we lose the optimality: it can be replaced by any constant larger than 4, giving the upper bound

$$\Gamma(t,x,y) \leq \frac{C_\delta}{(\lambda t)^{n/2}} e^{-|x-y|^2/((4+\delta)\Lambda t)}, \tag{4.10}$$

with any $\delta > 0$.

The general strategy of the proof is similar to that of the uniform bound without the Gaussian factor in Theorem 2.2 with several important modifications. Rather than consider only the function $\phi(t,x)$ we will use the exponential moments of $\phi(t,x)$. Fix $\alpha \in \mathbb{R}^n$ and consider the function

$$\phi_\alpha(t,x) = e^{-\alpha\cdot x}\psi_\alpha(t,x).$$

Here $\psi_\alpha(t,x)$ is the solution of the initial value problem with exponentially weighted initial data:

$$\begin{aligned} &\frac{\partial \psi_\alpha}{\partial t} = \nabla\cdot(a(x)\nabla\psi_\alpha), \quad t > 0, \quad x \in \mathbb{R}^n, \\ &\psi_\alpha(0,x) = g(x)e^{\alpha\cdot x}. \end{aligned} \tag{4.11}$$

The key point is that L^∞ bounds for ϕ_α will give us decay estimates on the function $\phi(t,x)$ itself with a judiciously chosen α. We will show the following proposition.

Proposition 4.3 *There exists a constant $C > 0$ that depends only on the dimension n so that*

$$\|\phi_\alpha(t)\|_{L^\infty} \leq \frac{C}{(\lambda t)^{n/2}} e^{2\alpha^2 t\Lambda}\|g\|_{L^1}. \tag{4.12}$$

Exercise 4.4 Verify by a direct computation that the conclusion of Proposition 4.3 holds for the standard heat equation.

Let us explain how Theorem 4.2 follows from Proposition 4.3. Consider the operator P_t^α that maps $g(x)$ to $\phi_\alpha(t,x)$. It is given explicitly by

$$P_t^\alpha g(x) = e^{-\alpha\cdot x}\int \Gamma(t,x,y)g(y)e^{\alpha\cdot y}dy = \int K(t,x,y)g(y)dy, \tag{4.13}$$

with the integral kernel

$$K(t,x,y) = \Gamma(t,x,y)e^{\alpha\cdot(y-x)}. \tag{4.14}$$

Proposition 4.3 says that the operator P_t^α obeys the bound

$$\|P_t^\alpha g\|_{L^\infty} \le \frac{C}{(\lambda t)^{n/2}} e^{2\alpha^2 t\Lambda} \|g\|_{L^1}. \tag{4.15}$$

Therefore, the (non-negative) integral kernel $K(t,x,y)$ of the operator P_t^α satisfies the L^∞-bound

$$K(t,x,y) \le \frac{C}{(\lambda t)^{n/2}} e^{2\alpha^2 t\Lambda}, \tag{4.16}$$

and Green's function itself satisfies

$$\Gamma(t,x,y) \le \frac{C}{(\lambda t)^{n/2}} e^{2\alpha^2 t\Lambda} e^{\alpha\cdot(x-y)}. \tag{4.17}$$

As this estimate holds for all $\alpha \in \mathbb{R}^n$, we can take, in particular,

$$\alpha = \frac{1}{4t\Lambda}(y-x) \tag{4.18}$$

and get the desired Gaussian upper bound

$$\Gamma(t,x,y) \le \frac{C}{(\lambda t)^{n/2}} e^{-|x-y|^2/(8\Lambda t)}. \tag{4.19}$$

Thus, the Gaussian bound on the function $\Gamma(t,x,y)$ is a consequence of the L^∞ bound (4.12) on the functions ϕ_α. Our task, therefore, is to prove the $L^1 \to L^\infty$ decay estimate (4.15) for the operator P_t^α. In the proof of Theorem 2.2 we have obtained such bound for the solution operator S_t for the original Cauchy problem (4.2):

$$S_t g(x) = \int \Gamma(t,x,y) g(y) dy. \tag{4.20}$$

This was done by first establishing the $L^1 \to L^2$ bound on S_t using the Nash inequality, and then using the fact that S_t is self-adjoint, and duality to deduce the $L^2 \to L^\infty$ bound on S_t. The final step was to use the semi-group property

$$S_t = S_{t/2} \circ S_{t/2},$$

that gives the $L^1 \to L^\infty$ estimate for S_t as the product of $L^1 \to L^2$ and $L^2 \to L^\infty$ bounds for $S_{t/2}$. Here, the strategy is reversed: we will first show the $L^2 \to L^\infty$ bound and then use duality and semi-group property of P_t^α to obtain the $L^1 \to L^\infty$ bound for P_t^α.

The operators P_t^α share a lot of common properties with the solution operator S_t. They are not symmetric like S_t but the adjoint operator $P_t^{\alpha*}$ is obtained by simply switching the sign of α:

$$P_t^{\alpha*} = P_t^{-\alpha}. \tag{4.21}$$

Indeed, recall that, as we have seen in the proof of Theorem 2.1, the operator S_t is symmetric, meaning that

$$\Gamma(t,x,y) = \Gamma(t,y,x). \tag{4.22}$$

Therefore, the operator $P_t^{\alpha*}$ has the form

$$P_t^{\alpha*}f(x) = \int K(t,y,x)f(y)dy = e^{\alpha\cdot x}\int \Gamma(t,y,x)f(y)e^{-\alpha\cdot y}dy \qquad (4.23)$$
$$= e^{\alpha\cdot x}\int \Gamma(t,x,y)f(y)e^{-\alpha\cdot y}dy.$$

In other words, (4.21) holds.

Continuing our analogy with S_t, the operators P_t^α form a semi-group:

$$P_t^\alpha = P_{t-s}^\alpha \circ P_s^\alpha, \quad 0 \le s \le t. \qquad (4.24)$$

In order to verify (4.24) we will use the semigroup property of Green's function:

$$\Gamma(t,x,z) = \int \Gamma(t-s,x,y)\Gamma(s,y,z)dy. \qquad (4.25)$$

We deduce from this property that

$$\begin{aligned}(P_{t-s}^\alpha \circ P_s^\alpha)g(x) &= \int e^{-\alpha\cdot x}\Gamma(t-s,x,y)[P_s^\alpha g](y)e^{\alpha\cdot y}dy \\ &= \int e^{-\alpha\cdot(x-y)}\Gamma(t-s,x,y)\Gamma(s,y,z)e^{-\alpha\cdot(y-z)}g(z)dzdy \qquad (4.26)\\ &= \int e^{-\alpha\cdot(x-z)}\left(\int \Gamma(t-s,x,y)\Gamma(s,y,z)dy\right)g(z)dz \\ &= \int e^{-\alpha\cdot(x-z)}\Gamma(t,x,y)g(z)dz = P_t g(x),\end{aligned}$$

which is (4.24).

As we have mentioned, we will prove directly the $L^2 \to L^\infty$ bound rather than the $L^1 \to L^2$ bound as we did in the proof of Theorem 2.2.

Lemma 4.5 *There exists a constant $C > 0$ that depends only on the dimension n so that*

$$\|\phi_\alpha(t)\|_{L^\infty} \le \frac{C}{(\lambda t)^{n/4}}e^{2\alpha^2 t\Lambda}\|g\|_{L^2}, \qquad (4.27)$$

that is,

$$\|P_t^\alpha\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/4}}e^{2\alpha^2 t\Lambda}. \qquad (4.28)$$

Here is how the conclusion of Proposition 4.3 follows from Lemma 4.5. As $P_t^{\alpha*} = P_t^{-\alpha}$, the adjoint operator also satisfies the $L^2 \to L^\infty$ estimate (4.28) (with α replaced by $(-\alpha)$ which makes no difference):

$$\|P_t^{\alpha*}g\|_{L^\infty} \le \frac{C}{(\lambda t)^{n/4}}e^{2\alpha^2 t\Lambda}\|g\|_{L^2}. \qquad (4.29)$$

Therefore, for any function $g \in L^1$ and $f \in L^2$ we have

$$\int (P_t^\alpha g(x))f(x)dx = \int g(x)P_t^{\alpha*}f(x)dx \le \|g\|_{L^1}\|P_t^{\alpha*}f\|_{L^\infty} \le \frac{C}{(\lambda t)^{n/4}}e^{2\alpha^2 t\Lambda}\|g\|_{L^1}\|f\|_{L^2}, \qquad (4.30)$$

hence

$$\|P_t^\alpha g\|_{L^2} \le \frac{C}{(\lambda t)^{n/4}} e^{2\alpha^2 t\Lambda} \|g\|_{L^1}, \tag{4.31}$$

or

$$\|P_t^\alpha\|_{L^1\to L^2} \le \frac{C}{(\lambda t)^{n/4}} e^{2\alpha^2 t\Lambda}. \tag{4.32}$$

As the operators P_t^α form a semi-group, we have

$$P_t^\alpha = P_{t/2}^\alpha \circ P_{t/2}^\alpha. \tag{4.33}$$

Hence, as in the proof of Theorem 2.1 we get the bound

$$\|P_t^\alpha\|_{L^1\to L^\infty} \le \|P_{t/2}\|_{L^1\to L^2} \|P_{t/2}\|_{L^2\to L^\infty} \le \frac{C}{(\lambda t)^{n/2}} e^{2\alpha^2 t\Lambda}, \tag{4.34}$$

which proves Proposition 4.3.

The proof of Lemma 4.5

The most technical part of the proof of the upper bound in Theorem 4.2 is the $L^2 \to L^\infty$ bound for the operators P_t^α in Lemma 4.5. We will get a family of differential inequalities for the norms

$$M_p(t) = \|\phi_\alpha(t)\|_{L^{2p}}, \quad 1 \le p < +\infty, \tag{4.35}$$

of the form

$$\frac{dM_p}{dt} \le -\frac{C\lambda}{2p} \frac{M_p^{1+4p/n}}{M_{p/2}^{4p/n}} + \alpha^2 p \Lambda M_p, \tag{4.36}$$

together with the "boundary condition" at $p = 1$:

$$M_1(t) = \|\phi_\alpha(t)\|_{L^2} \le e^{\alpha^2 \Lambda t} \|g\|_{L^2}, \quad t \ge 0. \tag{4.37}$$

The second step will be to will use an ODE argument to get bounds on $M_p(t)$ in terms of $M_1(t)$ and finish the proof.

Let us show how (4.36) is obtained. The function ϕ_α satisfies the Cauchy problem

$$\begin{aligned} &\frac{\partial \phi_\alpha}{\partial t} = e^{-\alpha\cdot x} \nabla \cdot (a(x) \nabla (e^{\alpha\cdot x}\phi_\alpha)), \\ &\phi_\alpha(0,x) = g(x). \end{aligned} \tag{4.38}$$

Multiplying this equation by ϕ_α^{2p-1} gives

$$\frac{1}{2p}\frac{d}{dt}\int |\phi_\alpha(t,x)|^{2p} dx = \int e^{-\alpha\cdot x} \phi_\alpha^{2p-1} \nabla \cdot (a(x)\nabla(e^{\alpha\cdot x}\phi_\alpha)) dx. \tag{4.39}$$

Let us now rewrite the dissipation term in the right side as follows:

$$\begin{aligned} D &:= \int e^{-\alpha\cdot x}\phi_\alpha^{2p-1} \nabla\cdot(a(x)\nabla(e^{\alpha\cdot x}\phi_\alpha))dx = \int e^{-\alpha\cdot x}\phi_\alpha^{2p-1}(a(x)\alpha\cdot\nabla(e^{\alpha\cdot x}\phi_\alpha))dx \\ &-(2p-1)\int e^{-\alpha\cdot x}\phi_\alpha^{2p-2}(a(x)\nabla\phi_\alpha\cdot\nabla(e^{\alpha\cdot x}\phi_\alpha))dx = \int (a(x)\alpha\cdot\alpha)\phi_\alpha^{2p}dx \\ &-(2p-2)\int \phi_\alpha^{2p-1}(a(x)\alpha\cdot\nabla\phi_\alpha)dx - (2p-1)\int \phi_\alpha^{2p-2}(a(x)\nabla\phi_\alpha\cdot\nabla\phi_\alpha)dx. \end{aligned} \tag{4.40}$$

We will use Young's inequality for the middle term in the last identity above:

$$|(a(x)\alpha\cdot\nabla\phi)|\le\frac{|\phi_\alpha|}{2}(a(x)\alpha\cdot\alpha)+\frac{1}{2|\phi_\alpha|}(a(x)\nabla\phi_\alpha\cdot\nabla\phi_\alpha). \tag{4.41}$$

This gives

$$\begin{aligned} D&\le p\int(a(x)\alpha\cdot\alpha)\phi_\alpha^{2p}dx-p\int\phi_\alpha^{2p-2}(a(x)\nabla\phi_\alpha\cdot\nabla\phi_\alpha)dx\\ &\le p\Lambda|\alpha|^2\int|\phi_\alpha|^{2p}dx-\frac{\lambda}{p}\int|\nabla(\phi_\alpha^p)|^2dx. \end{aligned} \tag{4.42}$$

We will now use the Nash inequality for the function $\phi_\alpha^p(x)$:

$$\int|\nabla(\phi_\alpha)^p|^2dx\ge C_n\left(\int|\phi_\alpha|^{2p}dx\right)^{1+2/n}\left(\int|\phi_\alpha|^pdx\right)^{-4/n}. \tag{4.43}$$

Using this in (4.42) leads to the dissipation bound:

$$D\le p\Lambda|\alpha|^2\int|\phi_\alpha|^{2p}dx-\frac{C\lambda}{p}\left(\int|\phi_\alpha|^{2p}dx\right)^{1+2/n}\left(\int|\phi_\alpha|^pdx\right)^{-4/n}. \tag{4.44}$$

Going back to identity (4.39) and writing it in terms of the moments M_p gives

$$\frac{1}{2p}\frac{d}{dt}(M_p^{2p})\le p\Lambda|\alpha|^2M_p^{2p}-\frac{C\lambda}{p}\frac{M_p^{2p(1+2/n)}}{M_{p/2}^{p(4/n)}}, \tag{4.45}$$

or

$$\frac{dM_p}{dt}\le p\Lambda|\alpha|^2M_p-\frac{C\lambda}{p}\frac{M_p^{1+4p/n}}{M_{p/2}^{4p/n}}, \tag{4.46}$$

which is the differential inequality we were looking for. It is not closed as the right side involves not only M_p but also on $M_{p/2}$. The constant C here depends only on dimension n. One consequence of (4.46) is an exponentially growing in time bound

$$M_p(t)\le e^{p\Lambda|\alpha|^2t}\|g\|_{L^{2p}}, \tag{4.47}$$

which we will use later with $p=1$.

We now use the differential inequalities (4.46) to bound the moments $M_p(t)$ in terms of $M_{p/2}(t)$. Let us first take out the exponential factor: set

$$G_p(t)=M_p(t)e^{-p\Lambda|\alpha|^2t}. \tag{4.48}$$

Then (4.46) implies that G_p satisfies

$$\frac{dG_p}{dt}\le-\frac{C\lambda}{p}\frac{G_p^{1+4p/n}}{M_{p/2}^{4p/n}}e^{-p\Lambda|\alpha|^2t}e^{p\Lambda|\alpha|^2(1+4p/n)t}=-\frac{C\lambda}{p}\frac{G_p^{1+4p/n}}{M_{p/2}^{4p/n}}e^{4p^2\Lambda|\alpha|^2t/n}, \tag{4.49}$$

hence

$$\frac{n}{4p}\frac{d}{dt}(G_p^{-4p/n}) \geq \frac{C\lambda}{p}\frac{1}{M_{p/2}^{4p/n}}e^{4p^2\Lambda|\alpha|^2 t/n}. \tag{4.50}$$

It would be convenient to proceed if we knew that $M_{p/2}(t)$ were increasing in time. Let us see what we may expect in this regard: consider the standard heat kernel (corresponding to $g(x) = \delta(x)$)

$$G_0(t,x) = \frac{e^{-x^2/(4t)}}{(4\pi t)^{n/2}},$$

and compute

$$\begin{aligned} |M_p^{(0)}(t)|^{2p} &= \int e^{-2p\alpha\cdot x}G_0^{2p}(t,x)dx = \int e^{-2p\alpha\cdot x}\frac{e^{-px^2/(2t)}}{(4\pi t)^{pn}}dx \\ &= e^{2p\alpha^2 t}\int \exp\{-\frac{p}{2}(\frac{x}{\sqrt{t}} + 2\alpha\sqrt{t})^2\}\frac{dx}{(4\pi t)^{pn}} = C_p\frac{e^{2p\alpha^2 t}}{t^{pn-n/2}}. \end{aligned} \tag{4.51}$$

Observe that while $M_p^{(0)}(t)$ is not monotonic in time, it becomes monotonic if we multiply it by $t^{p(n-1)/(4p)}$. This motivates the following: set

$$\bar{M}_p(t) = \max_{0\leq s\leq t}\left[s^{(p-1)n/(4p)}M_p(s)\right], \tag{4.52}$$

so that $\bar{M}_p(t)$ is non-decreasing in time, and

$$\frac{1}{M_{p/2}(t)^{4p/n}} \geq \frac{t^{p-2}}{\bar{M}_{p/2}(t)^{4p/n}}. \tag{4.53}$$

Using this in inequality (4.50) gives, with another constant C that depends only on dimension n:

$$\frac{1}{G_p(t)^{4p/n}} \geq C\lambda\int_0^t \frac{s^{p-2}}{\bar{M}_{p/2}(s)^{4p/n}}e^{4p^2\Lambda|\alpha|^2 s/n}ds. \tag{4.54}$$

As the function $\bar{M}_p(s)$ is non-decreasing in s we deduce that

$$\frac{1}{G_p(t)^{4p/n}} \geq \frac{C\lambda}{\bar{M}_{p/2}(t)^{4p/n}}\int_0^t s^{p-2}e^{4p^2\Lambda|\alpha|^2 s/n}ds. \tag{4.55}$$

The integral in the right side can be evaluated explicitly for integer p but we will only estimate it:

$$\begin{aligned} &\int_0^t s^{p-2}e^{4p^2\Lambda|\alpha|^2 s/n}ds = \left(\frac{nt}{4p^2\Lambda|\alpha|^2}\right)^{p-1}\int_0^{4p^2\Lambda|\alpha|^2/n} s^{p-2}e^{ts}ds \\ &\geq \left(\frac{nt}{4p^2\Lambda|\alpha|^2}\right)^{p-1}\int_{(1-1/p^2)4p^2\Lambda|\alpha|^2/n}^{4p^2\Lambda|\alpha|^2/n} s^{p-2}e^{ts}ds \\ &\geq \left(\frac{nt}{4p^2\Lambda|\alpha|^2}\right)^{p-1}e^{(1-1/p^2)4p^2\Lambda|\alpha|^2 t/n}\int_{(1-1/p^2)4p^2\Lambda|\alpha|^2/n}^{4p^2\Lambda|\alpha|^2/n} s^{p-2}ds \\ &= \frac{t^{p-1}}{p-1}e^{(1-1/p^2)4p^2\Lambda|\alpha|^2 t/n}(1-(1-\frac{1}{p^2})^{p-1}). \end{aligned} \tag{4.56}$$

This estimate can be improved if we replace the lower limit of integration in (4.56) not by $(1-1/p^2)$ times the upper limit but by $(1-\delta/p^2)$ times the upper limit with an appropriately chosen $\delta > 0$. This improves the final constant in the estimates and gives the more precise version (4.10) of the Gaussian upper bound[5] but we will not pursue this avenue here as our hands are already full with technicalities. As a slight simplification, the last factor above satisfies

$$1-(1-\frac{1}{p^2})^{p-1} \le \frac{K}{p}, \quad \text{for all } p \ge 1, \tag{4.57}$$

with a universal constant K. Going back to (4.55) we obtain

$$\frac{1}{G_p(t)^{4p/n}} \ge \frac{\lambda}{\bar{M}_{p/2}(t)^{4p/n}} \frac{Kt^{p-1}}{p^2} e^{(1-1/p^2)4p^2\Lambda|\alpha|^2 t/n}. \tag{4.58}$$

We re-write this inequality in terms of $M_p(t)$:

$$\begin{aligned} M_p(t) &\le C^{n/(4p)} \bar{M}_{p/2}(t) \left(\frac{p^2}{\lambda t^{p-1}}\right)^{n/(4p)} e^{p\Lambda\alpha^2 t} e^{-(1-1/p^2)4p^2\Lambda|\alpha|^2 t/(4p)} \\ &= C^{n/(4p)} \bar{M}_{p/2}(t) \left(\frac{p^2}{\lambda t^{p-1}}\right)^{n/(4p)} e^{\Lambda|\alpha|^2 t/p}. \end{aligned} \tag{4.59}$$

Multiplying both sides by $t^{(p-1)n/(4p)}$ gives

$$M_p(t) t^{(p-1)n/(4p)} \le C^{n/(4p)} \bar{M}_{p/2}(t) \left(\frac{p^2}{\lambda}\right)^{n/(4p)} e^{\Lambda|\alpha|^2 t/p}. \tag{4.60}$$

Therefore, for any $0 \le s \le t$ we have

$$M_p(s) s^{(p-1)n/(4p)} \le C^{n/(4p)} \bar{M}_{p/2}(s) \left(\frac{p^2}{\lambda}\right)^{n/(4p)} e^{\Lambda|\alpha|^2 s/p} \le C^{n/(4p)} \bar{M}_{p/2}(t) \left(\frac{p^2}{\lambda}\right)^{n/(4p)} e^{\Lambda|\alpha|^2 t/p}. \tag{4.61}$$

Taking the supremum over all $0 \le s \le t$ we arrive at

$$\bar{M}_p(t) \le C^{n/(4p)} \bar{M}_{p/2}(t) \left(\frac{p^2}{\lambda}\right)^{n/(4p)} e^{\Lambda|\alpha|^2 t/p}. \tag{4.62}$$

Taking $p = 2^k$ we deduce that for all $k \ge 1$ we have

$$\bar{M}_{2^k}(t) \le \frac{C}{\lambda^{n/4}} \bar{M}_1(t) e^{\Lambda|\alpha|^2 t} = C M_1(t) e^{\Lambda|\alpha|^2 t}, \tag{4.63}$$

since

$$\prod_{k=1}^{\infty} (2^k)^{1/2^k} = \exp\left[(\log 2) \sum_{k=1}^{\infty} \frac{k}{2^k}\right] < +\infty. \tag{4.64}$$

Now, we are almost done: (4.63) means that

$$\|\phi_\alpha\|_{L^{2^k}} \le \frac{C e^{\alpha^2\Lambda t} e^{\Lambda|\alpha|^2 t}}{\lambda^{n/4} t^{(2^k-1)n/(4\cdot 2^k)}} \|g\|_{L^2}, \tag{4.65}$$

[5] In particular, the constant 8 in (4.10) can be turned into $4+\delta$ for any $\delta > 0$ which is nearly optimal.

for all $k \geq 1$. Passing to the limit $k \to +\infty$, it follows that

$$\|\phi_\alpha\|_{L^\infty} \leq \frac{Ce^{2\alpha^2\Lambda t}}{(\lambda t)^{n/4}}\|g\|_{L^2}, \tag{4.66}$$

with a constant $C > 0$ that depends only on the dimension n, as we have claimed. This competes the proof of Lemma 4.3 and Theorem 2.2.

4.2 The proof of the lower bound

In this section we prove the lower Gaussian bound for Green's function.

Theorem 4.6 *There exists a constant $C > 0$ that depends only on the ellipticity constants λ, Λ and dimension n so that*

$$\Gamma(t,x,y) \geq \frac{1}{Ct^{n/2}}e^{-C(x-y)^2/t}. \tag{4.67}$$

We will not try to track the dependence of the constant on λ and Λ as we did in the proof of Theorem 4.2, though that can also be done albeit at the expense of rather long expressions. Thus, throughout this proof we will denote by C various constants that depend on λ, Λ and dimension n. The main ingredient in the proof of Theorem 4.6 is the uniform lower bound on $\Gamma(t,x,y)$ on the set

$$\{|x-y| \leq \sqrt{t}\}.$$

Theorem 4.7 *There exists a constant C that depends only on the dimension n and ellipticity constants λ and Λ so that*

$$\Gamma(t,x,y) \geq \frac{1}{Ct^{n/2}}, \tag{4.68}$$

for all x, y such that $|x-y| \leq \sqrt{t}$.

The uniform lower bound in Theorem 4.7 is actually sufficient to produce the Gaussian decay in Theorem 4.6, and this is what we show first. Without loss of generality we may assume that $y = 0$. We need to show that

$$\Gamma(t,x,y) \geq \frac{1}{Ct^{n/2}}e^{-Cx^2/t} \tag{4.69}$$

Theorem 4.7 implies that we only need to consider $|x| \geq \sqrt{t}$. Let $x \in \mathbb{R}^n$, $t > 0$ and k be the smallest integer larger than $4|x|^2/t$:

$$k - 1 \leq \frac{4|x|^2}{t} < k. \tag{4.70}$$

Consider a sequence of balls

$$B_j = B\Big(\frac{jx}{k}, \frac{\sqrt{t}}{2\sqrt{k}}\Big) = \Big\{y: \ \big|y - \frac{j}{k}x\big| \leq \frac{\sqrt{t}}{2\sqrt{k}}\Big\}, \ \ j = 1, \dots, k-1. \tag{4.71}$$

As

$$\frac{|x|}{k} < \frac{\sqrt{t}}{2\sqrt{k}}, \tag{4.72}$$

each pair of consecutive balls B_j and B_{j+1} overlap, and, moreover, the center of B_{j+1} lies inside B_j and vice versa. In particular, the origin $y = 0$ lies inside B_1, and the point x lies inside B_{k-1}. Then, given any collection of points $\xi_l \in B_l$ they satisfy the following properties:

$$|\xi_1| \leq \frac{\sqrt{t}}{\sqrt{k}}, \quad |x - \xi_{k-1}| \leq \frac{\sqrt{t}}{\sqrt{k}}, \tag{4.73}$$

and

$$|\xi_{l+1} - \xi_l| \leq \frac{\sqrt{t}}{\sqrt{k}}, \text{ for all } 1 \leq k \leq k-2. \tag{4.74}$$

The semigroup property of Green's function $\Gamma(t, x, y)$ implies that

$$\begin{aligned}\Gamma(t,x,0) &= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \Gamma\big(\frac{t}{k}, x, \xi_{k-1}\big)\Gamma\big(\frac{t}{k}, \xi_{k-1}, \xi_{k-2}\big) \dots \Gamma\big(\frac{t}{k}, \xi_1, 0\big) d\xi_1 \dots d\xi_{k-1} \qquad (4.75)\\ &\geq \int_{B_{k-1}} d\xi_{k-1} \int_{B_{k-2}} d\xi_{k-2} \dots \int_{B_1} d\xi_1 \Gamma\big(\frac{t}{k}, x, \xi_{k-1}\big)\Gamma\big(\frac{t}{k}, \xi_{k-1}, \xi_{k-2}\big) \dots \Gamma\big(\frac{t}{k}, \xi_1, 0\big).\end{aligned}$$

The uniform bound in Theorem 4.7 together with the bounds on distances (4.73) and (4.74) imply that

$$\Gamma\big(\frac{t}{k}, \xi_j, \xi_{j-1}\big) \geq \frac{k^{n/2}}{Ct^{n/2}}, \quad \text{for all } 2 \leq j \leq k-1, \tag{4.76}$$

as well as

$$\Gamma\big(\frac{t}{k}, x, \xi_{k-1}\big) \geq \frac{k^{n/2}}{Ct^{n/2}}, \quad \Gamma\big(\frac{t}{k}, \xi_1, 0\big) \geq \frac{k^{n/2}}{Ct^{n/2}}. \tag{4.77}$$

Using these estimates in (4.75) leads to

$$\Gamma(t,x,0) \geq |B_1|^{k-1} \left(\frac{k^{n/2}}{Ct^{n/2}}\right)^k = C_n^{k-1} \left(\frac{\sqrt{t}}{\sqrt{k}}\right)^{n(k-1)} \left(\frac{k^{n/2}}{Ct^{n/2}}\right)^k = \frac{C_n^{k-1} k^{n/2}}{C^k t^{n/2}} = DK_0^k \left(\frac{k}{t}\right)^{n/2}, \tag{4.78}$$

with constants K_0 and D that depend on λ and Λ. As $4x^2/t < k < 8x^2/t$, we conclude that

$$\Gamma(t,x,0) \geq \frac{1}{Ct^{n/2}} e^{-Cx^2/t}, \tag{4.79}$$

with a constant C that depends on λ, Λ and dimension n. Therefore, Theorem 4.6 is a consequence of Theorem 4.7.

Proof of the uniform lower bound in Theorem 4.7

We now prove the lower bound in Theorem 4.7, that is, we show that

$$\Gamma(t,x,y) \geq \frac{1}{Ct^{n/2}}, \tag{4.80}$$

for all x, y such that $|x-y| \le \sqrt{t}$. The reason why this estimate holds is, roughly speaking, the following. Solution of the Cauchy problem

$$\phi_t = \nabla \cdot (a(x)\nabla\phi), \tag{4.81}$$

with $\phi(0,x) = \phi_0(x) \ge 0$ conserves mass:

$$\int \phi(t,x)dx = \int \phi_0(x)dx. \tag{4.82}$$

The Gaussian upper bound in Theorem 4.2 means that the total mass outside of the ball $B_N = \{|x| \ge N\sqrt{t}\}$ is small for large N, so that

$$\int_{|x|\le N\sqrt{t}} \phi(t,x)dx \ge \frac{1}{2}\int \phi_0(x)dx, \tag{4.83}$$

for a sufficiently large N. If we imagine that $\phi(t,x)$ is more or less equally distributed over the ball B_N, we obtain the lower bound (4.80).

In order to simplify slightly the notation let us make the following observation. Let $\phi(t,x)$ be the solution of

$$\frac{\partial\phi}{\partial t} = \nabla \cdot (a(x)\nabla\phi), \tag{4.84}$$

and set $\phi_L(t,x) = \phi(L^2t), Lx)$. The function $\phi_L(t,x)$ satisfies

$$\frac{\partial\phi_L(t,x)}{\partial t} = L^2\frac{\partial\phi(L^2t,Lx)}{\partial t} = L^2\nabla\cdot(a\nabla\phi)(L^2t,Lx) = \nabla\cdot(a_L(x)\nabla\phi_L(t,x)), \tag{4.85}$$

which is an equation

$$\frac{\partial\phi_L(t,x)}{\partial t} = \nabla\cdot(a_L(x)\nabla\phi_L(t,x)), \tag{4.86}$$

of the same form as (4.84) but with the diffusion matrix $a(x)$ replaced[6] by $a_L(x) = a(Lx)$. Let us investigate how the Green's functions $\Gamma(t,x,y)$ and $\Gamma_L(t,x,y)$ for (4.84) and (4.86) are related. We know from the above that if $\phi(t,x)$ solves (4.84) with the initial data $\phi(0,x) = \phi_0(x)$, and $\phi_L(t,x)$ solves (4.86) with the initial data $\phi_L(0,x) = \phi_0(Lx)$, then $\phi_L(t,x) = \phi(L^2t,Lx)$. In other words, we have the identity

$$\int \Gamma(L^2t,Lx,y)\phi_0(y)dy = \int \Gamma_L(t,x,y)\phi_0(Ly)dy = \frac{1}{L^n}\int \Gamma_L(t,x,\frac{y}{L})\phi_0(y)dy, \tag{4.87}$$

for all initial data $\phi_0 \in L^1$. As ϕ_0 is an arbitrary function, we deduce the following scaling relation

$$\Gamma(L^2t,Lx,Ly) = \frac{1}{L^n}\Gamma_L(t,x,y). \tag{4.88}$$

Therefore, in order to show that

$$\Gamma(t,x,y) \ge \frac{1}{Ct^{n/2}}, \text{ for all } |x-y| \le \sqrt{t}, \tag{4.89}$$

[6]As a slight digression, we mention an important question of what happens when is L large, meaning that we observe the original solution $\phi(t,x)$ after long times $t \sim L^2$ and on large scales $x \sim L$. This is the scope of the homogenization theory [8] that is particularly well developed when $a(x)$ is either periodic or random in x.

it is sufficient to show that there exists a constant C that does not depend on L so that

$$\Gamma_L(1,x,y) \geq \frac{1}{C}, \text{ for all } |x-y| \leq 1. \tag{4.90}$$

The matrices $a(x)$ and $a_L(x)$ have the same ellipticity constants. Hence, we can reformulate Theorem 4.7 as the statement that there exists a constant $C > 0$ that depends only on the ellipticity constants and dimension so that

$$\Gamma(1,x,y) \geq \frac{1}{C}, \text{ for all } |x-y| \leq 1, \tag{4.91}$$

and this is what we will prove.

The key ingredient in the proof is, once again, an integral bound.

Lemma 4.8 *For every $\tau > 0$ there exists a constant B that depends only on λ, Λ, τ and n so that we have*

$$\int e^{-\pi|y|^2} \log \Gamma(\tau,x,y)dy \geq -B_\tau, \tag{4.92}$$

for all x such that $|x| \leq 1$.

Let us explain why this lemma is sufficient to prove the lower bound (4.91). Take any x and y so that $|x-y| \leq 1$. Without loss of generality we may assume that $|x| \leq 1$ and $y = 0$. The semi-group property implies that

$$\Gamma(1,x,0) = \int \Gamma(1/2,x,\xi)\Gamma(1/2,\xi,0)d\xi \geq \int \Gamma(1/2,x,\xi)\Gamma(1/2,\xi,0)e^{-\pi|\xi|^2}d\xi. \tag{4.93}$$

Applying Jensen's inequality, recalling that $\Gamma(t,\xi,0) = \Gamma(t,0,\xi)$, and using Lemma 4.8 gives

$$\log \Gamma(1,x,0) \geq \int e^{-\pi|\xi|^2} \log \Gamma(1/2,x,\xi)d\xi + \int e^{-\pi|\xi|^2} \log \Gamma(1/2,\xi,0)d\xi \geq -2B_{1/2}, \tag{4.94}$$

so that (4.91) holds.

The proof of Lemma 4.8

The very last step in the proof of Theorem 4.1 is to prove Lemma 4.8. Fix x such that $|x| \leq 1$, take any $\varepsilon > 0$, and set $u(t,y) = \Gamma(t,y,x) + \varepsilon$, and

$$G(t) = \int e^{-\pi|y|^2} \log u(t,y)dy. \tag{4.95}$$

The role of ε here is simply to make the integral above "clearly convergent" – otherwise, there may be a hypothetical problem at infinity where $u(t,y)$ is very small. All our bounds will be uniform in ε. If we momentarily set $\varepsilon = 0$ then

$$\int u(t,y)dy = 1,$$

for all $t > 0$, Jensen's inequality implies that

$$0 = \log\Big(\int u(t,y)dy\Big) \geq \log\Big(\int u(t,y)e^{-\pi|y|^2}dy\Big) \geq \int (\log u(t,y))e^{-\pi|y|^2}dy = G(t). \quad (4.96)$$

Our goal is to show that, even with $\varepsilon > 0$, $G(t)$ is maybe negative but bounded from below for each $t > 0$, uniformly in $|x| \leq 1$, and $\varepsilon > 0$. Note that if $\varepsilon = 0$ then $G(0)$ is not very well defined but $G(s) \to -\infty$ as $s \downarrow 0$ since $u(0,y) = \delta(y-x)$. Therefore an estimate from below for $G(s)$ that is uniform in ε is not an a priori obvious fact. Let us obtain a differential inequality for $G(t)$:

$$\frac{dG}{dt} = \int \frac{1}{u(t,y)} \nabla\cdot(a(y)\nabla u(t,y))e^{-\pi|y|^2}dy = -\int a(y)\nabla u(t,y)\cdot\nabla\Big(\frac{e^{-\pi|y|^2}}{u(t,y)}\Big)dy$$
$$= 2\pi\int a(y)\nabla(\log u(t,y))\cdot y e^{-\pi|y|^2}dy + \int (a(y)\nabla(\log u(t,y))\cdot\nabla(\log u(t,y))e^{-\pi|y|^2}dy.$$

Let us rewrite the integrands above:

$$a(y)\nabla(\log u)\cdot\nabla(\log u) + 2\pi a(y)\nabla(\log u)\cdot y = \frac{1}{2}a(y)\nabla(\log u)\cdot\nabla(\log u) \quad (4.97)$$
$$+\frac{1}{2}a(y)(\nabla(\log u) + 2\pi y)\cdot(\nabla(\log u) + 2\pi y) - 2\pi^2(a(y)y\cdot y).$$

The first term in the right side is positive, which is good for us. Dropping the first term in the second line above gives

$$\frac{dG}{dt} \geq -2\pi^2\int (a(y)y\cdot y)e^{-\pi|y|^2}dy + \frac{1}{2}\int a(y)\nabla(\log u(t,y))\cdot\nabla(\log u(t,y))e^{-\pi|y|^2}dy$$
$$\geq -A + \frac{\lambda}{2}\int |\nabla(\log u(t,y))|^2 e^{-\pi|y|^2}dy, \quad (4.98)$$

with a constant A that depends only on the ellipticity constants of the matrix $a(x)$. Therefore, the function $G(t) + At$ is non-decreasing for $t > 0$, which is the right direction. It is not, however, sufficient since at the moment we do not have an ε-independent lower bound for $G(t)$ at any time, so saying that, for instance, $G(1) > -A + G(0)$ will not be of much use. What we will use is that the positive term in the right side of (4.98) is quadratic in $\log u$.

The next step is to recall that for the Gaussian probability measure

$$d\mu = e^{-\pi|y|^2}dy,$$

we have the Poincaré inequality in the whole space

$$\int_{\mathbb{R}^n} (\log w(y) - \langle w\rangle_\mu)^2 d\mu(y) \leq C\int_{\mathbb{R}} |\nabla(\log w(y))|^2 d\mu(y), \quad (4.99)$$

with

$$\langle w\rangle_\mu = \int_{\mathbb{R}^n} w d\mu = \int_{\mathbb{R}^n} w e^{-\pi|y|^2}dy.$$

A good reference for such generalized Poincaré inequalities is the book [44].

Exercise 4.9 Let $d\mu(x) = e^{-\pi|x|^2}dx$. Show that there exists a constant $C > 0$ so that for any function $\phi \in H^1(\mathbb{R}; d\mu)$ we have

$$\int |\phi(x) - \langle\phi\rangle|^2 d\mu(x) \le C \int |\nabla\phi|^2 d\mu(x), \tag{4.100}$$

with

$$\langle\phi\rangle = \int \phi(x) d\mu(x).$$

In our situation, this inequality takes the form

$$\int (\log u(t,y) - G(t))^2 e^{-\pi|y|^2} dy \le C \int |\nabla(\log u(t,y))|^2 e^{-\pi|y|^2} dy. \tag{4.101}$$

Therefore, we have

$$\frac{dG}{dt} \ge -A + B \int (\log u(t,y) - G(t))^2 e^{-\pi|y|^2} dy, \tag{4.102}$$

with the constants A and B that depend only on the ellipticity constants of the matrix $a(x)$. The function

$$p(u) = \frac{(\log u - D)^2}{u}$$

is decreasing in u for $u > e^{2+D}$ (here D is an arbitrary constant). In addition, we know from the upper bound on $\Gamma(t,x,y)$ that there exists a constant K_τ so that $u(s,y) \le K_\tau$ for all $y \in \mathbb{R}^n$ and all $\tau/2 \le t \le \tau$. Therefore, for all $\tau/2 \le t \le \tau$ we have

$$\begin{aligned} \frac{dG}{dt} &\ge -A + B \int_{S_t} \frac{(\log u(t,y) - G(t))^2}{u(t,y)} u(t,y) e^{-\pi|y|^2} dy \\ &\ge -A + B \frac{(\log K - G(t))^2}{K} \int_{S_t} u(t,y) e^{-\pi|y|^2} dy. \end{aligned} \tag{4.103}$$

Here S_t is the set

$$S_t = \{u(t,y) \ge e^{2+G(t)}\}.$$

If $G(t)$ is very negative (which is what we are trying to avoid), the set S_t is very large. The integral over S_t may be estimated as follows, using the fact that $u(t,y) \le e^{2+G(t)}$ for $y \notin S_t$, and $u(t,y) \ge e^{2+G(t)}$ for $y \in S_t$:

$$\begin{aligned} \int_{S_t} u(t,y) e^{-\pi|y|^2} dy &\ge \int_{S_t} (u(t,y) - e^{2+G(t)}) e^{-\pi|y|^2} dy \ge \int_{\mathbb{R}^n} (u(t,y) - e^{2+G(t)}) e^{-\pi|y|^2} dy \\ &= \int_{\mathbb{R}^n} u(t,y) e^{-\pi|y|^2} dy - e^{2+G(t)}. \end{aligned} \tag{4.104}$$

Next, as

$$\int_{\mathbb{R}^n} \Gamma(t,x,y) dy = 1, \tag{4.105}$$

the upper Gaussian bounds on $\Gamma(t,x,y)$ imply that for any $\tau > 0$ there exists a constant c_0 (that depends on τ) so that

$$\int_{\mathbb{R}^n} \Gamma(t,x,y) e^{-\pi|y|^2/2} dy \geq c_0, \tag{4.106}$$

for all $\tau/w \leq t \leq \tau$. The same upper bound on $\Gamma(t,x,y)$, together with (4.106) implies that there exists R (that also depends on τ) so that

$$\int_{|y|\geq R} u(t,y) e^{-\pi|y|^2/2} dy \leq \frac{c_0}{2}, \tag{4.107}$$

also for all $\tau/2 \leq t \leq \tau$. This is the crucial step in the proof: the upper bound necessitates the lower bound on $\Gamma(t,x,y)$. Returning to (4.104) we get

$$\int_{S_t} u(t,y) e^{-\pi|y|^2} dy \geq e^{-\pi|R|^2/2} \int_{|y|\leq R} u(t,y) e^{-\pi|y|^2/2} dy - e^{2+G(t)} \geq \frac{c_0 e^{-\pi|R|^2}}{2} - e^{2+G(t)}. \tag{4.108}$$

Inequality (4.103) now becomes

$$\frac{dG}{dt} \geq -A + B \frac{(\log K - G(t))^2}{K} \left[\frac{c_0 e^{-\pi|R|^2}}{2} - e^{2+G(t)} \right], \quad \frac{\tau}{2} \leq t \leq \tau. \tag{4.109}$$

Assume now that $G(\tau) < -M$ for some large M. The function $G(s) + As$ is non-decreasing in time, hence

$$G(t) \leq G(\tau) + A\tau - At < -M/2, \tag{4.110}$$

for all $t \in [\tau/2, \tau]$ provided that $M > 100A\tau$. Suppose that M is so large that (4.110) implies that

$$e^{-\pi R^2} > 10 e^{2+G(s)},$$

for all $\tau/2 \leq s \leq \tau$. Then, still under the assumption $G(\tau) < -M$, (4.109) implies that

$$\frac{dG}{dt} \geq -A + B \frac{(\log K - G(t))^2}{K} \frac{c_0 e^{-\pi|R|^2}}{5}, \quad \text{for } \frac{\tau}{2} \leq t \leq \tau. \tag{4.111}$$

However, if M is much larger than all other constants appearing in (4.111), and $G(t) \leq -M/2$ for all $t \in [\tau/2, \tau]$, it follows from the last inequality that

$$\frac{dG}{dt} \geq c G(t)^2, \quad \text{for } \frac{\tau}{2} \leq t \leq \tau, \tag{4.112}$$

with the constant c that still depends only on τ and the ellipticity constants of the matrix $a(x)$. However, this quadratic inequality blows up in a finite "backward" time, so if $G(t)$ satisfies (4.112), and $G(\tau/2) > -\infty$, it is impossible that $G(\tau) < -M$ for too large M (that depends explicitly on constant c). This gives an a priori lower bound on $G(\tau)$ that depends only on τ and the ellipticity constants of the matrix $a(x)$ and is uniform in $\varepsilon > 0$. In order to remove the need for the regularization $\varepsilon > 0$ note that we have shown

$$\int e^{-\pi|y|^2} \log(\Gamma(\tau,x,y) + \varepsilon) dy > -B_\tau. \tag{4.113}$$

As the function $\Gamma(\tau,x,y)$ is uniformly bounded from above by a constant $K(\tau)$, it follows from (4.113) that

$$\int e^{-\pi|y|^2}\log_-(\Gamma(\tau,x,y)+\varepsilon)dy > -B'_\tau, \tag{4.114}$$

with some constant B'_τ. Here $\log_- u = 0$ if $u > 1$ and $\log_- u = \log u$ if $u \in (0,1)$. Fatou's lemma now shows that

$$\int e^{-\pi|y|^2}\log_-\Gamma(\tau,x,y)dy > -B'_\tau. \tag{4.115}$$

This completes the proof of Theorem 4.1!

5 Gaussian bounds on the heat kernel imply everything

We will now show, once again following Fabes and Stroock [27], that the bounds on the heat kernel imply "all" classical regularity results on the parabolic equations in the divergence form. The physical reason for this implication is simple. Parabolic and elliptic equations tend to equilibrate locally, as can be seen, for instance, from the mean value property for harmonic functions. A potential enemy of this tendency is the "outside influence" – for example, if the solution is wild outside a ball, it may spoil the equilibrating properties inside the ball too, The heat kernel bounds provide two remedies against the outside influence – first, the upper Gaussian bounds impose limits on the influence of what happens outside the ball $B(x,R)$ on the solution inside a slightly smaller ball $B(x,\delta R)$ with $\delta < 1$. On the other hand, the lower bounds on the heat kernel show that local influence is quite strong – the combination of the two allows to prove local regularity results.

5.1 A lower bound for Green's functions on a bounded domain

As before, we will denote by $\Gamma(t,x,y)$ the Green's function for the Cauchy problem

$$\begin{aligned} &\phi_t = \nabla\cdot(a(x)\nabla\phi), \quad t>0, \quad x\in\mathbb{R}^n, \\ &\phi(0,x) = \phi_0(x). \end{aligned} \tag{5.1}$$

That is, solution of (5.1) can be written as

$$\phi(t,x) = \int_{\mathbb{R}^n}\Gamma(t,x,y)\phi_0(y)dy. \tag{5.2}$$

In order to bound the "outside influence" on $\phi(t,x)$ in a ball $B(\xi,R)$ we will consider the worst case scenario (assume for the sake of intuition that $\phi_0(x)\geq 0$), setting the Dirichlet boundary condition on the boundary $\partial B(\xi,R)$. The maximum principle implies the true solution satisfies $\phi(t,x) > 0$ on $\partial B(\xi,R)$, hence in that way we will account for the "strongest outside cooling influence".

To formalize this idea, we will make use of the Green's function for the Cauchy problem on bounded domains. Let $B(\xi,R)\subset\mathbb{R}^n$ be a ball of radius R centered at a point $\xi\in\mathbb{R}^n$. We will denote by $\Gamma_{\xi,R}(t,x,y)$ the Green's function for the problem

$$\begin{aligned}
&\psi_t = \nabla\cdot(a(x)\nabla\psi), \quad t>0, \quad x\in B(\xi,R),\\
&\psi(0,x)=\psi_0(x), \quad x\in B(\xi,R)\\
&\psi(t,y)=0 \text{ for } y\in\partial B(\xi,R).
\end{aligned} \tag{5.3}$$

The function $\psi(t,x)$ has a representation

$$\psi(t,x)=\int_{B(\xi,R)}\Gamma_{\xi,R}(t,x,y)\psi_0(y)dy. \tag{5.4}$$

Our immediate task will be to find uniform lower bounds for $\Gamma_{\xi,R}(t,x,y)$ strictly inside the ball $B(\xi,R)$, that is, in a slightly smaller ball $B(\xi,\delta R)$ – we can not possibly expect such bounds all the way to the boundary as $\Gamma_{\xi,R}$ vanishes there.

It is instructive to write an equation for the function $\psi(t,y)$ in the whole space, in the sense of distributions. For any smooth test function $\eta\in\mathcal{S}(\mathbb{R}^n)$ (the Schwartz class) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\psi(t,x)\nabla\cdot(a(x)\nabla\eta(x))dx=\int_{B(\xi,R)}\psi(t,x)\nabla\cdot(a(x)\nabla\eta(x))dx\\
&=\int_{\partial B(x,R)}\psi(t,y)(a(y)\nabla\eta(y)\cdot\nu)dy-\int_{B(\xi,R)}(a(x)\nabla\eta(x)\cdot\nabla\psi(t,x))dx.
\end{aligned} \tag{5.5}$$

Here ν is the outward normal to the sphere $\partial B(\xi,R)$. The first term in the right side vanishes because of the Dirichlet boundary conditions, leading to

$$\begin{aligned}
&\int_{\mathbb{R}^n}\psi(t,x)\nabla\cdot(a(x)\nabla\eta(x))dx=-\int_{B(\xi,R)}(a(x)\nabla\eta(x)\cdot\nabla\psi(t,x))dx\\
&=-\int_{\partial B(x,R)}\eta(y)(a(y)\nabla\psi(t,y)\cdot\nu)dy+\int_{B(\xi,R)}\eta(x)\nabla\cdot(a(x)\nabla\psi(t,x))dx\\
&=-\int_{\partial B(x,R)}\eta(y)(a(y)\nabla\psi(t,y)\cdot\nu)dy+\int_{B(\xi,R)}\eta(x)\psi_t(t,x)dx.
\end{aligned} \tag{5.6}$$

Therefore, $\psi(t,x)$ satisfies the following problem in the whole space:

$$\begin{aligned}
&\psi_t=\nabla\cdot(a(x)\nabla\psi)+(a(x)\nabla\psi(t,x)\cdot\nu)\delta_{\partial B(\xi,R)}(x), \quad t>0, \quad x\in\mathbb{R}^n,\\
&\psi(0,x)=\psi_0(x), \quad x\in\mathbb{R}^n.
\end{aligned} \tag{5.7}$$

Recall that if $\psi_0(x)\geq 0$ then $\psi(t,x)\geq 0$ inside $B(\xi,R)$. Therefore, as $\psi(t,x)=0$ on $\partial B(\xi,R)$, we have

$$(a(x)\nabla\psi\cdot\nu)\leq 0 \text{ for } x\in\partial B(\xi,R).$$

Hence the source in (5.6) is negative, as it should be from the physical considerations – the boundary has the cooling effect. Duhamel's principle implies that $\psi(t,x)$ can be written as

$$\psi(t,x)=\int_{B(\xi,R)}\Gamma(t,x,y)\psi_0(y)dy+\int_0^t\int_{\partial B(\xi,R)}\Gamma(t-s,x,y)(a(y)\nabla\psi(s,y)\cdot\nu(y))dy. \tag{5.8}$$

If we take the initial data $\psi_0(y) = \delta(y-z)$ with some $z \in B(\xi, R)$, we get from (5.8):

$$\Gamma_{\xi,R}(t,x,z) = \Gamma(t,x,z) + \int_0^t \int_{\partial B(\xi,R)} \Gamma(t-s,x,y)(a(y)\nabla\Gamma_{\xi,R}(s,y,z)\cdot\nu(y))dyds. \tag{5.9}$$

The maximum principle implies that

$$(a(y)\nabla\Gamma_{\xi,R}(s,y,z)\cdot\nu(y)) < 0,$$

so we may rewrite (5.10) as

$$\Gamma_{\xi,R}(t,x,z) = \Gamma(t,x,z) - \int_0^t \int_{\partial B(\xi,R)} \Gamma(t-s,x,y)d\mu(y)ds. \tag{5.10}$$

Here we have defined the measure

$$d\mu(s,y) = -(a(y)\nabla\Gamma_{\xi,R}(s,y,z)\cdot\nu(y))dy.$$

In order to estimate the effect of the cold boundary in (5.10), we go back to the equation for $\Gamma_{\xi,R}$:

$$\begin{aligned} &\frac{\partial\Gamma_{\xi,R}}{\partial t} = \nabla\cdot(a(x)\nabla\Gamma_{\xi,R}), \quad t>0, \quad x\in B(\xi,R), \\ &\Gamma_{\xi,R}(0,x) = \delta(x-z), \quad x \in B(\xi,R) \\ &\Gamma_{\xi,R}(t,y) = 0 \text{ for } y \in \partial B(\xi,R), \end{aligned} \tag{5.11}$$

and integrate over the ball $B(\xi,R)$ and in time, we get

$$\int\limits_{B(\xi,R)} \Gamma_{\xi,R}(t,x,z)dx - 1 = \int\limits_0^t \int\limits_{\partial B(\xi,R)} (a(y)\nabla\Gamma_{\xi,R}(s,y,z)\cdot\nu(y))dyds = -\int\limits_0^t \int\limits_{\partial B(\xi,R)} d\mu(s,y)ds. \tag{5.12}$$

We conclude that

$$\int_0^t \int_{\partial B(\xi,R)} d\mu(s,y)ds < 1, \tag{5.13}$$

for all $t > 0$. This is a very important point – we have a bound on the effect of the cold boundary over time.

Take now any $\delta \in (0,1)$ and assume that both x and z lie in the "slightly smaller" ball $B(\xi, \delta R)$. Then the upper Gaussian bound on $\Gamma(t,x,y)$ that we have already proved, together with (5.10) and (5.13) imply that

$$\Gamma_{\xi,R}(t,x,z) \geq \Gamma(t,x,z) - \sup_{0\leq\tau\leq t} \frac{C}{\tau^{n/2}} e^{-(1-\delta)^2R^2/(C\tau)}. \tag{5.14}$$

Next, the lower Gaussian bound on $\Gamma(t,x,y)$ gives

$$\Gamma_{\xi,R}(t,x,z) \geq \frac{1}{Ct^{n/2}} e^{-C|x-z|^2/t} - \sup_{0\leq\tau\leq t} \frac{C}{\tau^{n/2}} e^{-(1-\delta)^2R^2/(C\tau)}. \tag{5.15}$$

Therefore, there exists $r \in (0, 1-\delta)$ that depends only on δ so that for all $0 < t \le r^2R^2$ and $|z-x| < rR$, with $z, x \in B(\xi, \delta R)$, we have

$$\Gamma_{\xi,R}(t,x,z) \ge \frac{1}{2Ct^{n/2}} e^{-C|x-z|^2/t}. \tag{5.16}$$

The constant C depends only on δ but not on R.

The fact that we obtained first a lower bound on $\Gamma_{\xi,R}(t,x,z)$ only for nearby points x and z and at short times is very natural – by virtue of being inside $B(\xi, \delta R)$, these points are separated from the boundary of $B(x, R)$ (where the Dirichelt boundary condition is imposed) by the distance $(1-\delta)R$ which is larger than $|x-z|$. Therefore, it is reasonable to expect that the "warming" influence of z at x at short times is stronger than the combined "cooling" influence of all boundary points on $\partial B(x, R)$ that are too far away to compete.

In order to extend this estimate to all of the smaller ball $B(\xi, \delta R)$ and all times in an interval of the form $\gamma R^2 \le t \le R^2$ with some $\gamma > 0$, we use a simpler version of the argument in the proof of Theorem 4.7. Take any $x, z \in B(\xi, \delta R)$ and $t \le R^2$. Consider a sequence of balls $B_j = B(\xi_j, rR/3)$, $j = 1, \dots, k-1$, such that $\xi_1 = z$, $x \in B_{k-1}$ and each next center $\xi_{j+1} \in B_j$. By possibly increasing the number of balls we can also ensure that $t/k \le r^2R^2$. The total number k of the required balls is bounded by

$$k \le 1 + \max\left[\frac{|x-z|}{10rR}, \frac{t}{R^2r^2}\right] \le 1 + \max\left[C(\delta), \frac{t}{R^2r^2}\right]. \tag{5.17}$$

Therefore, as $t \le R^2$, we conclude that k is bounded by a constant K_δ that only depends on δ:

$$k \le K_\delta \text{ for } t \le R^2. \tag{5.18}$$

The semigroup property of Green's function $\Gamma_{\xi,R}(t,x,z)$ implies that we may iterate:

$$\begin{aligned}
\Gamma_{\xi,R}(t,x,z) &= \int\limits_{B(\xi,R)} \dots \int\limits_{B(\xi,R)} \Gamma_{\xi,R}\big(\frac{t}{k}, x, \xi_{k-1}\big)\Gamma_{\xi,R}\big(\frac{t}{k}, \xi_{k-1}, \xi_{k-2}\big) \dots \Gamma_{\xi,R}\big(\frac{t}{k}, \xi_1, z\big) d\xi_1 \dots d\xi_{k-1} \\
&\ge \int\limits_{B_{k-1}} d\xi_{k-1} \int\limits_{B_{k-2}} d\xi_{k-2} \dots \int\limits_{B_1} d\xi_1 \Gamma_{\xi,R}\big(\frac{t}{k}, x, \xi_{k-1}\big)\Gamma_{\xi,R}\big(\frac{t}{k}, \xi_{k-1}, \xi_{k-2}\big) \dots \Gamma_{\xi,R}\big(\frac{t}{k}, \xi_1, z\big).
\end{aligned} \tag{5.19}$$

If $t \ge \gamma R^2$ then for any $z, z' \in B(\xi, \delta R)$ we have

$$\frac{(z-z')^2}{t} \le \frac{\delta^2 R^2}{\gamma R^2} = \frac{\delta^2}{\gamma}.$$

Hence, in the range $\gamma R^2 \le t \le R^2$, the uniform bound (5.16) together with the uniform upper bound (5.18) on k, implies that

$$\Gamma\big(\frac{t}{k}, \xi_j, \xi_{j-1}\big) \ge \frac{1}{C't^{n/2}}, \quad \text{for all } 2 \le j \le k, \tag{5.20}$$

with a constant C' that depends only on γ and δ. Similarly, we have

$$\Gamma\big(\frac{t}{k}, x, \xi_k\big) \ge \frac{1}{Ct^{n/2}}, \quad \Gamma\big(\frac{t}{k}, \xi_1, z\big) \ge \frac{1}{Ct^{n/2}}. \tag{5.21}$$

Using these estimates in (5.19) leads to

$$\Gamma(t,x,z) \geq |B_1|^{k-1}\left(\frac{1}{Ct^{n/2}}\right)^k = C_n^{k-1}(rR)^{n(k-1)}\left(\frac{1}{Ct^{n/2}}\right)^k = \frac{DK_0^k}{R^n}\left(\frac{r^2R^2}{t}\right)^{nk/2}, \quad (5.22)$$

with the constants K_0 and D that only depend on δ. As $\gamma R^2 < t \leq R^2$, and k obeys the upper bound (5.18) we simply get

$$\Gamma(t,x,z) \geq \frac{C}{R^n}, \quad \gamma R^2 < t \leq R^2, \quad x,z \in B(\xi,\delta R). \quad (5.23)$$

Let us summarize this result.

Theorem 5.1 *For each $\delta > 0$ and $\gamma > 0$ there exists c_0 that depends only on the ellipticity constants of the matrix $a(x)$, dimension n, δ and γ so that for all $\xi \in \mathbb{R}^n$ and all $R > 0$ we have a lower bound*

$$\Gamma_{\xi,R}(t,x,z) \geq \frac{c_0}{R^n}, \quad (5.24)$$

for all $x, z \in B(\xi,\delta R)$ and all $\gamma R^2 \leq t \leq R^2$.

It is this corollary of the Gaussian heat kernel bounds that will be crucial in the proof of parabolic regularity properties below: it limits the outside influence!

5.2 A decay of oscillation estimate

The lower bound on the Green's function in a ball that we have obtained above implies a decay of oscillation estimate. Consider a parabolic cylinder

$$D(s,\xi;R) = \{s - R^2 \leq t \leq s, \quad |x-\xi| \leq R\}, \quad (5.25)$$

and the corresponding oscillation of a function u over $D(s,\xi;R)$:

$$\mathrm{Osc}(u;s,\xi,R) = \sup\{|u(t,x,) - u(t',x')| : \quad (t,x),(t',x') \in D(s,\xi;R)\}. \quad (5.26)$$

Theorem 5.2 *For each $\delta \in (0,1)$ there exists $\rho < 1$ that depends only on dimension n, the ellipticity constants of the matrix $a(x)$ and δ so that any solution $u \in C^\infty([s-R^2,s]\times\bar{B}(\xi,R))$ of*

$$u_t = \nabla\cdot(a(x)\nabla u), \quad s - R^2 < t < s, \quad x \in B(\xi,R), \quad (5.27)$$

satisfies

$$\mathrm{Osc}(u;s,\xi,\delta R) \leq \rho\mathrm{Osc}(u;s,\xi,R). \quad (5.28)$$

Proof. Let $m(r)$ and $M(r)$ denote the minimum and maximum of u over the parabolic cylinder $D(s,\xi;r)$. Consider the set

$$S = \left\{x \in B(\xi,\delta R) : \quad u(s-R^2,x) \geq \frac{M(R)+m(R)}{2}\right\},$$

where the solution is "large" at time $s-R^2$. Assume first that this set itself is relatively large: $|S| \geq |B(\xi, \delta R)|/2$. Together with the bounds on the Dirichlet Green's function this will be enough to show that at any time separated from $s - R^2$:

$$s - \delta^2 R^2 < t \leq s,$$

solution inside the ball $B(\xi, \delta R)$ will be "not too small". That is, since $u(s - R^2, x)$ is large on a big set inside $B(x, \delta R)$, after a short time it will be "large enough" on all of $B(\xi, \delta R)$. To this end, note that the function

$$u_1(t, x) = u(t, x) - m(R)$$

satisfies the same equation as $u(t, x)$, and is positive for all $x \in D(s, \xi, R)$, in particular, $u_1(t, x) \geq 0$ for all $s - R^2 \leq t \leq s$ and all $x \in \partial B(\xi, R)$. The function

$$u_2(x) = \int_{B(\xi,R)} (u(s - R^2, y) - m(R))\Gamma_{\xi,R}(t - (s - R^2), x, y)dy$$

also satisfies the same parabolic equation but $u_2(t, x) = 0$ for all $s - R^2 \leq t \leq s$ and all $x \in \partial B(\xi, R)$, and, in addition, $u_1(s - R^2, x) = u_2(s - R^2, x)$ for all $x \in B(\xi, R)$. It follows from the maximum principle that $u_1(t, x) \geq u_2(t, x)$ for all $(t, x) \in D(s, \xi, R)$. Therefore, for $(t, x) \in D(s, \xi, \delta R)$ we have

$$\begin{aligned} u(t, x) - m(R) &\geq \int_{B(\xi,R)} (u(s - R^2, y) - m(R))\Gamma_{\xi,R}(t - (s - R^2), x, y)dy \quad (5.29) \\ &\geq \int_S \Gamma_{\xi,R}(t - (s - R^2), x, y)dy \geq \frac{M(R) - m(R)}{2} \frac{c_\delta}{R^n}|S| \\ &= \varepsilon(M(R) - m(R)). \end{aligned}$$

We used Theorem 5.1 in the last step above, since $t - (s - R^2) \geq (1 - \delta^2)R^2$, and also the assumption $|S| \geq |B(\xi, \delta R)|/2$. The constant ε does not depend on R or u. It follows that

$$m(\delta R) \geq m(R) + \varepsilon(M(R) - m(R)),$$

and thus

$$M(\delta R) - m(\delta R) \leq M(R) - m(\delta R) \leq (1 - \varepsilon)(M(R) - m(R)). \quad (5.30)$$

On the other hand, if S is small: $|S| \leq |B(\xi, \delta R)|$ we would simply consider the difference $M(R) - u(t, x)$ for any $s - \delta^2 R^2 < t \leq s$, and $x \in B(\xi, \delta R)$:

$$\begin{aligned} M(R)) - u(t, x) &\geq \int_{B(\xi,R)} (M(R) - u(s - R^2, y))\Gamma_{\xi,R}(t - (s - R^2), x, y)dy \quad (5.31) \\ &\geq \frac{M(R) - m(R)}{2} \int_{S^c} \Gamma_{\xi,R}(t - (s - R^2), x, y)dy \geq \frac{M(R) - m(R)}{2} \frac{c_\delta}{R^n}|S^c| \\ &= \varepsilon(M(R) - m(R)), \end{aligned}$$

which also implies (5.30). □

5.3 The Hölder regularity

The decay of oscillations implies the Hölder regularity of solutions – we will use an iterative local blow-up argument. Consider some $s > 0$, $\xi \in \mathbb{R}^n$ and R such that $s - R^2 > 0$, and assume that $u(t,x)$ satisfies the parabolic equation in the parabolic cylinder $D(s,\xi,R)$:

$$u_t = \nabla \cdot (a(x)\nabla u), \quad x \in B(\xi,R), \quad s - R^2 \le t \le s. \tag{5.32}$$

We are going to show that if $u(t,x)$ is bounded in $D(s,\xi,R)$ then $u(t,x)$ has to satisfy Hölder a priori bounds in a smaller set

$$D_\delta(s,\xi) = \{s - (1-\delta^2)R^2 \le t \le s, \quad |x - \xi| \le (1-\delta)R\},$$

for any $\delta \in (0,1)$. The main point is that if we step slightly inside $D(s,\xi,R)$, away from the boundary $|x-\xi| = R$, and from the initial time $t = s - R^2$, then the Hölder norm of u depends only[7] on the L^∞ norm of u in the slightly bigger set $D(s,\xi,R)$. Accordingly, take some t, t' so that

$$s - (1-\delta^2)R^2 \le t' \le t \le s,$$

and $x, x' \in B(\xi,(1-\delta)R)$. Let us also denote

$$l = \sqrt{t-t'} + |x - x'|,$$

so that

$$(t',x') \in [t - l^2, t] \times B(x,l), \tag{5.33}$$

and set

$$M = \sup\{|u(r,y)| : \quad s - R^2 \le r \le s, \quad |y - \xi| \le R\}.$$

Note that

$$|u(t,x) - u(t',x')| \le \mathrm{Osc}(u;t,x,l), \tag{5.34}$$

because of (5.33). Iterating (5.34), going to larger and larger parabolic cylinders, with the help of Theorem 5.2 gives

$$\begin{aligned}|u(t,x) - u(t',x')| &\le \mathrm{Osc}(u;t,x,l) \le \rho \mathrm{Osc}(u;t,x,\frac{l}{\delta}) \le \dots \le \rho^{m-1}\mathrm{Osc}(u;t,x,\frac{l}{\delta^m})\\ &\le 2M\rho^{m-1}.\end{aligned} \tag{5.35}$$

We may iterate (blow-up the cylinder) as long as we stay inside $D(s,\xi,R)$:

$$t - \frac{l^2}{\delta^{2m}} > s - R^2, \tag{5.36}$$

and

$$B(x, \frac{l}{\delta^m}) \subset B(\xi,R). \tag{5.37}$$

[7] And of course, also on δ – on how far away from the boundary $|x-\xi| = R$ and from the the initial time $s - R^2$ we are.

For (5.36) to hold, as $t > s-(1-\delta^2)R^2$, it suffices to have

$$s-(1-\delta^2)R^2-\frac{l^2}{\delta^{2m}} > s-R^2, \tag{5.38}$$

that is,

$$\frac{l}{\delta^{m+1}} < R. \tag{5.39}$$

On the other hand, as $|x-\xi| \le (1-\delta)R$, for (5.37) to hold it is enough to ensure

$$(1-\delta)R+\frac{l}{\delta^m} < R, \tag{5.40}$$

which is nothing but (5.39) again. Let us choose the largest m so that (5.39) holds, that is

$$\delta^{m+2} \le \frac{l}{R} < \delta^{m+1},$$

then (5.35) gives

$$|u(t,x)-u(t',x')| \le 2M\rho^{m-1} \le C(\delta,\rho)M\exp\Big\{\frac{\log\rho}{\log\delta}\log\Big(\frac{l}{R}\Big)\Big\} \le CM\Big(\frac{l}{R}\Big)^{\beta}, \tag{5.41}$$

with the constants C and β that depend only on δ and λ (recall that ρ itself depends only on δ and λ). Therefore, we have shown that there exist a constant $C>0$ and $\beta>0$ that depend only on δ and λ so that if $u(t,x)$ satisfies

$$u_t = \nabla\cdot(a(x)\nabla u), \quad x \in B(\xi,R), \quad s-R^2 \le t \le s, \tag{5.42}$$

then for any t,t' so that $s-(1-\delta^2)R^2 \le t' \le t \le s$ and $x,x' \in B(\xi,(1-\delta)R)$ we have

$$|u(t,x)-u(t',x')| \le C(\lambda,\delta)M\left(\frac{|x-x'|+\sqrt{t-t'}}{R}\right)^{\beta}, \tag{5.43}$$

which is the desired Hölder estimate. Of course, the constant $C(\delta,\lambda)$ blows up as $\delta \downarrow 0$, as expected – the initial condition is assumed to be only locally bounded, not Hölder! But for any $t>0$ any solution is Hölder continuous both in time and space.

5.4 The Harnack inequality

The last step in milking the heat kernel bounds is to prove the Harnack inequality. Let $u(t,x) \ge 0$ (non-negativity is a crucial assumption here) be the solution of

$$u_t = \nabla\cdot(a(x)\nabla u), \quad x \in \bar{B}(x,R), \quad s-R^2 \le t \le s. \tag{5.44}$$

Theorem 5.3 *(The Harnack inequality) Let $0<\alpha<\beta<1$ and $0<\delta<1$ be given. There exist a constant M that depends on the dimension n, ellipticity constants of the matrix $a(x)$, and α, β, and δ, but not on R and u such that for all $s-\beta R^2 \le t \le s-\alpha R^2$ and $y \in B(x,\delta R)$ we have*

$$u(t,y) \le Mu(s,x). \tag{5.45}$$

Physically, this means that a hot point at distance r away will heat $u(s,x)$ after a time r^2 passes. The reason are the Hölder bounds – if $u(t,y)$ is large, it is "not too small" in a neighborhood $B(y,r_0)$ of y. The fact that $u \geq 0$ everywhere means that we may bound $u(s,x)$ from below if we simply restrict u to be zero away from $B(y,r_0)$ at time t, and consider the corresponding Cauchy problem starting at time t with this cut-off initial data and the Dirichlet boundary condition at $\partial B(x,R)$. The lower bounds on the Dirichlet Green's function will imply a lower bound on $u(s,x)$.

The above was the perspective that "u at an earlier time bounds u at later time from below". Alternatively, we may think that the Harnack inequality says that "u at a later time bounds u at an earlier time from above". To see it from that point of view, it is convenient to set $(s,x)=(0,0)$ and $R=1$ – the general case follows by the usual shifting and scaling argument. We may also assume that $u(0,0)=1$. The general reason is as follows: as $u \geq 0$, we may get from the bounds in Theorem 5.1 on the Dirichlet Green's function $\Gamma_{0,1}$ in the unit ball that for any $t<0$ the measure of the set of points y such that $u(t,y)>M$ can not be too large for large M – otherwise, we would have $u(0,0)>1$. On the other hand, if this set is small, then the oscillation of $u(t,y)$ around any point where $u(t,y)>2M$ is large. Iterating backward in time will produce larger and larger oscillation and show that u is unbounded on the time interval $[-1,0]$ which would be a contradiction.

Let us now formalize the above argument. Given any $r\in[-1,-\alpha]$ let $v(t,x)$ satisfy

$$v_t = \nabla\cdot(a(x)\nabla v), \quad x\in \bar{B}(0,1), \quad r\leq t\leq 0, \tag{5.46}$$

with the initial condition $v(r,x)=u(r,x)$ and the Dirichlet boundary condition $v(t,y)=0$ for $|y|=1$. As $u(t,y)\geq 0$ on the boundary $\partial B(0,1)$, we know that[8]

$$1 = u(0,0) \geq \int_{B(0,1)} \Gamma_{0,1}(-r,0,y)u(r,y)dy. \tag{5.47}$$

Then, Theorem 5.1 implies that there exists $\varepsilon>0$ that depends on α so that for all $M>0$, and all $-1\leq r\leq -\alpha$, we have

$$1 = u(0,0) \geq \int_{B(0,1)} \Gamma_{0,1}(-r,0,y)u(r,y)dy \geq \varepsilon M|S(r,M)|, \tag{5.48}$$

where

$$S(t,M) = \{y\in B(0,(1+\delta)/2):\ u(t,y)\geq M\}.$$

We conclude that

$$|S(t,M)| \leq \frac{1}{\varepsilon M}, \tag{5.49}$$

for all $t\in[-1,-\alpha]$ and $M>0$. Suppose that $y\in S(t,M)$, and l is such that $t-4l^2\geq -1$, and $B(y,2l)\in \bar{B}(0,(1+\delta)/2)$. If $B(y,l)$ is contained in the set $S(t,\sigma M)$ with some $\sigma<1$, then

$$c_n l^n \leq |S(t,\sigma M)| \leq \frac{1}{\varepsilon\sigma M}. \tag{5.50}$$

[8]Here $\Gamma_{0,1}$ denotes the Green's function for the parabolic Dirichlet problem on the unit ball $B(0,1)$, in accordance with the notation of Theorem 5.1.

Let us choose $\sigma = (1-\rho)/2 < 1$, with ρ as in the decay of oscillation estimate (5.28) in Theorem 5.2, and

$$l = \left(\frac{2}{c_n \varepsilon \sigma M}\right)^{1/n}.$$

Then (5.50) is false, meaning that $B(y,l)$ is not contained inside $S(t,\sigma M)$, and there exists a point $y_1 \in B(y,l)$ such that $u(t,y_1) < \sigma M$. It follows that

$$\text{Osc}(u;t,y,l) \geq u(t,y) - u(t,y_1) \geq (1-\sigma)M. \tag{5.51}$$

Applying Theorem 5.2 we deduce that

$$\text{Osc}(u;t,y,2l) \geq \frac{1}{\rho}\text{Osc}(u;t,y,l) \geq \frac{(1-\sigma)}{\rho}M = KM. \tag{5.52}$$

In particular, there exists $t' \in [t-4l^2, t]$ and $y' \in B(y,2l)$ such that

$$u(t',y') \geq KM.$$

Note that if we set

$$\sigma = \frac{1-\rho}{2},$$

then

$$K = \frac{1-\sigma}{\rho} = \frac{1+\rho}{2\rho} > 1.$$

Let us now proceed inductively using the above argument. Assume that there is $t_0 \in [-\beta,-\alpha]$ and $y_0 \in B(0,\delta)$ such that $u(t_0,y_0) \geq M_0$. Then we may find a point t_1, y_1 such that (with a constant c that does not depend on M_0 but rather ε, ρ, etc.)

$$t_0 - \frac{4c}{M_0^{2/n}} \leq t_1 \leq t_0, \quad |y_1 - y_0| \leq \frac{2c}{M_0^{2/n}}$$

so that $u(t_1,y_1) \geq M_1 = KM_0 > M_0$. Iterating, we obtain a sequence of points t_m, y_m so that

$$t_m - \frac{4c}{M_m^{2/n}} \leq t_{m+1} \leq t_m, \quad |y_{m+1} - y_m| \leq \frac{2c}{M_m^{2/n}}$$

so that $u(t_{m+1},y_{m+1}) \geq M_{m+1} = KM_m = K^m M_0$. Since $K > 1$, if M_0 is sufficiently large (depending on ε, ρ, α and β), the sequence t_m, y_m converges to a point $\bar{t}, \bar{y}$ in $[-\beta,-\alpha] \times B(0,\delta)$. But then $u(\bar{t},\bar{y})$ is unbounded which is contradiction. This gives a bound on M_0, proving Theorem 5.3.

Chapter 3

Advection-diffusion and mixing

We have seen in Chapter 2 (Theorem 2.2 in that Chapter) that solutions of the parabolic Cauchy problem

$$\begin{aligned} &\phi_t + u(t,x)\cdot\nabla\phi = \nabla\cdot(a(x)\nabla\phi), \qquad (0.1)\\ &\phi(0,x) = \phi_0(x), \end{aligned}$$

with a divergence-free flow $u(t,x)$, $\nabla\cdot u(t,x)=0$, satisfy a uniform bound

$$\|\phi(t,)\|_{L^\infty} \le \frac{C}{t^{n/2}}\|\phi_0\|_{L^1}, \qquad (0.2)$$

with a constant C that does not depend on the flow u. Similarly, solutions of an elliptic Dirichlet problem

$$-\nabla\cdot(a(x)\nabla\phi) + u(x)\cdot\nabla\phi = f(x), \quad x\in\Omega \qquad (0.3)$$

$$u = 0 \text{ on } \partial\Omega, \qquad (0.4)$$

with a divergence-free $u(x)$ satisfy a uniform bound (see Theorem 3.1 in Chapter 2)

$$\|\phi\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)}, \qquad (0.5)$$

for all $p > n/2$, with the constant C that does not depend on the flow u (as long as u is incompressible) but only on the ellipticity constants of the matrix $a(x)$. The natural question then is what can be the effect of a strong incompressible flow? In this chapter we will consider various situations when the presence of a strong flow dramatically changes the qualitative behavior of solutions.

1 Boundary and interior layers in cellular flows

The first question is whether uniform bounds such as (0.2) or (0.5) hold for derivatives of u, or, more generally, Hölder constants, or the constant in the Harnack inequality. The answer is no, not in general, and cellular flows considered in this section give an example when solutions have sharp transitions.

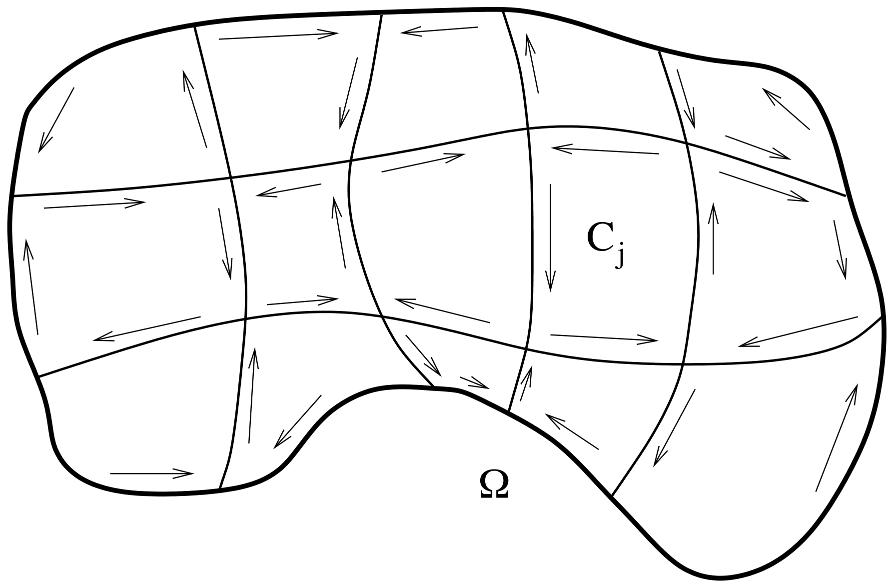

Figure 1.1: The domain Ω is partitioned by flow separatrices into cells $\mathcal{C}_j$.

We consider the steady advection-diffusion problem

$$\Delta\phi^\varepsilon - \frac{1}{\varepsilon} u \cdot \nabla\phi^\varepsilon = 0 \tag{1.1}$$

in a simply connected bounded domain $\Omega \subset \mathbb{R}^2$. The flow u is incompressible: $\nabla \cdot u = 0$, and non-penetrating through the boundary of Ω: $u \cdot n = 0$ at $\partial\Omega$ (see Figure 1.1). The small parameter $\varepsilon = \mathrm{Pe}^{-1} \ll 1$ is the inverse of the Péclet number. Equation (1.1) is supplemented by the Dirichlet boundary data:

$$\phi^\varepsilon(x) = g(x), \quad x \in \partial\Omega. \tag{1.2}$$

As the flow u is incompressible, a stream function $H(x_1, x_2)$ exists so that

$$u = \nabla^\perp H = (H_{x_2}, -H_{x_1}).$$

The flow is directed along the level set of H at each $x \in \Omega$. The canonical example to keep in mind is $H(x_1, x_2) = \sin x_1 \sin x_2$, and $\Omega = [0, 2\pi] \times [0, 2\pi]$. Furthermore, since the normal component of u at the boundary $\partial\Omega$ vanishes, $\partial\Omega$ has to be contained in a level set of H: $\partial\Omega \subseteq \{H = H_0\}$. Hence, either Ω is bounded by a closed streamline of the flow u or by a collection of separatrices of u that connect a finite number of singular points of H lying on the level set $\{H = H_0\}$. The latter case is of the most interest to us. We will assume without loss of generality that the critical value $H_0 = 0$. All the critical points of H are assumed to be non-degenerate. Then the set Ω is a union of finite number of flow cells $\mathcal{C}_j$ bounded by separatrices of u, where $H(x) = 0$, as in Figure 1.1. There is one extremum of the function $H(x)$ (a minimum or a maximum) inside each cell – these are neutrally stable points of the flow. The streamlines of the flow (level sets of the stream function) are assumed to be sufficiently regular inside each flow cell away from the saddle points of $H(x)$, and are

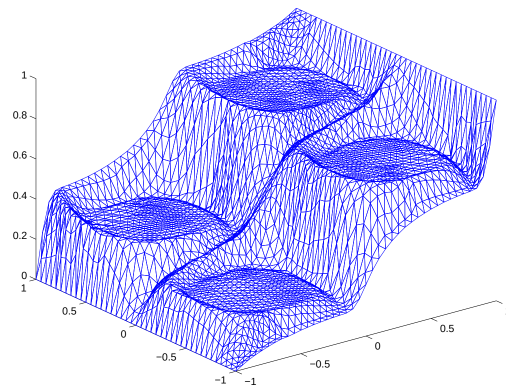

Figure 1.2: The temperature distribution for periodic cellular flows computed. $u = \nabla^{\perp} H$, $H = \sin(\pi x)\sin(\pi y)$; four cells, $\mathrm{Pe} = 20$.

topologically concentric around the extremum point inside each cell. We will also assume that the boundary data $g(x) \neq \text{const}$, and $g(x)$ is sufficiently smooth.

The problem of the qualitative behavior of solutions of (1.1)-(1.2) has been studied in various areas where passive scalar advection arises, such as oceanography, meteorology, etc. The effect in which we are interested is the non-trivial coupling of diffusion and strong advection at a high Péclet number. If we neglect diffusion and approximate (1.1) by

$$u \cdot \nabla \phi = 0 \tag{1.3}$$

we see that ϕ has to be approximately constant along the flow lines of u. This is, however, impossible near the boundary where the data is prescribed, creating one source of large gradients – solution wants to be uniform on $\partial\Omega$ but it can not! Another region where (1.3) need not hold is near the stationary points of u. We will see that this leads to a boundary layer along the boundafry $\partial\Omega$ but also to an interior transition layer along the separatrices: see Figure 1.2 for a numerical illustration. The width l of the boundary layer should be such that the diffusion term in (1.1) balances the advection term. If $\phi(x)$ has such a boundary layer then $\Delta\phi \sim 1/l^2$ in the boundary layer while $u \cdot \nabla\phi \sim O(1)$. Therefore, we should have from (1.1):

$$\frac{1}{l^2} \sim \frac{1}{\varepsilon},$$

meaning that $l = O(\varepsilon^{1/2})$. The total dissipation rate would then scale in the corresponding fashion: $|\nabla\phi^\varepsilon| \sim 1/l$ in the boundary layer, and its width is l, hence

$$\int_\Omega |\nabla\phi^\varepsilon(x)|^2 dx \sim l \cdot \frac{1}{l^2} \sim O(\varepsilon^{-1/2}). \tag{1.4}$$

We describe in this section some results that confirm that picture. Note that above qualitative picture: existence of the boundary layers of width $\varepsilon^{1/2}$, as well as the estimate (1.4) immediately preclude the Harnack inequality, oscillation bounds and Hölder estimates that would be uniform in the flow: they all blow-up as $\varepsilon \downarrow 0$!

Theorem 1.1 *Assume that $\partial\Omega$ is a piecewise smooth curve, $u(x)$ is smooth, and the boundary data g in (1.2) is also smooth. There exists a constant $C = C(u, g, \Omega) > 0$ that is independent of $\varepsilon > 0$ so that*

$$\frac{1}{C\sqrt{\varepsilon}} \le \int_\Omega |\nabla\phi(x)|^2 dx \le \frac{C}{\sqrt{\varepsilon}}. \tag{1.5}$$

We will not prove Theorem 1.1 in its full scope – see [28, 50] for the proof, but rather prove the upper bound and a weaker lower bound that still blows up as $\varepsilon \to 0$ but at a much slower rate. Convergence of solution to a constant inside a cell is quantified as follows. Let

$$\mathcal{D}_j(h) = \{x \in \mathcal{C}_j : \ |H(x)| \ge h\}, \quad h > 0,$$

be a domain strictly inside the flow cell $\mathcal{C}_j$, at distance $O(h)$ away from the separatrices.

Theorem 1.2 *There exist constants K_j^ε so that we have inside each cell $\mathcal{C}_j$*

$$\sup_{x \in \mathcal{D}(N\sqrt{\varepsilon})} \left|\phi^\varepsilon(x) - K_j^\varepsilon\right| \le \frac{C}{N^{3/2}}. \tag{1.6}$$

Again, we will prove here only a weaker statement that still implies that solution becomes approximately constant inside each cell, at distance l_ε that vanishes as $\varepsilon \to 0$ (we will estimate $l_\varepsilon \ll \varepsilon^{1/6}$ rather than the true result $l_\varepsilon \sim \sqrt{\varepsilon}$.

1.1 A uniform upper bound

We prove in this section the uniform upper bound on the total dissipation rate in the inequality (1.5) in Theorem 1.1.

Theorem 1.3 *There exists a constant $C = C(u, g, \Omega)$ so that*

$$\int_\Omega |\nabla\phi(x)|^2 dx \le \frac{C}{\sqrt{\varepsilon}}. \tag{1.7}$$

Proof. The proof is by integration by parts. Let ψ^ε be a function to be specified later. We multiply (1.1) by the function $q^\varepsilon = \phi^\varepsilon - \psi^\varepsilon$ and obtain after integration by parts:

$$\varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial\phi^\varepsilon}{\partial\nu} dl - \varepsilon \int_\Omega (\nabla\phi^\varepsilon - \nabla\psi^\varepsilon) \cdot \nabla\phi^\varepsilon dx - \int_\Omega (\phi^\varepsilon - \psi^\varepsilon) u \cdot \nabla\phi^\varepsilon dx = 0.$$

Using incompressibility of the flow u we get

$$\varepsilon \int_\Omega |\nabla \phi^\varepsilon|^2 dx \le \varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial \phi^\varepsilon}{\partial \nu} dl + \varepsilon \int_\Omega |\nabla \psi^\varepsilon|^2 dx + \frac{\varepsilon}{4} \int_\Omega |\nabla \phi^\varepsilon|^2 dx \tag{1.8}$$
$$+\frac{\alpha}{2} \int_\Omega |\psi^\varepsilon|^2 dx + \frac{1}{2\alpha} \int_\Omega |u \cdot \nabla \phi^\varepsilon|^2 dx,$$

with the constant α to be chosen. In order to estimate the last term in (1.8), we multiply (1.1) by $u \cdot \nabla \phi^\varepsilon$ and integrate to get

$$\int_\Omega |u \cdot \nabla \phi^\varepsilon|^2 dx = \varepsilon \int_\Omega (u \cdot \nabla \phi^\varepsilon) \Delta \phi^\varepsilon dx = \varepsilon \int_{\partial\Omega} (u \cdot \nabla \phi^\varepsilon) \frac{\partial \phi^\varepsilon}{\partial \nu} dl - \varepsilon \int_\Omega \nabla (u \cdot \nabla \phi^\varepsilon) \cdot \nabla \phi^\varepsilon dx.$$

Next, we rewrite the last term above as:

$$\nabla (u \cdot \nabla \phi^\varepsilon) \cdot \nabla \phi^\varepsilon = \frac{\partial u_k}{\partial x_i} \frac{\partial \phi^\varepsilon}{\partial x_k} \frac{\partial \phi^\varepsilon}{\partial x_i} + u_k \frac{\partial^2 \phi^\varepsilon}{\partial x_k \partial x_i} \frac{\partial \phi^\varepsilon}{\partial x_i} = \frac{\partial u_k}{\partial x_i} \frac{\partial \phi^\varepsilon}{\partial x_k} \frac{\partial \phi^\varepsilon}{\partial x_i} + \frac{1}{2} u \cdot \nabla (|\nabla \phi^\varepsilon|^2). \tag{1.9}$$

Once again using incompressibility of u, we obtain from the above

$$\int_\Omega |u \cdot \nabla \phi^\varepsilon|^2 dx = \varepsilon \int_{\partial\Omega} (u \cdot \nabla \phi^\varepsilon) \frac{\partial \phi^\varepsilon}{\partial \nu} dl - \frac{1}{2} \varepsilon \int_\Omega (u \cdot \nabla \left(|\nabla \phi^\varepsilon|^2\right)) dx - \varepsilon \int_\Omega \frac{\partial u_n}{\partial x_m} \frac{\partial \phi^\varepsilon}{\partial x_m} \frac{\partial \phi^\varepsilon}{\partial x_n} dx$$
$$\le \varepsilon \int_{\partial\Omega} (u \cdot \nabla \phi^\varepsilon) \frac{\partial \phi^\varepsilon}{\partial \nu} dl + \varepsilon M \int_\Omega |\nabla \phi^\varepsilon|^2 dx, \tag{1.10}$$

where $M = \|\nabla u\|_{L^\infty(\Omega)}$. We insert (1.10) into (1.8) to get

$$\frac{3\varepsilon}{4} \int_\Omega |\nabla \phi^\varepsilon|^2 dx \le \varepsilon \int_{\partial\Omega} q^\varepsilon \frac{\partial \phi^\varepsilon}{\partial \nu} dl + \varepsilon \int_\Omega |\nabla \psi^\varepsilon|^2 dx + \frac{\alpha}{2} \int_\Omega |\psi^\varepsilon|^2 dx$$
$$+ \frac{\varepsilon}{2\alpha} \left(\int_{\partial\Omega} (u \cdot \nabla \phi^\varepsilon) \frac{\partial \phi^\varepsilon}{\partial \nu} dl + M \int_\Omega |\nabla \phi^\varepsilon|^2 dx \right).$$

With the choice $\alpha = 2M$ the above becomes

$$\frac{\varepsilon}{2} \int_\Omega |\nabla \phi^\varepsilon|^2 dx \le \varepsilon \int_{\partial\Omega} \left[q^\varepsilon + \frac{1}{4M} (u \cdot \nabla \phi^\varepsilon) \right] \frac{\partial \phi^\varepsilon}{\partial \nu} dl + \varepsilon \int_\Omega |\nabla \psi^\varepsilon|^2 dx + M \int_\Omega |\psi^\varepsilon|^2 dx.$$

The function q^ε (and thus $\psi^\varepsilon = \phi^\varepsilon - q^\varepsilon$) is arbitrary so far. We now require that

$$q^\varepsilon + \frac{1}{4M} (u \cdot \nabla \phi^\varepsilon) = 0$$

on the boundary $\partial\Omega$. However, $\partial\Omega$ is a streamline of u so that $u \cdot \nabla \phi^\varepsilon = u \cdot \nabla g$ is a given function. That imposes a boundary condition on the function ψ^ε:

$$\psi^\varepsilon(x) = g(x) + \frac{1}{4M} (u \cdot \nabla g(x)) \quad \text{for } x \in \partial\Omega. \tag{1.11}$$

Therefore, for all test functions ψ^ε that satisfy the boundary condition (1.11) we have

$$\frac{\varepsilon}{2} \int_\Omega |\nabla \phi^\varepsilon|^2 dx \le \varepsilon \int_\Omega |\nabla \psi^\varepsilon|^2 dx + 4M \int_\Omega |\psi^\varepsilon|^2 dx. \tag{1.12}$$

Instead of optimizing over ψ^ε, we simply may choose a function ψ^ε so that it satisfies the boundary conditions (1.11), vanishes identically at distances larger that $\sqrt{\varepsilon}$ away from $\partial\Omega$ and satisfies the uniform bounds $\|\psi^\varepsilon\|_{L^\infty(\Omega)} \leq C$, $\|\nabla\psi^\varepsilon\|_{L^\infty(\Omega)} \leq C/\sqrt{\varepsilon}$. For such a test function we have

$$\int_\Omega |\psi^\varepsilon(x)|^2 dx \leq C\sqrt{\varepsilon},$$

and

$$\int_\Omega |\nabla\phi^\varepsilon|^2 dx \leq \frac{C}{\sqrt{\varepsilon}}.$$

Using these bounds in (1.12) we obtain the upper bound (1.7). $\square$

1.2 Oscillation on streamlines

We now show that the function ϕ^ε, solution of

$$\begin{aligned}&\Delta\phi^\varepsilon - \frac{1}{\varepsilon}u\cdot\nabla\phi^\varepsilon = 0,\\ &\phi^\varepsilon(x) = g(x), \quad x \in \partial\Omega,\end{aligned} \tag{1.13}$$

does not oscillate too much along the streamlines of the flow $u(x)$. We will not prove the sharpest possible results such as in Theorem 1.2 or the lower bound in Theorem 1.1 since that would require a more involved argument. Nevertheless, we will show that solution of (1.13) becomes approximately equal to a constant inside each flow cell $\mathcal{C}_j$, with narrow interior layers between cells where solution changes its value from one constant to another. This will already preclude Hölder estimates on $\phi^\varepsilon(x)$ that would be uniform in the flow – the gradient of the solution tends to infinity as $\varepsilon \to 0$ near the skeleton of separatrices.

In order to establish existence of the interior layers, take a function $\eta(x) = p(H(x)/h)$ with a smooth function p such that $p(s) = 0$ for $0 \leq |s| \leq 1/2$ and $p(s) = 1$ for $|s| \geq 3/4$. Then $\eta(x)$ vanishes near the boundary (we take h small) and near all of the separatrices (recall that $H(x) = 0$ on the boundary and on the skeleton of separatrices), and

$$u\cdot\nabla\eta = 0. \tag{1.14}$$

In addition, we have the bounds:

$$|\nabla\eta(x)| \leq \frac{C}{h}, \qquad \left|\frac{\partial^2\eta}{\partial x_i\partial x_j}\right| \leq \frac{C}{h^2}. \tag{1.15}$$

Let us multiply (1.13) by $\eta(u\cdot\nabla\phi^\varepsilon)$ and integrate over Ω:

$$\int_\Omega \eta(x)|u\cdot\nabla\phi^\varepsilon|^2 dx = \varepsilon\int_\Omega \eta(x)(u\cdot\nabla\phi^\varepsilon)\Delta\phi^\varepsilon dx. \tag{1.16}$$

Since $\eta(x)$ vanishes on the boundary, we have:

$$\begin{aligned}&\int_\Omega \eta(x)|u\cdot\nabla\phi^\varepsilon|^2 dx = -\varepsilon\int_\Omega \nabla(\eta(x)(u\cdot\nabla\phi^\varepsilon))\cdot\nabla\phi^\varepsilon dx\\ &= -\varepsilon\int_\Omega \eta(x)\nabla(u\cdot\nabla\phi^\varepsilon)\cdot\nabla\phi^\varepsilon - \varepsilon\int_\Omega (u\cdot\nabla\phi^\varepsilon)(\nabla\eta\cdot\nabla\phi^\varepsilon)dx.\end{aligned} \tag{1.17}$$

We rewrite the first term in the right side as in (1.9):

$$\nabla(u\cdot\nabla\phi^\varepsilon)\cdot\nabla\phi^\varepsilon=\frac{\partial u_k}{\partial x_i}\frac{\partial\phi^\varepsilon}{\partial x_k}\frac{\partial\phi^\varepsilon}{\partial x_i}+\frac{1}{2}u\cdot\nabla(|\nabla\phi^\varepsilon|^2). \tag{1.18}$$

Incompressibility of u and (1.14) imply that

$$\int_\Omega\eta(x)[u\cdot\nabla(|\nabla\phi^\varepsilon|^2)]dx=0. \tag{1.19}$$

Using this in (1.16) together with, once again, incompressibility of u gives

$$\begin{aligned}&\int_\Omega\eta(x)|u\cdot\nabla\phi^\varepsilon|^2dx=-\varepsilon\int_\Omega\eta(x)\frac{\partial u_k}{\partial x_i}\frac{\partial\phi^\varepsilon}{\partial x_k}\frac{\partial\phi^\varepsilon}{\partial x_i}dx-\varepsilon\int_\Omega(u\cdot\nabla\phi^\varepsilon)(\nabla\eta\cdot\nabla\phi^\varepsilon)dx\\&\le CM\varepsilon\int_\Omega|\nabla\phi^\varepsilon|^2dx+\frac{C\varepsilon}{h}\int_\Omega|\nabla\phi^\varepsilon|^2dx.\end{aligned} \tag{1.20}$$

Theorem 1.3 implies now that

$$\int_\Omega\eta(x)|u\cdot\nabla\phi^\varepsilon|^2dx\le\frac{C\sqrt{\varepsilon}}{h}. \tag{1.21}$$

This estimate tells us (in an averaged way) that the oscillation of ϕ^ε along the streamlines in the interior of the flow cells where $|H(x)|\ge h$ is small, as soon as $h\gg\sqrt{\varepsilon}$. Let us pass from the averaged statement to a point wise estimate, albeit losing some precision. Consider a closed streamline $H(x)=h$ in one cell $\mathcal{C}_j$:

$$\mathcal{L}_j(h)=\{x\in\mathcal{C}_j:\quad H(x)=h\}.$$

Then for any $x,x'\in\mathcal{L}_j(h)$ we have

$$|\phi^\varepsilon(x)-\phi^\varepsilon(x')|\le\oint_{\mathcal{L}_j(h)}|\tau\cdot\nabla\phi^\varepsilon|dl, \tag{1.22}$$

where $\tau=u/|u|$ is the unit tangent vector along the streamline. We deduce that[1]

$$|\phi^\varepsilon(x)-\phi^\varepsilon(x')|\le\big(\min_{\mathcal{L}_j(h)}|u|\big)^{-1}\oint_{\mathcal{L}_j(h)}|u\cdot\nabla\phi^\varepsilon|dl. \tag{1.23}$$

The flow amplitude $|u|$ vanishes only at the critical points of the stream-function $H(x)$. Let h_c be such that $|H(x)|\ge 2h_c$ at all minima and maxima[2] of H so that the only critical points of $H(x)$ in the region $D_0=\{|H(x)|\le h_c\}$ are the saddles of $H(x)$. As $H(x)=0$ at all saddle points, the function $H(x)$ vanishes quadratically near the saddles while $u(x)$ vanishes linearly around these points. Therefore, for the minimum of $|u|$ we have an estimate:

$$|u(x)|\ge C\sqrt{|H(x)|},\quad x\in D_0. \tag{1.24}$$

[1]This is a very rough estimate – in reality u is small only when the level set $\mathcal{L}(h)$ passes near a saddle point of H, and is $O(1)$ everywhere else along the level set.

[2]All maxima and minima of $H(x)$ are located inside the cells and away from the sepratricies – the only critical points of $H(x)$ on the skeleton of separatrices are saddle points of $H(x)$.

Then (1.23) implies

$$|\phi^\varepsilon(x)-\phi^\varepsilon(x')|\le \frac{C}{\sqrt{h}}\oint_{\mathcal{L}_j(h)}|u\cdot\nabla\phi^\varepsilon|dl,\quad \text{for any } x,x'\in\mathcal{L}_j(h),\ |h|\le h_c. \tag{1.25}$$

Using the Cauchy-Schwartz inequality gives

$$|\phi^\varepsilon(x)-\phi^\varepsilon(x')|^2\le I(h):=\frac{C}{h}\oint_{\mathcal{L}_j(h)}|u\cdot\nabla\phi^\varepsilon|^2dl,\quad \text{for any } x,x'\in\mathcal{L}_j(h),\ |h|\le h_c, \tag{1.26}$$

since the length $|\mathcal{L}_j(h)|$ of a level set is uniformly bounded by a constant. Let us assume without loss of generality that $H(x)\ge 0$ in the cell $\mathcal{C}_j$, and average this bound over an interval $h_0\le h\le 2h_0$, with $h_0\le h_c/2$: We get

$$\frac{1}{h_0}\int_{h_0}^{2h_0}I(h)dh\le\frac{C}{h_0^2}\int_{h_0}^{2h_0}\oint_{\mathcal{L}_j(h)}|u\cdot\nabla\phi^\varepsilon|^2dldh=\frac{C}{h_0^2}\int_{D_j(h_0)}|u\cdot\nabla\phi^\varepsilon|^2|\nabla H(x)|dx, \tag{1.27}$$

with the domain

$$D_j(h_0)=\{x\in\mathcal{C}_j:\quad h_0\le H(x)\le 2h_0\}.$$

We deduce that

$$\frac{1}{h_0}\int_{h_0}^{2h_0}I(h)dh\le\frac{C}{h_0^2}\int_{D_j(h_0)}|u\cdot\nabla\phi^\varepsilon|^2dx\le\frac{C}{h_0^2}\int_\Omega\eta_{h_0}(x)|u\cdot\nabla\phi^\varepsilon|^2dx\le\frac{C\sqrt{\varepsilon}}{h_0^3}. \tag{1.28}$$

It follows that there exists a level set $\mathcal{L}_j(h)$, with $h_0\le h\le 2h_0$ so that

$$I(h)\le\frac{C\sqrt{\varepsilon}}{h_0^3},$$

which, together with (1.26) implies that

$$|\phi^\varepsilon(x)-\phi^\varepsilon(x')|^2\le\frac{C\sqrt{\varepsilon}}{h_0^3},\quad \text{for any } x,x'\in\mathcal{L}_j(h), \tag{1.29}$$

that is, the oscillation over $\mathcal{L}_j(h)$ is small provided that $h_0\gg\varepsilon^{1/6}$.

Moreover, the function $\phi^\varepsilon(x)$ satisfies the elliptic equation (1.13). It follows that for any h the maximum and minimum of $\phi^\varepsilon(x)$ over the domain

$$\{x\in\mathcal{C}_j:\ |H(x)|\ge h\}$$

bounded by the closed curve $\mathcal{L}_j(h)$ is attained on its boundary $\mathcal{L}_j(h)$. In particular, for any h and h' such that $|h'|>|h|$, the maxima

$$M_j(h)=\sup_{x\in\mathcal{L}_j(h)}\phi^\varepsilon(x),\quad M_j(h')=\sup_{x\in\mathcal{L}_j(h')}\phi^\varepsilon(x),$$

and minima

$$m_j(h)=\inf_{x\in\mathcal{L}_j(h)}\phi^\varepsilon(x),\quad m_j(h')=\sup_{x\in\mathcal{L}_j(h')}\phi^\varepsilon(x),$$

satisfy

$$m_j(h) \le m_j(h') \le M_j(h') \le M_j(h).$$

Therefore, since (1.29) holds for some $h \in [h_0, 2h_0]$, we deduce that

$$|\phi^\varepsilon(x) - \phi^\varepsilon(x')|^2 \le \frac{C\sqrt{\varepsilon}}{h_0^3}, \quad \text{for any } x, x' \in \mathcal{L}_j(h) \text{ and } h \ge 2h_0. \tag{1.30}$$

As a consequence, there exists a constant K_j^ε so that

$$|\phi^\varepsilon(x) - K_j^\varepsilon|^2 \le \frac{C\sqrt{\varepsilon}}{h_0^3}, \quad \text{for any } x \in \mathcal{C}_j \text{ such that } h \ge 2h_0. \tag{1.31}$$

In particular, if we take $h_0 = \varepsilon^p$ with $0 < p < 1/6$, we see that

$$|\phi^\varepsilon(x) - K_j^\varepsilon|^2 \le C\varepsilon^{1/2-3p}, \quad \text{for any } x \in \mathcal{C}_j \text{ such that } h \ge \varepsilon^p. \tag{1.32}$$

This gives an upper bound on the width of the boundary layer as $O(\varepsilon^{1/6})$. Let us repeat that this is a very crude estimate – the true width of the boundary layer is $O(\sqrt{\varepsilon})$ but the proof of this requires more delicate arguments that are too particular to reproduce them here.

As a consequence, we deduce that

$$\int_\Omega |\nabla\phi^\varepsilon(x)|^2 dx \ge \frac{C_p}{\varepsilon^p}, \quad \text{for any } p \in (0, 1/6). \tag{1.33}$$

We also conclude that Hölder estimates for $\phi^\varepsilon(x)$ can not hold uniformly in ε even away from the boundary – $\phi^\varepsilon(x)$ changes across the separatrices over very short distances.

Exercise 1.4 One may wonder if all K_j^ε are always equal – this would disprove the above assertion about the failure of the Hölder estimates to be uniform in ε. Consider the flow with the same stream-function $H(x_1, x_2) = \sin x_1 \sin x_2$ in three domains: $\Omega_1 = [0, 2\pi] \times [0, 2\pi]$, $\Omega_2 = [2\pi, 4\pi] \times [0, 2\pi]$ and $\Omega_3 = [0, 4\pi] \times [0, 2\pi]$. Impose the periodic boundary conditions in x_2 and the following three boundary conditions in x_1:

$$\begin{aligned} \phi_1^\varepsilon(0, x_2) &= 0, & \phi_1^\varepsilon(2\pi, x_2) &= 2\pi, \\ \phi_2^\varepsilon(2\pi, x_2) &= 2\pi, & \phi_2^\varepsilon(4\pi, x_2) &= 4\pi, \\ \phi_3^\varepsilon(0, x_2) &= 0, & \phi_3^\varepsilon(4\pi, x_2) &= 4\pi. \end{aligned}$$

Show that each $\phi_j^\varepsilon(x)$, $j = 1, 2, 3$, can be written as $\phi_j^\varepsilon(x) = x_1 + \psi_j^\varepsilon(x)$, with a periodic function $\psi_j^\varepsilon(x)$. Use symmetries of the stream-function to express $\phi_2^\varepsilon(x)$ in terms of $\phi_1^\varepsilon(x)$, and $\phi_3^\varepsilon(x)$ in terms for $\phi_1^\varepsilon(x)$ and $\phi_2^\varepsilon(x)$, and explain why the values of $\phi_3^\varepsilon(x)$ inside each of the flows cells have to differ by $O(1)$.

2 Behavior of the Dirichlet eigenvalues

2.1 Eigenvalues of the Laplacian

Let us recall some very basic facts about the principal Dirichlet eigenvalues for the Laplacian on a bounded domain [25]. For any smooth bounded domain Ω there exists an eigenvalue λ_1

(called the principal eigenvalue) that corresponds to a positive eigenfunction $\phi_1 > 0$ in Ω:

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad x \in \Omega, \tag{2.1}$$
$$\phi_1 = 0 \text{ on } \partial\Omega.$$

Moreover, λ_1 is the smallest of all eigenvalues of the Dirichlet Laplacian on Ω, λ_1 is a simple eigenvalue and all other eigenfunctions of the Laplacian change sign in Ω. The principal eigenvalue is given by the Rayleigh quotient:

$$\lambda_1 = \inf_{\substack{\psi\in H_0^1(\Omega)\\ \|\psi\|_2=1}} \int_\Omega |\nabla\psi|^2 dx. \tag{2.2}$$

The principal eigenvalue determines the long time decay of solutions of the parabolic initial value problem in the following way. Consider the initial value problem

$$\psi_t = \Delta\psi, \quad t > 0, x \in \Omega, \tag{2.3}$$
$$\psi(t,x) = 0 \text{ on } \partial\Omega,$$
$$\psi(0,x) = g(x).$$

As $\phi_1(x) > 0$ in Ω, and, as follows from the Hopf lemma, $\partial\phi_1/\partial\nu < 0$ on $\partial\Omega$, we can find a constant $C > 0$ so that $\psi(1,x) \le C\phi_1(x)$ – we can not quite have such estimate at $t = 0$ since $g(x)$ may not satisfy the Dirichlet boundary conditions. The maximum principle implies that

$$\psi(t,x) \le Ce^{-\lambda_1(t-1)}\phi_1(x), \tag{2.4}$$

for $t > 1$, and, similarly,

$$-\psi(t,x) \ge -Ce^{-\lambda_1 t}\phi_1(x), \tag{2.5}$$

so that

$$|\psi(t,x)| \le Ce^{-\lambda_1 t}\phi_1(x), \quad t \ge 1. \tag{2.6}$$

Therefore, all solutions of the Cauchy problem decay at the exponential rate determined by λ_1.

2.2 The Dirichlet eigenvalues with a drift

Let us now consider the Dirichlet principal eigenvalue problem in a smooth bounded domain Ω, for a diffusion with a strong incompressible flow:

$$-\Delta\phi + \frac{1}{\varepsilon}u \cdot \nabla\phi = \lambda_1(\varepsilon)\phi, \quad \phi(x) > 0 \text{ in } \Omega, \tag{2.7}$$
$$\phi = 0 \text{ on } \partial\Omega.$$

We assume here that $\nabla \cdot u = 0$. Though the operator in the left side is not self-adjoint (so that its eigenvalues are not necessarily real), the Krein-Rutman theory for positive operators (see Chapter VIII of [22]) implies that it has a unique eigenvalue $\lambda_1(\varepsilon)$ that corresponds to a positive eigenfunction $\phi_1(x)$. This eigenvalue is real and simple, has the smallest real part of all eigenvalues, and is called the principal eigenvalue. As for the Laplacian, the maximum principle implies that the principal eigenvalue determines the long time decay of the solutions of the corresponding Cauchy problem:

$$\begin{aligned}
&\psi_t + \frac{1}{\varepsilon} u \cdot \nabla \psi = \Delta \psi, \quad t > 0, x \in \Omega, \\
&\psi(t,x) = 0 \text{ on } \partial\Omega, \\
&\psi(0,x) = g(x).
\end{aligned} \tag{2.8}$$

Let us recall the probabilistic interpretation of the solutions of (2.8). Consider the stochastic differential equation

$$dX_t = -\frac{1}{\varepsilon} u(X_t)dt + \sqrt{2}dW_t, \quad X_0 = x, \tag{2.9}$$

starting at a point $x \in \Omega$, and let τ be the first time that X_t hits the boundary $\partial\Omega$. Then solution of the Cauchy problem (2.8) can be expressed in terms of the diffusion X_t as

$$\psi(t,x) = \mathbb{E}_x[g(X_{\min(t,\tau))}], \tag{2.10}$$

with the convention that $g(X_\tau) = 0$. Intuitively, if the trajectories of the incompressible flow are "sufficiently mixing", for any point x_0 in the interior of Ω the trajectory of $u(x)$ that passes through x_0 eventually comes close to the "cold' boundary $\partial\Omega$. Therefore, such flow, when sufficiently fast, will force solutions of (2.10) very quickly to pass very close to $\partial\Omega$, and at that time diffusion will force X_t to exit Ω with a very high probability. This should make the exit time τ of the solutions of (2.9) smaller than t with a high probability, making $\psi(t,x)$ given by (2.10) very small since $g(X_\tau) = 0$. Physically, this means that a sufficiently mixing flow, together with diffusion, should dramatically increase the cooling of the interior by the boundary. One way to quantify this effect is as follows: solutions of the Cauchy problem (2.8) have a long time asymptotics

$$\psi(t,x) \sim e^{-\lambda_1(\varepsilon)t}\phi(x), \text{ for } t \gg 1,$$

as in (2.6). If they, indeed, decay faster as $\varepsilon \to 0$, we should have $\lambda_1(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ for "mixing" flows. Note that usually the mixing properties of a flow are defined in terms of its dynamical systems properties, by how solutions of the ODE

$$\dot{X} = u(X),$$

behave. Here, we are asking a PDE question – hence, the first problem is to define what "mixing" means for us. This is quantified by the following beautiful result due to Berestycki, Hamel and Nadirashvili [11]. We denote by $\mathcal{I}_0$ the set of all first integrals of u, solutions of

$$u \cdot \nabla \bar{\phi} = 0 \text{ a.e. in } \Omega, \tag{2.11}$$

in $H_0^1(\Omega)$.

Theorem 2.1 *The principal eigenvalue $\lambda_1(\varepsilon)$ of (2.7) tends to $+\infty$ as $\varepsilon \to 0$ if and only if the flow u has a first integral in $H_0^1(\Omega)$. Moreover, if u has a first integral in $H_0^1(\Omega)$, then*

$$\lambda_1(\varepsilon) \to \bar{\lambda} := \min_{w \in \mathcal{I}_0} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w|^2 dx} \ \textit{as } \varepsilon \to 0, \tag{2.12}$$

and the minimum in the right side is achieved.

A couple of comments are in order. First, notice that the only information about the Laplacian operator in (2.7) that survives in the statement of the theorem is in the condition that the first integral lies in $H_0^1(\Omega)$ – the regularity requirement comes exactly from the Laplacian. Irregular first integrals do not prevent strong decay of solutions of the Cauchy problem. Second, the strong flow essentially forces the eigenfunction to be close to a first integral, and then the variational principle (2.3) for the Laplacian operator is replaced by essentially the same expression (2.12) except that the set of test functions is restricted to the first integrals.

Proof of Theorem 2.1

First, assume that there exists a sequence $\varepsilon_n \to 0$ such that $\lambda_1(\varepsilon_n)$ are bounded. Consider the associated sequence of positive eigenfunctions $\phi_n(x)$ normalized so that $\|\phi_n\|_{L^2(\Omega)} = 1$. Then, as

$$\int_\Omega |\nabla \phi_n(x)|^2 dx = \lambda_1(\varepsilon_n) \int_\Omega |\phi_n(x)|^2 dx = \lambda_1(\varepsilon_n), \tag{2.13}$$

there exists a subsequence n_k so that the sequence ϕ_{n_k} converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function $\bar{w}(x) \in H_0^1(\Omega)$. Moreover, multiplying (2.7) by ε_{n_k} and passing to the limit $k \to +\infty$ gives

$$u \cdot \nabla \bar{w} = 0, \quad \text{weakly in } H_0^1(\Omega),$$

hence $\bar{w}$ is a first integral of u in $H_0^1(\Omega)$. Furthermore, as convergence of ϕ_{n_k} to $\bar{w}$ is strong in $L^2(\Omega)$ and weak in $H_0^1(\Omega)$, it also follows from (2.13) and Fatou's lemma that

$$\liminf_{n \to +\infty} \lambda_1(\varepsilon_n) \geq \frac{\int_\Omega |\nabla \bar{w}(x)|^2 dx}{\int_\Omega |\bar{w}(x)|^2 dx}. \tag{2.14}$$

Next, we claim that if u has a first integral w in $H_0^1(\Omega)$, then for any $\varepsilon \in \mathbb{R}$ and any $w \in \mathcal{I}_0$ we have

$$0 \leq \lambda_1(\varepsilon) \leq \frac{\int_\Omega |\nabla w(x)|^2 dx}{\int_\Omega |w(x)|^2 dx}. \tag{2.15}$$

In order to show that (2.15) holds, we take any $w \in \mathcal{I}_0$, multiply (2.7) by $w^2/(\phi + \delta)$ with $\delta > 0$ fixed:

$$-\int_\Omega \frac{w^2}{\phi + \delta} \Delta \phi \, dx + \int_\Omega \frac{w^2}{\phi + \delta} (u \cdot \nabla \phi) dx = \lambda_1(\varepsilon) \int_\Omega \frac{w^2}{\phi + \delta} \phi \, dx. \tag{2.16}$$

Integrating by parts in the first term gives

$$\begin{aligned} -\int_\Omega \frac{w^2}{\phi + \delta} \Delta \phi \, dx &= \int_\Omega \nabla \phi \cdot \nabla \left(\frac{w^2}{\phi + \delta} \right) dx = \int_\Omega \frac{2w(\phi + \delta) \nabla \phi \cdot \nabla w - w^2 |\nabla \phi|^2}{(\phi + \delta)^2} dx \\ &\leq \int_\Omega |\nabla w|^2 dx. \end{aligned}$$

The second term in the left side of (2.16) vanishes because $\nabla \cdot u = 0$ and w is a first integral:

$$\int_\Omega \frac{w^2}{\phi + \delta} (u \cdot \nabla \phi) \, dx = \int_\Omega w^2 (u \cdot \nabla (\log \phi + \delta)) \, dx = \int_\Omega 2w \log(\phi + \delta)(u \cdot \nabla w) dx = 0.$$

The boundary terms above vanish since $w \in H_0^1(\Omega)$ (it vanishes on the boundary). We conclude that

$$\lambda_1(\varepsilon)\int_\Omega \frac{w^2}{\phi+\delta}\phi dx \le \int_\Omega |\nabla w|^2 dx, \tag{2.17}$$

for any $w \in \mathcal{I}_0$. Passing to the limit $\delta \to 0$ gives (2.15). It remains to notice that (2.14) and (2.15) together imply the Rayleigh quotient formula (2.12).

3 Relaxation enhancement

3.1 Relaxation enhancing flows

As we have discussed, one interpretation of the eigenvalue enhancement estimate in Theorem 2.1 is in terms of the long time decay rate of the solution of the Cauchy problem

$$\begin{aligned} &\psi_t + \frac{1}{\varepsilon}u\cdot\nabla\psi = \Delta\psi, \quad t>0, x\in\Omega, \\ &\psi(t,x) = 0 \text{ on } \partial\Omega, \\ &\psi(0,x) = g(x), \end{aligned} \tag{3.1}$$

in Ω with the Dirichlet boundary condition. Its solution has the long time asymptotics

$$\psi(t,x) \sim e^{-\lambda_1(\varepsilon)t}\phi(x)$$

for $t \gg 1$, where $\phi(x)$ is the principal eigenfunction of the operator

$$-\Delta\phi + \frac{1}{\varepsilon}u\cdot\nabla\phi = \lambda_1(\varepsilon)\phi,$$

with the Dirichlet boundary conditions, and the exponential rate (the principal eigenvalue) $\lambda_1(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ if and only if the flow u has no first integrals in $H_0^1(\Omega)$.

In this section, we focus on similar questions in the case of a compact manifold without boundary or the Neumann boundary conditions. Then, the principal eigenvalue is simply zero and corresponds to the constant eigenfunction. One may instead study the second eigenvalue but that is not simple since we do not even know a priori that the second eigenvalue is real, and finding estimates for the real part of a complex eigenvalue that corresponds to an eigenfunction that also need not be real would not be an easy task. To the best of our knowledge, there is very little known about such estimates with the Neumann boundary conditions (or a manifold without boundary): see [41] p. 251 for a discussion. Neither probabilistic methods nor PDE methods of [11] seem to yield results easily in this situation. Moreover, even if the spectral gap estimate were available, generally it only provides a long time dynamical information, and how fast the long time limit is achieved may depend on ε.

On the other hand, our general interest is in the speed of convergence of the solution to its average, the relaxation speed, and there are ways to measure it other than in terms of the spectrum. Therefore, rather than address the spectral questions, we will reformulate our questions purely in terms of the Cauchy problem. Let us mention that there are various results on Gaussian and other estimates on the heat kernel corresponding to the incompressible drift and diffusion on manifolds such as the work of Norris [49] and Franke [35], but these estimates lead to upper bounds on the convergence rate to the equilibrium which essentially do not improve as $\varepsilon \to 0$, and thus do not quite address the effect of a strong flow – they are more in the spirit of Theorems 2.2 and 3.1 of Chapter 2.

Let Ω be a smooth compact n-dimensional Riemannian manifold. We consider solutions of the passive scalar equation

$$\phi^\varepsilon_t + \frac{1}{\varepsilon}u(x)\cdot\nabla\phi^\varepsilon - \Delta\phi^\varepsilon = 0, \quad \phi^\varepsilon(0,x) = \phi_0(x). \tag{3.2}$$

Here Δ is the Laplace-Beltrami operator on Ω, u is a divergence free vector field, ∇ is the covariant derivative, and $\varepsilon \in \mathbb{R}$ is a parameter regulating the strength of the flow: its amplitude $A = 1/\varepsilon$. For the sake of concreteness we will assume that Ω is a torus, and (3.2) is supplemented by periodic boundary conditions but the results of this section apply verbatim to advection-diffusion equations on a compact manifold without boundary, or with Neumann boundary conditions on a compact domain $\Omega \subset \mathbb{R}^n$, and with very slight modifications to the Robin boundary conditions on such domains – see [18] for the full cornucopia.

As time tends to infinity, the solution $\phi^\varepsilon(t,x)$ tends to its average,

$$\overline{\phi}(t) \equiv \frac{1}{|\Omega|}\int_\Omega \phi^\varepsilon(t,x)\,d\mu = \frac{1}{|\Omega|}\int_\Omega \phi_0(x)\,dx. \tag{3.3}$$

Here $|\Omega|$ is the volume of Ω. To see that, first, integrating (3.2) over M and using incompressibility of u gives

$$\frac{d}{dt}\int_\Omega \phi(t,x)dx = 0,$$

hence $\bar\phi(t) = \bar\phi(0)$ is preserved in time. Next, multiplying (3.2) by $\phi^\varepsilon(t,x) - \bar\phi$, and using incompressibility of $u(x)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\phi(t,x) - \bar\phi|^2 dx = -\int_\Omega |\nabla\phi^\varepsilon(t,x)|^2 dx. \tag{3.4}$$

The Poincaré inequality implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\phi(t,x) - \bar\phi|^2 dx \le -C_p\int_\Omega |\phi^\varepsilon(t,x) - \bar\phi|^2 dx, \tag{3.5}$$

whence

$$\|\phi(t,\cdot) - \bar\phi\|_{L^2(\Omega)} \le e^{-C_p t}\|\phi_0 - \bar\phi\|_{L^2(\Omega)}. \tag{3.6}$$

Exercise 3.1 Use the same strategy as in the proof of Theorem 2.2 of Chapter 2 to strengthen this result and show that

$$\|\phi(t,\cdot) - \bar\phi\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to +\infty. \tag{3.7}$$

We would like to understand how the speed of convergence to the average in (3.7) depends on the properties of the flow and determine which flows are particularly efficient in enhancing the relaxation process. We will use the following "fixed time" (no long time limit!) definition as a measure of the flow efficiency in improving the relaxation of the solution to a uniform state.

Definition 3.2 *An incompressible flow u on Ω is called relaxation enhancing if for every $\tau > 0$ and $\delta > 0$, there exist $\varepsilon(\tau, \delta)$ such that for any $\varepsilon < \varepsilon(\tau, \delta)$ and any $\phi_0 \in L^2(\Omega)$ with $\|\phi_0\|_{L^2(\Omega)} = 1$ we have*

$$\|\phi^\varepsilon(\tau, \cdot) - \overline{\phi}\|_{L^2(\Omega)} < \delta, \tag{3.8}$$

where $\phi^\varepsilon(t, x)$ is the solution of (3.2) and $\overline{\phi}$ the average of ϕ_0.

The choice of the L^2 norm in the definition is not essential and can be replaced by any L^p-norm with $1 \leq p \leq \infty$ – once again, using the same strategy as in the proof of Theorem 2.2 of Chapter 2, see [18] for extensions in this direction.

The motivation for this definition is the work of Fannjiang, Nonnemacher and Wolowski [29, 30, 31], where relaxation enhancement was studied in the discrete setting (see also [41] for related earlier references). In these papers a unitary evolution step (a certain measure preserving map on the torus) alternates with a dissipation step, which, for example, acts simply by multiplying the Fourier coefficients by damping factors. The absence of sufficiently regular eigenfunctions appears as a key for the enhanced relaxation in this particular class of dynamical systems. In [29, 30, 31], the authors also provide finer estimates of the dissipation time for particular classes of toral automorphisms (that is, they estimate how many steps are needed to reduce the L^2 norm of the solution by a factor of two if the dissipation strength is ε).

To understand why and when we expect relaxation enhancement, let us first look at the time-splitting approximation for (3.2). Assume that $\psi(t, x)$ solves the advection equation

$$\psi_t + \frac{2}{\varepsilon} u \cdot \nabla \psi = 0, \quad n\tau \leq t \leq (n + 1/2)\tau, \tag{3.9}$$

followed by the heat equation

$$\psi_t = 2\Delta\psi, \quad (n + 1/2)\tau \leq n\tau, \tag{3.10}$$

and then again (3.9) followed by (3.10), and so on. As the time step $\tau \to 0$ solution of this time-splitting scheme converges to the solution of (3.9). However, the smallness of τ that is required to make the error small depends on ε in a way that is very difficult to control efficiently. If we, in a rather cavalier fashion, fix the size of τ that is independent of ε then solution of the very first step is

$$\psi(\tau/2, x) = \phi_0(X(\tau/\varepsilon, x)), \tag{3.11}$$

where $X(t, x)$ is the trajectory

$$\dot{X}(t) = -u(X), \quad X(0) = x. \tag{3.12}$$

If the flow of (3.12) is sufficiently complex then the function $\psi(\tau/2, x)$ given by (3.11) has large gradients since the points $X(\tau/\varepsilon, x)$ and $X(\tau/\varepsilon, x')$ may be very far apart (meaning that the difference $\psi(\tau/2, x) - \psi(\tau/2, x')$ may be large) if ε is sufficiently small, even if x and x' are very close. This means that the initial data for the second step

$$\psi_t = 2\Delta\psi, \quad \tau/2 \leq \tau, \tag{3.13}$$

has a very large gradient. On the other hand, the dissipation identity for (3.13)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi-\bar{\psi}|^2 = -\int_\Omega |\nabla\psi|^2 dx \tag{3.14}$$

tells us that solutions with large gradient and zero average decay very fast. Therefore, we would deduce that for "sufficiently mixing" flows $u(x)$ solutions converge to their average very fast if ε is small. The problem with making this argument rigorous is that, as we have mentioned, for the convergence of the time-splitting scheme to the true solution we would need to take τ not fixed but $\tau \ll \varepsilon$, making the interaction of advection and diffusion non-trivial and very difficult to account for carefully. Neverthless, this intuition is correct. Here is the main result of this section.

Theorem 3.3 *([18]) A Lipschitz continuous incompressible flow $u \in Lip(\Omega)$ is relaxation enhancing if and only if the operator $u\cdot\nabla$ has no eigenfunctions in $H^1(\Omega)$, other than the constant function.*

The "sufficiently mixing" property of u is encoded in this theorem in the requirement that it does not have an eigenfunction in $H^1(\Omega)$. The reason for that condition can also be seen from the time-splitting scheme (3.11)-(3.13). The operator $u\cdot\nabla$ is skew-symmetric when u is divergence free:

$$\int_\Omega (u\cdot\nabla\eta(x))\eta(x)dx = 0, \tag{3.15}$$

for all $\eta \in H^1(\Omega)$. Therefore, all its eigenvalues $\lambda = i\omega$ are purely imaginary, and if $w \in H^1(\Omega)$ is an eigenfunction, then solution of (3.11)

$$\psi_t + \frac{2}{\varepsilon}u\cdot\nabla\psi = 0, \quad 0 \le t \le \tau/2 \tag{3.16}$$

with the initial data $\psi(0,x) = w(x)$ satisfies

$$\psi(t,x) = e^{2i\omega t/\varepsilon}w(x).$$

Therefore, $\|\psi(\tau/2,x)\|_{H^1(\Omega)} = \|w\|_{H^1(\Omega)}$ does not increase, hence the advection step does not prepare an irregular initial data for the heat equation in the second step, and there is no intuitive reason to expect relaxation enhancement when $\varepsilon \to 0$.

Furthermore, as in the eigenvalue enhancement in Theorem 2.1 the way the Laplacian operator (that is responsible for dissipation) enters the statement of Theorem 3.3 is in the requirement that the eigenfunction lies in the space $H^1(\Omega)$ – rough eigenfunctions (outside of $H^1(\Omega)$) do not prevent a flow $u(x)$ from being relaxation enhancing.

The discrepancy between Theorems 2.1 and 3.3 may seem surprising – after all, on the physical level the conditions for the relaxation enhancement and eigenvalue enhancement need not be very different but eigenvalue enhancement (with the Dirichlet boundary conditions) requires that the operator $u\cdot\nabla$ does not have first integrals while relaxation enhancement (with the periodic or Neumann boundary conditions) requires that this operator does not have eigenfunctions in $H^1(\Omega)$ with any eigenvalue (the first integral corresponds to a zero eigenvalue). This issue is resolved by the following

Proposition 3.4 *Let $u \in Lip(\Omega)$. If $\phi \in H^1(\Omega)$ is an eigenfunction of the operator $u \cdot \nabla$ corresponding to an eigenvalue $i\omega$, $\omega \in \mathbb{R}$, then $|\phi| \in H^1(\Omega)$ and it is the first integral of u, that is, $u \cdot \nabla|\phi| = 0$.*

Proof. The fact that $|\phi| \in H^1$ follows from the well-known properties of Sobolev functions (see, for example, [25]). If $\phi(x)$ satisfies

$$u \cdot \nabla\phi = i\omega\phi$$

then

$$u \cdot \nabla|\phi|^2 = u \cdot \nabla(\phi\bar{\phi}) = \phi(u \cdot \nabla\bar{\phi}) + \bar{\phi}(u \cdot \nabla\phi) = -i\omega\phi\bar{\phi} + i\omega\phi\bar{\phi} = 0,$$

hence $u \cdot \nabla|\phi| = 0$. □

Therefore, in the case of the Dirichlet boundary conditions, if $\phi \in H_0^1(\Omega)$ is an eigenfunction of the operator $u\cdot\nabla$ then $|\phi|$ is its first integral. Naturally, $|\phi|$ can not be equal identically to a constant since ϕ satisfies the Dirichlet boundary conditions (it lies in $H_0^1(\Omega)$). Moreover, if $\phi \in H_0^1(\Omega)$ is a first integral: $u \cdot \nabla\phi = 0$ then it is an eigenfunction (corresponding to eigenvalue $\lambda = 0$). Hence, for the Dirichlet boundary conditions the requirement that $u \cdot \nabla$ does not have a first integral in $H_0^1(\Omega)$ is equivalent to the condition that it does not have eigenfunctions in $H_0^1(\Omega)$.

On the other hand, existence of mean zero $H^1(\Omega)$ eigenfunctions (without imposing the Dirichlet boundary condition) need not guarantee the existence of a mean zero first integral, as can be seen from the following well-known example. Let $\alpha \in \mathbb{R}^n$ be a constant vector generating an irrational rotation on the n-dimensional torus Ω (that is, the components of α are independent over the field of rationals). The operator $\alpha \cdot \nabla$ has eigenvalues $2\pi i\alpha \cdot k$, where k are all possible vectors with integer components. The corresponding eigenfunctions are $w_k(x) = e^{2\pi ik\cdot x}$, $x \in \Omega$. Their absolute value is 1, which is a first integral of $\alpha\cdot\nabla$ but there are no other first integrals since α is irrational. Indeed, if there exists a function $\psi \in L^1(\Omega)$ such that

$$\psi(x + \alpha t) = \psi(x), \quad \text{for all } x \in \Omega \text{ and all } t \in \mathbb{R},$$

then the Fourier coefficients of the function ψ, defined by

$$\psi(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi ik\cdot x}\hat{\psi}_k, \quad \hat{\psi}_k = \int_\Omega e^{-2\pi ik\cdot y}\psi(y)dy,$$

should satisfy

$$\hat{\psi}_k = e^{2\pi ik\cdot\alpha t}\hat{\psi}_k, \quad \text{for all } k \in \mathbb{Z}^n, \text{ and all } t \in \mathbb{R}.$$

Therefore, either all $\hat{\psi}_k = 0$ for $k \neq 0$, or there exists $k \neq 0$ such that

$$k \cdot \alpha = 0.$$

The latter, however, is impossible since α is irrational. Hence, $\hat{\psi}_k = 0$ for all $k \neq 0$, and the only first integrals of u are constant functions if α is irrational. Thus, this flow is not relaxation enhancing (it has eigenfunctions in $H^1(\Omega)$) even though it has no first integrals other than a constant function.

Mixing and weakly mixing flows

An important class of relaxation enhancing flows is given by mixing and weakly mixing flows. Let us recall how they are defined. Any incompressible flow $u \in \mathrm{Lip}(\Omega)$ generates a unitary evolution group U^t on $L^2(\Omega)$, defined by

$$U^t f(x) = f(X(t;x)).$$

Here $X(t;x)$ is the measure preserving map associated with the flow, defined by (3.12):

$$\frac{dX}{dt} = -u(X), \quad X(0;x) = x. \tag{3.17}$$

The group U^t is a convenient tool to set up everything in $L^2(\Omega)$ rather than at the level of trajectories. For example, a function f is a first integral of u if

$$U^t f(x) = f(x), \text{ for all } x \in \Omega \text{ and all } t \in \mathbb{R}. \tag{3.18}$$

We say that a flow u is ergodic if its only first integrals are constant functions.

A flow is mixing if the following condition holds: for any two functions $f, h \in L^2(\Omega)$ we have

$$\lim_{t\to+\infty} \int_\Omega f(X(t;x))h(x)dx = \int_\Omega f(x)dx \int_\Omega h(x)dx. \tag{3.19}$$

The mixing condition (3.19) can be interpreted as follows. Let us start (3.17) at a random point x, equally distributed over the set Ω then the Lebesgue measure on Ω is invariant under the dynamics (3.17) since u is incompressible: for any measurable set A we have

$$\mathbb{P}(X(t) \in A) = \int_\Omega \chi_A(X(t;x))dx = \int_\Omega \chi_A(x)dx = |A|.$$

Consider two measurable sets $A \subset \Omega$ and $B \subset \Omega$, and the corresponding characteristic functions $h(x) = \chi_A(x)$ and $f(x) = \chi_B(x)$. Then (3.19) says that

$$P(X(t) \in B \text{ and } X(0) \in A) - |A| \cdot |B| \to 0, \text{ as } t \to +\infty, \tag{3.20}$$

that is, the events $\{X(0) \in A\}$ and $\{X(t) \in B\}$ become asymptotically (as $t \to +\infty$) independent – the fact that you end up in B does not depend on where you start.

Mixing implies ergodicity: if $U^t f(x) = f(x)$ for all $t \in \mathbb{R}$ then

$$\int_\Omega f(X(t;x))h(x)dx = \int_\Omega f(x)h(x)dx, \text{ for all } t > 0, \tag{3.21}$$

for all $h \in L^2(\Omega)$ which is incompatible with mixing unless f is a constant function.

A function $f \in L^2(\Omega)$ is an eigenfunction of the flow u if for any $t \in \mathbb{R}$ there exists $c(t)$ so that

$$U^t f(x) = c(t)f(t). \tag{3.22}$$

This definition is equivalent to the condition that

$$u \cdot \nabla f = \lambda f. \tag{3.23}$$

Indeed, the function $g(t,x) = U^t f(x)$ satisfies the advection equation

$$g_t + u \cdot \nabla g = 0, \quad g(0,x) = f(x). \tag{3.24}$$

therefore, if (3.23) holds then

$$f(X(t,x)) = e^{\lambda t} f(x).$$

On the other hand, if (3.22) holds then solution of (3.24) has the form

$$g(t,x) = c(t) f(x).$$

Inserting this expression into (3.24) gives

$$\dot{c}(t) f(x) + c(t) u(x) \cdot \nabla f(x) = 0.$$

Separation of variables now implies that there exists $\lambda \in \mathbb{C}$ such that (note that $c(0) = 1$ automatically)

$$c(t) = e^{-\lambda t},$$

and

$$u(x) \cdot \nabla f(x) = \lambda f(x).$$

Moreover, as the map $x \to X(t;x)$ is measure preserving for all $t \in \mathbb{R}$, $|c(t)| = 1$ for all t, whence λ is purely imaginary: $\lambda = i\omega$ with a real number ω.

An incompressible flow u is called weakly mixing if the corresponding operators U^t have only continuous spectrum, that is, the only eigenfunctions of U^t are constants. An equivalent definition is that (3.19) holds on average, that is:

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \left| \int_\Omega f(X(t;x)) h(x) dx - \int_\Omega f(x) dx \int_\Omega h(x) dx \right| dt = 0, \tag{3.25}$$

and the convergence in (3.20) holds for a set of times of density one.

Weakly mixing flows are ergodic: first integrals are eigenfunctions with eigenvalue zero but weakly mixing flows are not necessarily mixing (see, for instance, [19]). On the other hand, mixing flows are weakly mixing: essentially for the same reason that mixing flows are ergodic – if

$$U^t f = c(t) f, \text{ for all } t \in \mathbb{R},$$

then

$$\int_\Omega f(X(t;x)) h(x) dx = c(t) \int_\Omega f(x) h(x) dx, \text{ for all } t > 0, \tag{3.26}$$

for all $h \in L^2(\Omega)$ which is also incompatible with mixing unless f is a constant function.

A direct consequence of Theorem 3.3 is the following Corollary.

Corollary 3.5 *Any weakly mixing incompressible flow $u \in Lip(\Omega)$ is relaxation enhancing.*

There exist fairly explicit examples of weakly mixing flows [4, 32, 33, 42, 56, 48], some of which we describe below but delving into the detailed constructions would take us too far outside of the PDE realm.

3.2 Examples of relaxation enhancing flows

Before embarking on the proof of Theorem 3.3 we present in this section some examples of relaxation enhancing flows on a torus so as to assure the reader that this class is not empty. We first describe flows that have no eigenfunctions (they are weakly mixing) and then flows with very rough eigenfunctions none of which lie in $H^1(\Omega)$.

Weakly mixing incompressible flows on a torus

According to Theorem 3.3, a flow $u \in \mathrm{Lip}(\Omega)$ is relaxation enhancing if and only if it has no eigenfunctions in $H^1(\Omega)$. As we have mentioned, a natural class satisfying this condition is weakly mixing flows – which have no eigenfunctions in $L^2(\Omega)$ at all. Examples of weakly mixing flows on $\mathbb{T}^2$ go back to von Neumann [48] and Kolmogorov [42]. The flow in von Neumann's example is continuous; in the construction suggested by Kolmogorov the flow is smooth. The technical details of Kolmogorov's construction have been carried out in [56], a good review of these results is [39]. More recently, Fayad [32] generalized this example to show that weakly mixing flows are generic. To describe the result of [32] in more detail, let us recall that a vector $\alpha \in \mathbb{R}^n$ is called β-Diophatine if there exists a constant C such that for each $k \in \mathbb{Z}^n \setminus \{0\}$ we have

$$\inf_{p\in\mathbb{Z}} |\langle \alpha, k\rangle + p| \geq \frac{C}{|k|^{n+\beta}}.$$

The vector α is Liouvillean if it is not Diophantine for any $\beta > 0$. The Liouvillean numbers (and vectors) are the ones which can be very well approximated by rationals.

In order to construct a weakly mixing flow on a torus $\mathbb{T}^{n+1}$ we start with a very simple flow that is a local time change of a linear translation flow:

$$\frac{dX}{dt} = \frac{\alpha}{F(X,Y)}, \quad \frac{dY}{dt} = \frac{1}{F(X,Y)}, \quad X(0) = x, \quad Y(0) = y, \tag{3.27}$$

with a smooth positive function $F(x,y)$, $x \in \mathbb{T}^n$, $y \in \mathbb{T}$. Such flows have a unique invariant measure

$$d\mu = F(x,y)dxdy.$$

Indeed, for any smooth function $f(x,y)$ set

$$g(t,x,y) = U^t f(x,y) = f(X(t,x,y), Y(t,x,y)),$$

so that

$$\int_{\mathbb{T}^{n+1}} U^t f d\mu = \int_{\mathbb{T}^{n+1}} f(X(t,x,y), Y(t,x,y)) d\mu = \int_{\mathbb{T}^{n+1}} g(t,x,y) d\mu.$$

This function satisfies the first order advection equation

$$\frac{\partial g}{\partial t} - \frac{\alpha}{F(x,y)} \cdot \nabla_x g - \frac{1}{F(x,y)}\frac{\partial g}{\partial y} = 0, \quad g(0,x,y) = f(x,y). \tag{3.28}$$

Using this equation, we compute:

$$\frac{d}{dt}\int\limits_{\mathbb{T}^{n+1}} U^t f d\mu = \frac{d}{dt}\int\limits_{\mathbb{T}^{n+1}} g(t,x,y)d\mu = \int\limits_{\mathbb{T}^{n+1}} \left[\frac{\alpha}{F(x,y)}\cdot\nabla_x g(x,y)) \; + \frac{1}{F(x,y)}\frac{\partial g(x,y)}{\partial y}\right] d\mu$$
$$= \int\limits_{\mathbb{T}^{n+1}} \left[\alpha\cdot\nabla_x g(t,x,y) + \frac{\partial g(t,x,y)}{\partial y}\right] dxdy = 0. \qquad (3.29)$$

Therefore, μ is, indeed, an invariant measure for the flow (3.27). Let us denote by $C^r_+(\mathbb{T}^d)$ the set of positive C^r functions on the torus. Fayad's result is

Proposition 3.6 *([32]) Assume the irrational vector $\alpha \in \mathbb{R}^d$ is not β-Diophantine, for some $\beta > 0$. Then, for a dense G_δ set of functions F in $C^{\beta+n}_+(\mathbb{T}^{n+1})$ the flow* (3.27) *is weakly mixing (for the unique invariant measure $F(x,y)dxdy$).*

The flows given by this proposition have an invariant measure $F(x,y)dxdy$ and not the Lebesgue measure $dxdy$ – they are not quite divergence free, rather they satisfy

$$\nabla\cdot(Fu) = 0.$$

To obtain examples of relaxation enhancing flows, we need to modify these flows so that the resulting flow is divergence-free but the weakly mixing property is preserved. Assume for the sake of simplicity that we are working with a unit torus and that F given by Proposition 3.6 is normalized so that[3]

$$\int_{\mathbb{T}^{n+1}} F(x,y)dxdy = 1. \qquad (3.30)$$

A theorem of Moser [45] says that in this case there exists a measure preserving invertible transformation

$$Z:\ \mathbb{T}^{n+1}(F(x,y)dxdy) \to \mathbb{T}^{n+1}(dpdq),$$

that is as smooth as the function F. If we denote by

$$w(x,y) = (\alpha/F(x,y), 1/F(x,y))$$

the vector field in (3.27), then the vector field

$$u(p,q) = Z\circ w\circ Z^{-1}$$

is going to be incompressible (with respect to the standard Lebesgue measure $dpdq$). Moreover, the unitary evolutions generated by w and u in $L^2(F(x,y)dxdy)$ and $L^2(dpdq)$, respectively, are unitary equivalent and so have the same spectra. Therefore, the flow $u(p,q)$ is weakly mixing and thus relaxation enhancing.

[3]Obviously, if the flow (3.27) is weakly mixing for some F then it is also weakly mixing for all its positive multiples – multiplication of F by a constant amounts to a simple rescaling of time by the same constant.

Flows with rough eigenfunctions

We now describe an example of a different class of flows to which Theorem 3.3 applies. Namely, we will sketch a construction of a smooth incompressible flow $u(p, q)$, $\nabla \cdot u = 0$, on a torus $\mathbb{T}^2$ that has a purely discrete spectrum but none of the eigenfunctions are in $H^1(\mathbb{T}^2)$. The idea of the construction goes back to Kolmogorov [42]. We only sketch the construction, without presenting the full technical details [4, 39].

As before, we denote by U^t the flow on $L^2(\mathbb{T}^2)$ generated by u:

$$U^t f(x) = f(X(t; x)),$$

and $X(t, x)$ is the trajectory of

$$\frac{dX}{dt} = -u(X), \quad X(0; x) = x.$$

Proposition 3.7 *There exists a smooth incompressible (with respect to the Lebesgue measure) flow $u(x, y)$ on a two-dimensional torus $\mathbb{T}^2$ so that the corresponding unitary evolution U^t has a discrete spectrum on $L^2(\mathbb{T}^2)$ but none of the eigenfunctions of U^t are in $H^1(\mathbb{T}^2)$.*

Proof. The example will be given by a flow of the type (3.27):

$$\frac{dX}{dt} = \frac{\alpha}{F(X,Y)}, \quad \frac{dY}{dt} = \frac{1}{F(X,Y)}, \quad X(0) = x, \quad Y(0) = y, \tag{3.31}$$

with $d = 2$ and appropriately chosen α and $F(x, y)$. The idea of the construction is to find a flow of this form which can be mapped to a constant flow $(\alpha, 1)$ by a measure preserving map S with very low regularity properties. Since the eigenfunctions of the constant flow are explicitly computable, we can compute the eigenfunctions of the original flow. Due to the roughness of S, these will prove highly irregular. To obtain an incompressible flow (rather than a flow that preserves the measure $F(x, y)dxdy$), we will then proceed as in the previous example.

In order to find a required flow, we start with a smooth periodic function $Q \in C^\infty(\mathbb{S}^1)$ and an irrational number $\alpha \in \mathbb{R}$ so that the homology equation

$$R(\xi + \alpha) - R(\xi) = Q(\xi) - 1, \quad \xi \in \mathbb{S}^1, \tag{3.32}$$

has a solution $R(\xi)$ that is very rough. Note that for (3.32) to have a measurable solution the function $Q(\xi)$ should satisfy the normalization [4]

$$\int_0^1 Q(\xi) d\xi = 1. \tag{3.33}$$

The following Proposition is a particular case of Theorem 4.5 of [39].

Proposition 3.8 *Let α be a Liouvillean irrational number. There exists a $C^\infty(\mathbb{S}^1)$ function $Q(\xi)$ so that the homology equation (3.32) has a unique (up to an additive constant) measurable solution $R(\xi) : \mathbb{S}^1 \to \mathbb{R}$ such that for any $\lambda \in \mathbb{R}\backslash\{0\}$, the function $R_\lambda(\xi) = e^{i\lambda R(\xi)}$ is discontinuous everywhere.*

Note that without loss of generality we may assume that $Q(\xi)$ is positive – otherwise we choose M so that $Q(\xi) + M > 1$ and consider a rescaled function $Q_M(\xi) = (M + Q(\xi))/(M + 1)$. Then the function $R_M(\xi) = R(\xi)/(M + 1)$ is the solution of (3.32) with Q_M on the right side and, of course, $R_M(\xi)$ has the same set discontinuities as $R(\xi)$.

Given a Liouvillean irrational number α and a function $Q(\xi) > 0$ that satisfies the conclusion of Proposition 3.8 we define a function $F(x, y)$ on the torus $\mathbb{T}^2$ as follows. Choose $m > 0$ so that $m < \min Q(s)$ and a smooth cut-off function $\psi(y) \geq 0$ such that

$$\int_0^1 \psi(y)dy = 1, \tag{3.34}$$

and, in addition,

$$\psi(y) = 0 \text{ for } 0 \leq y \leq y_0 \text{ and } y_1 \leq y \leq 1 \text{ with } y_0 \text{ close to zero and } y_1 \text{ close to one.} \tag{3.35}$$

The choice of m ensures that the function

$$F(x, y) = m + \psi(y)(Q(x - \alpha y) - m), \quad 0 \leq x, y \leq 1 \tag{3.36}$$

is positive.

Next, we extend $F(x, y)$ periodically in both variables to the whole plane $\mathbb{R}^2$. The periodicity of $Q(x)$, (3.33), and (3.35) imply that the extension is smooth and, in addition, it has total mass over the torus equal to one:

$$\int_0^1 \int_0^1 F(x, y)dxdy = 1.$$

The normalization (3.34) implies that the functions F and Q are related by

$$\int_0^1 F(\xi + \alpha z, z)dz = \int_0^1 [m + \psi(z)(Q(\xi) - m)]dz = Q(\xi), \quad 0 \leq \xi \leq 1, \tag{3.37}$$

while for $\xi \geq 1$ we have

$$\int_0^1 F(\xi + \alpha z, z)dz = Q(\{\xi\}), \tag{3.38}$$

where $\{\xi\} = \xi - [\xi]$ is the fractional part of ξ ($[\xi]$ is the largest integer smaller or equal to ξ). Furthermore, we have

$$\int_n^{n+1} F(\xi + \alpha z, z)dz = \int_0^1 F(\xi + n\alpha + \alpha z, z)dz = Q(\{\xi + n\alpha\}). \tag{3.39}$$

Now, the required transformation $S : (x, y) \to (X, Y)$ is defined by [42, 57]

$$X(x, y) = x + \alpha(Y(x, y) - y), \qquad Y(x, y) = T(x - \alpha y, y) + R(x - \alpha y) \tag{3.40}$$

with the function $R(x)$ that satisfies the homology equation (3.32), and $T(x, y)$ defined by

$$T(x, y) = \int_0^y F(x + \alpha z, z)dz.$$

The function $T(x, y)$ is clearly periodic in x since $F(x, y)$ is periodic in x, while

$$
\begin{aligned}
T(x, y+1) &= \int_0^{1+y} F(x+\alpha z, z)dz = \int_0^1 F(x+\alpha z, z)dz + \int_1^{y+1} F(x+\alpha z, z)dz \\
&= Q(x) + \int_0^y F(x+\alpha+\alpha z, z)dz = T(x+\alpha, y) + Q(x) \qquad (3.41)
\end{aligned}
$$

because of (3.37).

Note that the transformation (3.40) implies that $x - \alpha y = X - \alpha Y$, and so it preserves the flow trajectories. Let us verify that S is well-defined as a mapping $\mathbb{T}^2 \to \mathbb{T}^2$: first, we have

$$
Y(x+1, y) = Y(x, y)
$$

since both $T(x, y)$ and $R(x)$ are periodic in x. For periodicity of $Y(x, y)$ in y we have, using the homology equation for the function R and (3.41):

$$
\begin{aligned}
Y(x, y+1) &= T(x-\alpha y-\alpha, y+1) + R(x-\alpha y-\alpha) \qquad (3.42)\\
&= T(x-\alpha y, y) + Q(x-\alpha y-\alpha) + R(x-\alpha y) - Q(x-\alpha y-\alpha) + 1 \\
&= T(x-\alpha y, y) + R(x-\alpha y) + 1 = Y(x, y) + 1.
\end{aligned}
$$

Finally, for $X(x, y)$ we have, clearly

$$
X(x+1, y) = X(x, y) + 1,
$$

as $Y(x, y)$ is periodic in x, and

$$
X(x, y+1) = x+\alpha(Y(x, y+1)-y-1) = x+\alpha(Y(x, y)+1-y-1) = x+\alpha(Y(x, y)-y) = X(x, y).
$$

Therefore, indeed, S is a well-defined mapping of $\mathbb{T}^2$ to itself.

Let us see what happens to the flow (3.31)

$$
\frac{dx}{dt} = \frac{\alpha}{F(x, y)}, \quad \frac{dy}{dt} = \frac{1}{F(x, y)}, \quad x(0) = x_0, \quad y(0) = y_0, \qquad (3.43)
$$

under this map. Note that

$$
x(t) - y(t) = x_0 - \alpha y_0,
$$

hence $Y(t)$ is given by

$$
Y(t) = T(x(t) - \alpha y(t), y(t)) + R(x(t) - \alpha y(t)) = T(x_0 - \alpha y_0, y(t)) + R(x_0 - \alpha y_0), \qquad (3.44)
$$

so that

$$
\begin{aligned}
\frac{dY}{dt} &= \frac{\partial T(x_0 - \alpha y_0, y(t))}{\partial y} \dot{y}(t) = F(x_0 - \alpha y_0 + \alpha y(t), y(t)) \frac{1}{F(x(t), y(t)} \qquad (3.45)\\
&= F(x(t) - \alpha y(t) + \alpha y(t), y(t)) \frac{1}{F(x(t), y(t)} = 1.
\end{aligned}
$$

On the other hand, for $X(t)$ we have

$$\frac{dX}{dt} = \dot{x}(t) + \alpha(\dot{Y}(t) - \dot{y}(t)) = \frac{\alpha}{F(x(t), y(t)} + \alpha - \frac{\alpha}{F(x(t), y(t)} = \alpha. \tag{3.46}$$

Therefore, the image of the flow (3.31) under S is simply the uniform flow:

$$\frac{dX}{dt} = \alpha, \quad \frac{dY}{dt} = 1. \tag{3.47}$$

We will denote $w_{unif} = (\alpha, 1)$.

Note that S is invertible with a measurable inverse. Indeed, we have

$$X - \alpha Y = x - \alpha y, \tag{3.48}$$

so that

$$Y = T(X - \alpha Y, y) + R(X - \alpha Y). \tag{3.49}$$

As the function F is positive, the function $T(x, y)$ is strictly increasing in y so that (3.49) has a unique solution $y(X, Y)$, and then (3.48) defines $x(X, Y)$ uniquely.

In addition, S is measure preserving in the following sense:

$$\int [S^* f](x, y) F(x, y) dx dy = \int f(S(x, y)) F(x, y) dx dy = \int f(X, Y) dX dY \tag{3.50}$$

for any function $f \in C(\mathbb{T}^2)$. In order to see that, let us introduce intermediate changes of variables: $S = S_3 \circ S_2 \circ S_1$, with $S_1 : (x, y) \to (z, y_1)$ with

$$z = x - \alpha y, \quad y_1 = y,$$

followed by $S_2 : (z, y_1) \to (Z, y_2)$

$$Z = z, \quad y_2 = T(z, y_1) + R(z),$$

and finally $S_3 : (Z, y_2) \to (X, Y)$, with

$$X = Z + \alpha y_2, \quad Y = y_2.$$

The corresponding Jacobians are:

$$J_1 = J_3 = 1, \quad J_2 = \frac{\partial T}{\partial y_1}(z, y_1) = F(z + \alpha y_1, y_1) = F(x, y).$$

Therefore, the Jacobian of S is, indeed,

$$J = J_1 J_2 J_3 = F(x, y),$$

hence (3.50) holds and S is measure-preserving.

Hence, S^* may be extended as an operator $L^2(dxdy) \to L^2(d\mu)$ with the preservation of the corresponding norms. It follows that the unitary evolutions U_w^t and U_{unif}^t generated by the flow w given by (3.43) and the uniform flow w_{unif}, respectively, are conjugated by means of the unitary transformation

$$S^*: \ L^2(\mathbb{T}^2, dXdY) \to L^2(\mathbb{T}^2, d\mu),$$

that is, we have

$$U^t_{unif} = [S^*]^{-1} U^t_w S^*.$$

It follows that U^t_w and U^t_{unif} have the same spectrum:

$$\lambda_{nl} = 2\pi in\alpha + 2\pi il, \quad l, n \in \mathbb{Z}.$$

It also follows that the eigenfunctions of the operator U_w may be written as

$$\begin{aligned}\psi^w_{nl}(x,y) &= e^{2\pi inX(x,y)+2\pi ilY(x,y)} = e^{2\pi in(x-\alpha y+\alpha Y(x,y))+2\pi ilY(x,y)} \\ &= e^{2\pi in(x-\alpha y)}e^{(2\pi in\alpha+2\pi il)(T(x-\alpha y,y)+R(x-\alpha y))} = \zeta(x,y)e^{(2\pi in\alpha+2\pi il)R(x-\alpha y)}\end{aligned} \tag{3.51}$$

with a smooth function $\zeta(x,y) \in C^\infty([0,1]^2)$. Note that the function

$$\zeta(x,y) = e^{2\pi in(x-\alpha y)}e^{(2\pi in\alpha+2\pi il)T(x-\alpha y,y)}$$

is not periodic in y even though the function $\psi^w_{nl}(x,y)$ is periodic. In order to verify that ψ^w_{nl} are not in $H^1(\mathbb{T}^2)$ it suffices to check that the function

$$\Theta_\lambda(x,y) = e^{i\lambda R(x-\alpha y)} = R_\lambda(x-\alpha y)$$

is not in $H^1([0,1]^2)$ for any real $\lambda \neq 0$. The function $R_\lambda(s)$ is defined in the Proposition 3.8 and is everywhere discontinuous. Since $\Theta_\lambda(x,y)$ is constant on the lines

$$x - \alpha y = \text{const},$$

if it were in $H^1([0,1]^2)$, it would force $R_\lambda(s)$ to be in $H^1(\mathbb{S}^1)$ and hence continuous but this function is discontinuous everywhere. Therefore, the eigenfunctions ψ^w_{nl} cannot be in $H^1(\mathbb{T}^2)$ unless $n = l = 0$.

Finally, to obtain an incompressible flow with rough eigenfunctions, we introduce a smooth transformation of the torus

$$\bar{S}: \ (x,y) \to (p,q)$$

by setting

$$p = \int_0^x \bar{F}(s)ds, \quad q = \frac{1}{\bar{F}(x)}\int_0^y F(x,z)dz, \text{ where } \bar{F}(x) = \int_0^1 F(x,z)dz.$$

Note that $\bar{F}(x)$ is periodic, and

$$p(x+1,y) = \int_0^{x+1} \bar{F}(s)ds = p(x,y) + \int_0^1 \bar{F}(s)ds = p(x,y)+1.$$

We also have $q(x+1,y) = q(x,y)$ and

$$q(x,y+1) = \frac{1}{\bar{F}(x)}\int_0^{y+1} F(x,z)dz = q(x,y)+1.$$

Therefore, indeed, $\bar{S}$ is a mapping of $\mathbb{T}^2$ to itself. Since $F(x,y)$ is positive, $\bar{S}$ is one-to-one. It is immediate to verify that it maps the measure $d\mu$ onto the Lebesgue measure $dpdq$ – the Jacobian of $\bar{S}$ is $F(x,y)$. Hence, the evolution group generated by the image $u(p,q)$ of the flow $w(x,y)$ will have the same discrete spectrum as U_w. In addition, the eigenfunctions ψ_{nl}^w of U_w are the images of the eigenfunctions ψ_{nl}^u of u under $\bar{S}^*$:

$$\psi_{nl}^w(x,y) = (\bar{S}^*\psi_{nl}^u)(x,y) = \psi_{nl}^u(\bar{S}(x,y)).$$

As the functions ψ_{nl}^w are not in $H^1(\mathbb{T}^2)$ and the map Z is smooth, it follows that all the eigenfunctions of the incompressible flow $u(p,q)$ are not in $H^1(\mathbb{T}^2)$. This finishes the proof of Proposition 3.7. □

3.3 An abstract criterion for relaxation enhancement

Theorem 3.3 (and its generalizations to other boundary conditions discussed in [18]) follows from a rather general abstract criterion, which we are now going to describe. We start with a self-adjoint, positive, unbounded operator Γ with a discrete spectrum on a separable Hilbert space H. Let

$$0 < \lambda_1 \le \lambda_2 \le \dots$$

be the eigenvalues of Γ, and e_j the corresponding orthonormal eigenvectors forming a basis in H. The (homogenous) Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors

$$\psi = \sum_j c_j e_j$$

such that

$$\|\psi\|^2_{H^m(\Gamma)} \equiv \sum_j \lambda_j^m |c_j|^2 < \infty.$$

Note that $H^2(\Gamma)$ is the domain $D(\Gamma)$ of Γ.

The crucial assumption is that $\lambda_n \to +\infty$ – this makes the set where the dissipation by Γ is not too large a compact subset of H.

We will also use a self-adjoint operator L such that, for any $\psi \in H^1(\Gamma)$ and $t > 0$ we have

$$\|L\psi\|_H \le C\|\psi\|_{H^1(\Gamma)} \text{ and } \|e^{iLt}\psi\|_{H^1(\Gamma)} \le B(t)\|\psi\|_{H^1(\Gamma)} \tag{3.52}$$

with both the constant C and the function $B(t) < \infty$ independent of ψ and $B(t) \in L^2_{\text{loc}}[0,\infty)$. Here e^{iLt} is the unitary evolution group generated by the self-adjoint operator L. Consider a solution $\phi^\varepsilon(t)$ of the Bochner differential equation

$$\frac{d}{dt}\phi^\varepsilon(t) = \frac{i}{\varepsilon}L\phi^\varepsilon(t) - \Gamma\phi^\varepsilon(t), \quad \phi^\varepsilon(0) = \phi_0. \tag{3.53}$$

Theorem 3.9 *Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum and let a self-adjoint operator L satisfy conditions* (3.52)*. Then the following two statements are equivalent:*

(i) For any $\tau, \delta > 0$ there exists $\varepsilon_0(\tau,\delta)$ such that for any $0 < \varepsilon < \varepsilon_0(\tau,\delta)$ and any $\phi_0 \in H$ with $\|\phi_0\|_H = 1$, the solution $\phi^\varepsilon(t)$ of the equation (3.53) *satisfies $\|\phi^\varepsilon(\tau)\|^2_H < \delta$.*

(ii) The operator L has no eigenvectors lying in $H^1(\Gamma)$.

In the setting of Theorem 3.3, L corresponds to the operator $iu \cdot \nabla$ (or, to be precise, the self-adjoint operator generating the unitary evolution group U^t which is equal to $iu \cdot \nabla$ on $H^1(\Omega)$), and Γ to $-\Delta$, with H the subspace of mean zero functions on $L^2(\Omega)$.

Theorem 3.9 provides a sharp answer to the general question of when a combination of fast unitary evolution and dissipation produces a significantly stronger dissipative effect than dissipation alone. It can be useful in any model describing a physical situation which involves fast unitary dynamics with dissipation (or, equivalently, unitary dynamics with weak dissipation). The proof of Theorem 3.9 uses ideas from quantum dynamics, in particularly the RAGE theorem (see e.g., [21]) describing evolution of a quantum state belonging to the continuous spectral subspace of a self-adjoint operator. The paper [18] contains more examples of operators with either no eigenfunctions, or with rough eigenfunctions, as we have done in the diffusion-advection context in Section 3.2.

One might wonder whether one of the two conditions in (3.52) does not imply the other. Taking $L = \Gamma$ shows that

$$\|e^{iLt}\psi\|_1 \leq B(t)\|\psi\|_1 \tag{3.54}$$

does not imply

$$\|L\psi\| \leq C\|\psi\|_1. \tag{3.55}$$

because in this case the evolution e^{iLt} is unitary on H^1 but the domain of L is $H^2 \subsetneq H^1$. The following example shows that (3.55) does not imply (3.54) either. Let $H \equiv L^2(0,1)$, and define the operator Γ by

$$\Gamma f(x) \equiv \sum_n e^{n^2} \hat{f}(n) e^{2\pi i n x}$$

for all $f \in H$ such that $e^{n^2}\hat{f}(n) \in \ell^2(\mathbb{Z})$, and take $Lf(x) \equiv xf(x)$. Then L is bounded on H and so (3.55) holds automatically, but

$$\left[e^{itL} f\right](x) = f(x)e^{itx}$$

so that

$$e^{2\pi i L} e^{2\pi i n x} = e^{2\pi i (n+1)x}.$$

It follows that $e^{2\pi i L}$ is not bounded on H^1 (and neither is e^{iLt} for any $t \neq 0$).

3.4 The proof of Theorem 3.9

The general idea of the proof is quite straightforward. The dissipation balance for our problem is

$$\frac{1}{2}\frac{d}{dt}(\|\phi^\varepsilon\|_H^2) = -\langle \Gamma\phi^\varepsilon, \phi^\varepsilon\rangle = -\|\phi^\varepsilon(t)\|^2_{H^1(\Gamma)}. \tag{3.56}$$

Therefore, if $\|\phi\|_{H^1(\Gamma)}$ is large on a time interval $[t_1, t_2]$ then the $\|\phi^\varepsilon(t)\|_H$ will drop significantly over this time. On the other hand, we will show that if $\|\phi^\varepsilon(\tau)\|_{H^1(\Gamma)}$ is small at some time τ then, as L does not have $H^1(\Gamma)$-eigenfunctions, the free evolution

$$\frac{d\phi^0}{dt} = \frac{i}{\varepsilon} L\phi^0, \quad \phi^0(\tau) = \phi^\varepsilon(\tau), \quad t \geq \tau, \tag{3.57}$$

will make the $H^1(\Gamma)$-norm of ϕ^0 very large in a time so short that the free evolution is close to the true evolution over this short time interval. This means that the $H^1(\Gamma)$-norm of $\phi^\varepsilon(t)$ will also he very large. Hence, even if the $H^1(\Gamma)$-norm of ϕ^ε drops, it will go back up again very quickly, forcing the dissipation to be large most of the time, and reducing the $\|\phi^\varepsilon(t)\|$ very efficiently. Making this argument rigorous will take us some time, no pun intended. A crucial role is played by the fact that the unit ball in the $H^1(\Gamma)$-norm is compact in H.

Preliminaries

We first collect some elementary facts and estimates for equation (3.53). Henceforth, we are going to denote the standard norm in the Hilbert space H by $\|\cdot\|$, the inner product in H by $\langle\cdot,\cdot\rangle$, the Sobolev spaces $H^m(\Gamma)$ simply by H^m and norms in these Sobolev spaces by $\|\cdot\|_m$. We have the following existence and uniqueness result.

Proposition 3.10 *Assume that for any $\psi \in H^1$, we have*

$$\|L\psi\| \le C\|\psi\|_1. \tag{3.58}$$

Then for any $T > 0$, there exists a unique solution $\phi(t)$ of the equation

$$\frac{d\phi(t)}{dt} = (iL - \Gamma)\phi(t), \quad \phi(0) = \phi_0 \in H^1.$$

This solution satisfies

$$\phi(t) \in L^2([0,T], H^2) \cap C([0,T], H^1), \;\; \dot\phi(t) \in L^2([0,T], H). \tag{3.59}$$

Exercise 3.11 Proposition 3.10 can be proved by standard methods using Galerkin approximations and then establishing uniqueness and regularity. Fill in the details of the argument.

Next we establish a few properties that are more specific to our particular problem. It will be more convenient for us, in terms of notation, to rescale time by the factor ε^{-1}, arriving at the equation

$$\frac{d\tilde\phi^\varepsilon(t)}{dt} = (iL - \varepsilon\Gamma)\tilde\phi^\varepsilon(t), \;\; \tilde\phi^\varepsilon(0) = \phi_0. \tag{3.60}$$

Lemma 3.12 *Assume that* (3.58) *holds, then for any initial data $\phi_0 \in H$, $\|\phi_0\| = 1$, the solution $\tilde\phi^\varepsilon(t)$ of* (3.60) *satisfies*

$$\varepsilon\int_0^\infty \|\tilde\phi^\varepsilon(t)\|_1^2 dt \le \frac{1}{2}. \tag{3.61}$$

Proof. Recall that if $\phi \in H^1(\Gamma)$, then $\Gamma\phi \in H^{-1}(\Gamma)$ and $\langle\Gamma\phi,\phi\rangle = \|\phi\|_1^2$. The fact that L is self-adjoint allows us to compute

$$\frac{d}{dt}\|\tilde\phi^\varepsilon\|^2 = \langle\tilde\phi^\varepsilon, \tilde\phi^\varepsilon_t\rangle + \langle\tilde\phi^\varepsilon_t, \tilde\phi^\varepsilon\rangle = -2\varepsilon\|\tilde\phi^\varepsilon\|_1^2. \tag{3.62}$$

Integrating in time and taking into account the normalization of ϕ_0, we obtain (3.61). □

An immediate consequence of (3.62) is the following result, that we state here as a separate lemma for convenience.

Lemma 3.13 *Suppose that for all times $t \in (a, b)$ we have $\|\tilde{\phi}^\varepsilon(t)\|_1^2 \geq N\|\tilde{\phi}^\varepsilon(t)\|^2$. Then the following decay estimate holds:*

$$\|\tilde{\phi}^\varepsilon(b)\|^2 \leq e^{-2\varepsilon N(b-a)}\|\tilde{\phi}^\varepsilon(a)\|^2.$$

Next we need an estimate on the growth of the difference between solutions corresponding to $\varepsilon > 0$ and $\varepsilon = 0$ in the H-norm.

Lemma 3.14 *Assume, in addition to (3.58), that for any $\psi \in H^1$ and $t > 0$ we have*

$$\|e^{iLt}\psi\|_1 \leq B(t)\|\psi\|_1 \tag{3.63}$$

for some $B(t) \in L^2_{\text{loc}}[0, \infty)$. Let $\phi^0(t), \phi^\varepsilon(t)$ be solutions of

$$\frac{d\phi^0(t)}{dt} = iL\phi^0(t), \quad \frac{d\tilde{\phi}^\varepsilon(t)}{dt} = (iL - \varepsilon\Gamma)\tilde{\phi}^\varepsilon(t),$$

satisfying $\phi^0(0) = \phi^\varepsilon(0) = \phi_0 \in H^1$. Then we have

$$\frac{d}{dt}\|\tilde{\phi}^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{1}{2}\varepsilon\|\phi^0(t)\|_1^2 \leq \frac{1}{2}\varepsilon B^2(t)\|\phi_0\|_1^2. \tag{3.64}$$

Proof. Note that $\phi^0(t) = e^{iLt}\phi_0$ by definition. Assumption (3.63) says that this unitary evolution is bounded in the $H^1(\Gamma)$ norm. The regularity guaranteed by conditions (3.58), (3.63) and Proposition 3.10 allows us to multiply the equation

$$\frac{d}{dt}(\tilde{\phi}^\varepsilon - \phi^0) = iL(\tilde{\phi}^\varepsilon - \phi^0) - \varepsilon\Gamma\tilde{\phi}^\varepsilon$$

by $\phi^\varepsilon - \phi^0$. We obtain

$$\frac{d}{dt}\|\tilde{\phi}^\varepsilon - \phi^0\|^2 \leq 2\varepsilon(\|\tilde{\phi}^\varepsilon\|_1\|\phi^0\|_1 - \|\tilde{\phi}^\varepsilon\|_1^2) \leq \frac{1}{2}\varepsilon\|\phi^0\|_1^2,$$

which is the first inequality in (3.64). The second inequality follows simply from the assumption (3.63). □

The following corollary is immediate.

Corollary 3.15 *Assume that (3.58) and (3.63) are satisfied, and the initial data $\phi_0 \in H^1$. Then the solutions $\tilde{\phi}^\varepsilon(t)$ and $\phi^0(t)$ defined in Lemma 3.14 satisfy*

$$\|\tilde{\phi}^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{1}{2}\varepsilon\|\phi_0\|_1^2 \int_0^\tau B^2(t)\,dt$$

for any time $t \leq \tau$.

Eigenvectors in $H^1(\Gamma)$ prohibit relaxation enhancement

One direction in the proof of Theorem 3.9 is much easier: existence of $H^1(\Gamma)$ eigenvectors of the operator L ensures existence of $\tau, \delta > 0$ and ϕ_0 with $\|\phi_0\| = 1$ such that $\|\phi^\varepsilon(\tau)\| > \delta$ for all ε – that is, if such eigenvectors exist, then the operator L is not relaxation enhancing.

Assume that the initial data $\phi_0 \in H^1$ for

$$\frac{d}{dt}\phi^\varepsilon(t) = \frac{i}{\varepsilon}L\phi^\varepsilon(t) - \Gamma\phi^\varepsilon(t), \quad \phi^\varepsilon(0) = \phi_0 \tag{3.65}$$

is an eigenvector of L corresponding to an eigenvalue E, normalized so that $\|\phi_0\| = 1$. Take the inner product of (3.65) with ϕ_0. We arrive at

$$\frac{d}{dt}\langle\phi^\varepsilon(t), \phi_0\rangle = \frac{iE}{\varepsilon}\langle\phi^\varepsilon(t), \phi_0\rangle - \langle\Gamma\phi^\varepsilon(t), \phi_0\rangle.$$

This and the assumption $\phi_0 \in H^1$ lead to

$$\left|\frac{d}{dt}\left(e^{-iEt/\varepsilon}\langle\phi^\varepsilon(t), \phi_0\rangle\right)\right| \le \frac{1}{2}\left(\|\phi^\varepsilon(t)\|_1^2 + \|\phi_0\|_1^2\right).$$

Note that the value of the expression being differentiated on the left hand side is equal to one at $t = 0$. By Lemma 3.12 (with a simple time rescaling) we have

$$\int_0^\infty \|\phi^\varepsilon(t)\|_1^2\, dt \le 1/2.$$

Therefore, for $t \le \tau = (2\|\phi_0\|_1^2)^{-1}$ we have $|\langle\phi^\varepsilon(t), \phi_0\rangle| \ge 1/2$. Thus, $\|\phi^\varepsilon(\tau)\| \ge 1/2$, uniformly in ε. $\Box$

The RAGE theorem and the time spent in high modes

The proof of the other direction in Theorem 3.9, that absence of eigenfunctions implies relaxation enhancement, is more subtle, and will require some preparation. We switch to the equivalent formulation (3.60), and drop the tilde (hoping that this will not cause any confusion). We need to show that if L has no H^1 eigenvectors, then for all $\tau, \delta > 0$ there exists $\varepsilon_0(\tau, \delta) > 0$ such that if $\varepsilon < \varepsilon_0$, then $\|\phi^\varepsilon(\tau/\varepsilon)\| < \delta$ whenever $\|\phi_0\| = 1$. The main idea of the proof can be naively described as follows. If the operator L has purely continuous spectrum or its eigenfunctions are rough then the H^1-norm of the free evolution (with $\varepsilon = 0$) is large most of the time[4]. On the other hand, we will show that for small ε the full evolution is close to the free evolution for a sufficiently long time. This clearly leads to dissipation enhancement.

Our first task is to get good control of the free evolution e^{iLt}. The first ingredient that we need to recall is the so-called RAGE theorem[5].

[4]In all fairness, we should point out that the mechanism of this effect is quite different for the continuous and point spectra.

[5]This theorem was first proved by Ruelle [54] and later generalized by Amrein and Georgescu in [3], and by Enss in [24].

Theorem 3.16 (RAGE) *Let L be a self-adjoint operator in a Hilbert space H. Let P_c be the spectral projection on its continuous spectral subspace. Let C be any compact operator. Then for any $\phi_0 \in H$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|Ce^{iLt} P_c \phi_0\|^2 \, dt = 0.$$

Clearly, the result can be equivalently stated for a unitary operator U, replacing e^{iLt} with U^t. The proof of the RAGE theorem can be found, for example, in [21]. This theorem is a generalization of the following classical theorem by Wiener.

Theorem 3.17 *Let $d\mu$ be a finite measure on $\mathbb{R}$ with the Fourier transform*

$$F(t) = e^{ixt} d\mu(x).$$

Then

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T |F(t)|^2 dt = \sum_{x\in\mathbb{R}} |\mu(\{x\})|^2. \tag{3.66}$$

Note that the sum in the right side of (3.66) is finite since μ is a finite measure.

A direct consequence of the RAGE theorem is the following lemma. Recall that we denote by $0 < \lambda_1 \le \lambda_2 \le \dots$ the eigenvalues of the operator Γ and by $e_1, e_2, \dots$ the corresponding orthonormal eigenvectors. Let us also denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $e_1, \dots, e_N$ and by

$$S = \{\phi \in H : \ \|\phi\| = 1\}$$

the unit sphere in H. The following lemma shows that if the initial data lies in the continuous spectrum of L then the L-evolution will spend most of time in the higher modes of Γ.

Lemma 3.18 *Let $K \subset S$ be a compact set. For any $N, \sigma > 0$, there exists $T_c(N, \sigma, K)$ such that for all $T \ge T_c(N, \sigma, K)$ and any $\phi \in K$, we have*

$$\frac{1}{T} \int_0^T \|P_N e^{iLt} P_c \phi\|^2 \, dt \le \sigma. \tag{3.67}$$

The key observation of Lemma 3.18 is that the time $T_c(N, \sigma, K)$ is uniform for all $\phi \in K$.

Proof. Since P_N is compact, RAGE theorem implies that for any vector $\phi \in S$, there exists a time $T_c(N, \sigma, \phi)$ that depends on the function ϕ such that (3.67) holds for $T > T_c(N, \sigma, \phi)$. To prove the uniformity in ϕ, note that the function

$$f(T, \phi) = \frac{1}{T} \int_0^T \|P_N e^{iLt} P_c \phi\|^2 \, dt$$

is uniformly continuous on S for all T (with constants independent of T):

$$|f(T,\phi)-f(T,\psi)| \leq \frac{1}{T}\int_0^T \left|\|P_N e^{iLt}P_c\phi\| - \|P_N e^{iLt}P_c\psi\|\right| \left(\|P_N e^{iLt}P_c\phi\| + \|P_N e^{iLt}P_c\psi\|\right) dt$$

$$\leq (\|\phi\|+\|\psi\|)\frac{1}{T}\int_0^T \|P_N e^{iLt}P_c(\phi-\psi)\|dt \leq 2\|\phi-\psi\|.$$

Now, existence of a uniform $T_c(N,\sigma,K)$ follows from compactness of K by standard arguments. □

H^1-norm of free solutions with rough eigenfunctions

We also need a lemma which controls from below the growth of the H^1 norm of free solutions corresponding to rough eigenfunctions. We denote by P_p the spectral projection on the pure point spectrum of the operator L.

Lemma 3.19 *Assume that not a single eigenvector of the operator L belongs to $H^1(\Gamma)$. Let $K \subset S$ be a compact set. Consider the set $K_1 \equiv \{\phi \in K : \|P_p\phi\| \geq 1/2\}$. Then for any $B > 0$ we can find $N_p(B,K)$ and $T_p(B,K)$ such that for any $N \geq N_p(B,K)$, any $T \geq T_p(B,K)$ and any $\phi \in K_1$, we have*

$$\frac{1}{T}\int_0^T \|P_N e^{iLt}P_p\phi\|_1^2\, dt \geq B. \tag{3.68}$$

Note that unlike in (3.67), we have the H^1 norm in (3.68).

Proof. The set K_1 may be empty, in which case there is nothing to prove. Otherwise, let us denote by E_j the eigenvalues of L (distinct, without repetitions) and by Q_j the orthogonal projection on the space spanned by the eigenfunctions corresponding to E_j. First, let us show that for any $B > 0$ there is $N(B,K)$ such that for any $\phi \in K_1$ we have

$$\sum_j \|P_N Q_j \phi\|_1^2 \geq 2B \tag{3.69}$$

if $N \geq N(B,K)$. It is clear that for each fixed ϕ with $P_p\phi \neq 0$ we can find $N(B,\phi)$ so that (3.69) holds, since by assumption $Q_j\phi$ does not belong to H^1 whenever $Q_j\phi \neq 0$. Assume that $N(B,\phi)$ cannot be chosen uniformly for $\phi \in K_1$. This means that for any n, there exists $\phi_n \in K_1$ such that

$$\sum_j \|P_n Q_j \phi_n\|_1^2 < 2B.$$

Since K_1 is compact, we can find a subsequence n_l such that ϕ_{n_l} converges to $\bar{\phi} \in K_1$ in H as $n_l \to \infty$. For any N and any $n_{l_1} > N$ we have

$$\sum_j \|P_N Q_j \bar{\phi}\|_1^2 \leq \sum_j \|P_{n_{l_1}} Q_j \bar{\phi}\|_1^2 \leq \liminf_{l\to\infty} \sum_j \|P_{n_{l_1}} Q_j \phi_{n_l}\|_1^2. \tag{3.70}$$

The last inequality follows by Fatou's Lemma from the convergence of ϕ_{n_l} to $\bar{\phi}$ in H and the fact that

$$\|P_{n_{l_1}} Q_j \psi\|_1 = \lambda_{n_{l_1}}^{1/2} \|Q_j \psi\| \leq \lambda_{n_{l_1}}^{1/2} \|\psi\|,$$

for any n_{l_1}. But now the expression in the right hand side of (3.66) is less than or equal to

$$\liminf_{l \to \infty} \sum_j \|P_{n_l} Q_j \phi_{n_l}\|_1^2 \leq 2B.$$

Thus ,

$$\sum_j \|P_N Q_j \bar{\phi}\|_1^2 \leq 2B \text{ for any } N,$$

a contradiction since $\bar{\phi} \in K_1$. Therefore, there exists $N(B, K)$ so that (3.69) holds for all $N \geq N(B, K)$ and all $\phi \in K_1$.

Next, take $\phi \in K_1$ and consider

$$\frac{1}{T} \int_0^T \|P_N e^{iLt} P_p \phi\|_1^2 \, dt = \sum_{j,l} \frac{e^{i(E_j - E_l)T} - 1}{i(E_j - E_l)T} \langle \Gamma P_N Q_j \phi, P_N Q_l \phi \rangle. \tag{3.71}$$

In (3.71), we set

$$\frac{e^{i(E_j - E_l)T} - 1}{i(E_j - E_l)T} \equiv 1 \text{ if } j = l.$$

Notice that the sum in (3.71) converges absolutely. Indeed,

$$P_N Q_j \phi = \sum_{i=1}^N \langle Q_j \phi, e_i \rangle e_i,$$

and $\langle \Gamma e_i, e_k \rangle = \lambda_i \delta_{ik}$, therefore

$$\langle \Gamma P_N Q_j \phi, P_N Q_l \phi \rangle = \sum_{i=1}^N \lambda_i \langle Q_j \phi, e_i \rangle \overline{\langle Q_l \phi, e_i \rangle}.$$

Hence, the sum in the right side of (3.71) does not exceed

$$\begin{aligned} \sum_{i=1}^N \lambda_i \sum_{j,l} |\langle Q_j \phi, e_i \rangle \langle Q_l \phi, e_i \rangle| &\leq \lambda_N \sum_{i=1}^N \sum_{j,l} \|Q_j \phi\| \|Q_l \phi\| |\langle Q_j \phi / \|Q_j \phi\|, e_i \rangle \langle Q_l \phi / \|Q_l \phi\|, e_i \rangle| \\ &\leq \lambda_N \sum_{i=1}^N \sum_{j,l} \|Q_l \phi\|^2 |\langle Q_j \phi / \|Q_j \phi\|, e_i \rangle|^2 \leq \lambda_N N, \end{aligned} \tag{3.72}$$

with the second step obtained using the Cauchy-Schwartz inequality, and the third by $\|\phi\| = \|e_i\| = 1$. Then for each fixed N, we have by the dominated convergence theorem that the expression in (3.71) converges to

$$\sum_j \|\Gamma^{1/2} P_N Q_j \phi\|^2 = \sum_j \|P_N Q_j \phi\|_1^2$$

as $T \to \infty$.

Now assume $N \geq N_p(B,K) \equiv N(B,K)$, so that (3.69) holds. We claim that we can choose $T_p(B,K)$ so that for any $T \geq T_p(B,K)$ we have

$$\left| \frac{1}{T}\int_0^T \|P_N e^{iLt} P_p \phi\|_1^2 \, dt - \sum_j \|P_N Q_j \phi\|_1^2 \right| = \left| \sum_{l \neq j} \frac{e^{i(E_j - E_l)T} - 1}{i(E_j - E_l)T} \langle \Gamma P_N Q_j \phi, P_N Q_l \phi \rangle \right| \leq B \tag{3.73}$$

for all $\phi \in K_1$. Indeed, this follows from convergence to zero for each individual ϕ as $T \to \infty$, compactness of K_1, and uniform continuity of the expression in the middle of (3.73) in ϕ for each T (with constants independent of T). The latter is proved by estimating the difference of these expressions for $\phi, \psi \in K_1$ and any T by

$$\sum_{l \neq j} |\langle \Gamma P_N Q_j \phi, P_N Q_l(\phi - \psi)\rangle| + |\langle \Gamma P_N Q_j(\phi - \psi), P_N Q_l \psi \rangle|,$$

which is then bounded by $2\lambda_N N \|\phi - \psi\|$ using the trick from (3.72). Combining (3.69) and (3.73) proves the lemma. □

Tracking the full dynamics with free evolution

We can now complete the proof of Theorem 3.9. Recall that given any $\tau, \delta > 0$, we need to show the existence of $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then solution of the rescaled problem

$$\frac{d\phi^\varepsilon(t)}{dt} = (iL - \varepsilon\Gamma)\phi^\varepsilon(t), \;\; \tilde{\phi}^\varepsilon(0) = \phi_0. \tag{3.74}$$

satisfies $\|\phi^\varepsilon(\tau/\varepsilon)\| < \delta$ for any initial datum $\phi_0 \in H$, $\|\phi_0\| = 1$. Let us outline the idea of the proof. Lemma 3.13 tells us that if the H^1 norm of the solution $\phi^\varepsilon(t)$ is large, relaxation is happening quickly. If, on the other hand, $\|\phi^\varepsilon(\tau_0)\|_1^2 \leq \lambda_M \|\phi^\varepsilon(\tau_0)\|^2$, where M is to be chosen depending on τ and δ, then the set of all unit vectors satisfying this inequality is compact, and so we can apply Lemma 3.18 and Lemma 3.19. Using these lemmas, we will show that even if the H^1 norm is small at some moment of time τ_0, it will be large on average in some time interval after τ_0. Enhanced relaxation will follow.

We now provide the details. Since Γ is an unbounded positive operator with a discrete spectrum, we know that its eigenvalues $\lambda_n \to \infty$ as $n \to \infty$. Let us choose M large enough, so that

$$e^{-\lambda_M \tau/80} < \delta.$$

Define the sets $K \equiv \{\phi \in S : \; \|\phi\|_1^2 \leq \lambda_M\} \subset S$ and as before, $K_1 \equiv \{\phi \in K : \; \|P_p \phi\| \geq 1/2\}$. It is easy to see that K is compact. Choose N so that $N \geq M$ and $N \geq N_p(5\lambda_M, K)$ from Lemma 3.19. Define

$$\tau_1 \equiv \max\left\{T_p(5\lambda_M, K), T_c(N, \tfrac{\lambda_M}{20\lambda_N}, K)\right\},$$

where T_p is from Lemma 3.19, and T_c from Lemma 3.18. Finally, choose $\varepsilon_0 > 0$ so that $\tau_1 < \tau/2\varepsilon_0$, and

$$\varepsilon_0 \int_0^{\tau_1} B^2(t)\, dt \leq \frac{1}{20\lambda_N}, \tag{3.75}$$

where $B(t)$ is the function from condition (3.63).

Take any $\varepsilon < \varepsilon_0$. If we have

$$\|\phi^\varepsilon(s)\|_1^2 \geq \lambda_M \|\phi^\varepsilon(s)\|^2$$

for all $s \in [0, \tau]$ then Lemma 3.13 implies that

$$\|\phi^\varepsilon(\tau/\varepsilon)\| \leq e^{-2\lambda_M \tau} \leq \delta,$$

by the choice of M and we are done. Otherwise, let τ_0 be the first time in the interval $[0, \tau/\varepsilon]$ such that

$$\|\phi^\varepsilon(\tau_0)\|_1^2 \leq \lambda_M \|\phi^\varepsilon(\tau_0)\|^2 \tag{3.76}$$

(it may be that $\tau_0 = 0$, of course). We claim that the following estimate holds for the decay of $\|\phi^\varepsilon(t)\|$ on the interval $[\tau_0, \tau_0 + \tau_1]$:

$$\|\phi^\varepsilon(\tau_0 + \tau_1)\|^2 \leq e^{-\lambda_M \varepsilon \tau_1/20} \|\phi^\varepsilon(\tau_0)\|^2. \tag{3.77}$$

For the sake of transparency, henceforth we will denote $\phi^\varepsilon(\tau_0) = \phi_0$. On the interval $[\tau_0, \tau_0 + \tau_1]$, consider the function $\phi^0(t)$ satisfying

$$\frac{d}{dt}\phi^0(t) = iL\phi^0(t), \quad \phi^0(\tau_0) = \phi_0.$$

Note that by the choice of ε_0, (3.75), (3.76), and Corollary 3.15, we have

$$\|\phi^\varepsilon(t) - \phi^0(t)\|^2 \leq \frac{\lambda_M}{40\lambda_N}\|\phi_0\|^2 \tag{3.78}$$

for all $t \in [\tau_0, \tau_0 + \tau_1]$. Split

$$\phi^0(t) = \phi_c(t) + \phi_p(t),$$

where $\phi_{c,p}$ also solve the free equation

$$\frac{d}{dt}\phi_{c,p}(t) = iL\phi_{c,p}(t),$$

but with initial data $P_c\phi_0$ and $P_p\phi_0$ at $t = \tau_0$, respectively. We will now consider two cases.

Case I. Assume that

$$\|P_c\phi_0\|^2 \geq \frac{3}{4}\|\phi_0\|^2,$$

or, equivalently, $\|P_p\phi_0\|^2 \leq \frac{1}{4}\|\phi_0\|^2$. Note that since $\phi_0/\|\phi_0\| \in K$ by the hypothesis, we can apply Lemma 3.18. Our choice of τ_1 implies that

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_c(t)\|^2\, dt \leq \frac{\lambda_M}{20\lambda_N}\|\phi_0\|^2. \tag{3.79}$$

By elementary considerations,

$$\|(I-P_N)\phi^0(t)\|^2 \geq \frac{1}{2}\|(I-P_N)\phi_c(t)\|^2 - \|(I-P_N)\phi_p(t)\|^2 \geq \frac{1}{2}\|\phi_c(t)\|^2 - \frac{1}{2}\|P_N\phi_c(t)\|^2 - \|\phi_p(t)\|^2.$$

Taking into account the fact that the free evolution e^{iLt} is unitary, $\lambda_N \geq \lambda_M$, our assumptions on $\|P_{c,p}\phi_0\|$ and (3.79), we obtain

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|(I-P_N)\phi^0(t)\|^2\,dt \geq \frac{1}{10}\|\phi_0\|^2. \tag{3.80}$$

Using (3.78), we conclude that

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|(I-P_N)\phi^\varepsilon(t)\|^2\,dt \geq \frac{1}{40}\|\phi_0\|^2. \tag{3.81}$$

This estimate implies that

$$\int_{\tau_0}^{\tau_0+\tau_1} \|\phi^\varepsilon(t)\|_1^2\,dt \geq \frac{\lambda_N\tau_1}{40}\|\phi_0\|^2. \tag{3.82}$$

Combining (3.82) with (3.62) yields

$$\|\phi^\varepsilon(\tau_0+\tau_1)\|^2 \leq \left(1-\frac{\lambda_N\varepsilon\tau_1}{20}\right)\|\phi^\varepsilon(\tau_0)\|^2 \leq e^{-\lambda_N\varepsilon\tau_1/20}\|\phi^\varepsilon(\tau_0)\|^2. \tag{3.83}$$

This finishes the proof of (3.77) in the first case since $\lambda_N \geq \lambda_M$.

Case II. Now suppose that $\|P_p\phi_0\|^2 \geq \frac{1}{4}\|\phi_0\|^2$. In this case $\phi_0/\|\phi_0\| \in K_1$, and we can apply Lemma 3.19. In particular, by the choice of N and τ_1, we have

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_p(t)\|_1^2\,dt \geq 5\lambda_M\|\phi_0\|^2. \tag{3.84}$$

Since (3.79) still holds because of our choice of τ_0 and τ_1, it follows that

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi_c(t)\|_1^2\,dt \leq \frac{\lambda_M}{20}\|\phi_0\|^2. \tag{3.85}$$

Note that the H-norm in (3.79) has been replaced in (3.85) by the H^1-norm at the expense of the factor of λ_N. Together, (3.84) and (3.85) imply

$$\frac{1}{\tau_1}\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi^0(t)\|_1^2\,dt \geq 2\lambda_M\|\phi_0\|^2. \tag{3.86}$$

Finally, (3.86) and (3.78) give

$$\int_{\tau_0}^{\tau_0+\tau_1} \|P_N\phi^\varepsilon(t)\|_1^2\,dt \geq \frac{\lambda_M\tau_1}{2}\|\phi_0\|^2 \tag{3.87}$$

since $\|P_N\phi^\varepsilon - P_N\phi^0\|_1^2 \leq \lambda_N\|\phi^\varepsilon - \phi^0\|^2$. As before, (3.87) implies

$$\|\phi^\varepsilon(\tau_0+\tau_1)\|^2 \leq e^{-\lambda_M\varepsilon\tau_1}\|\phi^\varepsilon(\tau_0)\|^2, \tag{3.88}$$

finishing the proof of (3.77) in the second case.

Summarizing, we see that if $\|\phi^\varepsilon(\tau_0)\|_1^2 \leq \lambda_M\|\phi^\varepsilon(\tau_0)\|^2$, then

$$\|\phi^\varepsilon(\tau_0+\tau_1)\|^2 \leq e^{-\lambda_M\varepsilon\tau_1/20}\|\phi^\varepsilon(\tau_0)\|^2. \tag{3.89}$$

On the other hand, for any interval $I = [a,b]$ such that $\|\phi^\varepsilon(t)\|_1^2 \geq \lambda_M\|\phi^\varepsilon(t)\|^2$ on I, we have by Lemma 3.13 that

$$\|\phi^\varepsilon(b)\|^2 \leq e^{-2\lambda_M\varepsilon(b-a)}\|\phi^\varepsilon(a)\|^2. \tag{3.90}$$

Combining all the decay factors gained from (3.89) and (3.90), and using $\tau_1 < \tau/2\varepsilon$, we find that there is $\tau_2 \in [\tau/2\varepsilon, \tau/\varepsilon]$ such that

$$\|\phi^\varepsilon(\tau_2)\|^2 \leq e^{-\lambda_M\varepsilon\tau_2/20} \leq e^{-\lambda_M\tau/40} < \delta^2$$

by our choice of M. Then (3.62) gives $\|\phi^\varepsilon(\tau/\varepsilon)\| \leq \|\phi^\varepsilon(\tau_2)\| < \delta$, finishing the proof of Theorem 3.9. □

4 Regularity and mixing of the trajectories

Following the work of Crippa and De Lellis [20], we will now consider how mixing, essentially in the sense of dynamical systems, are the trajectories of a dynamical system

$$\frac{dX(t,x)}{dt} = u(t, X(t,x)), \quad X(0,x) = x, \tag{4.1}$$

driven by an incompressible flow u: $\nabla_x \cdot u(t,x) = 0$. Crippa and De Lellis address much more general flows but for the sake of simplicity and to highlight just one of their results we will assume that u is divergence free, uniformly bounded and Lipschitz, and consider (4.1) on the two-dimensional torus $\mathbb{T}^2 = [0,1]\times[0,1]$, and on a fixed time-interval $0 \leq t \leq T$. Let $\Phi:\ \mathbb{T}^2 \to \mathbb{T}^2$ be the time T map of the flow:

$$\Phi(x) = X(T,x), \quad x \in \mathbb{T}^2. \tag{4.2}$$

We say that a set A with $|A| = 1/2$ (it occupies half of the torus) is mixed on scale ε by Φ if for any ball $B(x,\varepsilon)$ with $x \in \mathbb{T}^2$ we have

$$\frac{1}{4}|B(x,\varepsilon)| \leq |B(x,\varepsilon)\cap\Phi(A)| \leq \frac{3}{4}|B(x,\varepsilon)|, \tag{4.3}$$

that is, at least a quarter of points (in measure) in every ball "come from A" and at least a quarter of them come from the complement $A' = \mathbb{T}^2 \setminus A$. The constants 1/4 and 3/4 may be replaced in this definition by any δ and $1-\delta$, respectively, for $0 < \delta < 1/2$ (it would be too optimistic to expect this to hold with $\delta = 1/2$ since $|A| = 1/2$ and the total area of the torus is 1). For the sake of concreteness we take A to be the "bottom" half of the torus:

$$A = \{(x_1,x_2) \in \mathbb{T}^2, \ \ 0 \leq x_2 \leq 1/2\}, \tag{4.4}$$

that is A itself is "unmixed as it gets". The question we would like to study is how rough the flow u has to be in order to mix this set on a scale ε by time T.

First, we note that if we take a ball $B(y,\varepsilon)$ that contains a point $z=\Phi(x)$ with x from a smaller strip

$$A_1=\{(x_1,x_2)\in\mathbb{T}^2:\ 1/6\le x_2\le 1/3\},$$

then, since $B(y,\varepsilon)$ contains both the point z and an image of some point x' from the complement $A'=\mathbb{T}^2\setminus A$ (this is because Φ mixes A on scale ε), and since for any such x' the distance $|x-x'|>1/6$, we have

$$\frac{|\Phi(x)-\Phi(x')|}{|x-x'|}\le\frac{2\varepsilon}{1/6}=12\varepsilon.$$

Hence, the Lipschitz constant of the inverse map Φ^{-1} should satisfy

$$\mathrm{Lip}(\Phi^{-1})\ge\frac{1}{12\varepsilon}. \tag{4.5}$$

Moreover, Φ^{-1} is simply the time T map of the time-reversed flow $v(t,x)=-u(T-t,x)$:

$$\Phi^{-1}:\ x\to Y(T,x),$$

with

$$\frac{dY}{dt}=-u(T-t,Y),\quad Y(0)=x. \tag{4.6}$$

The gradient matrix $G=\nabla Y$, with the entries $G_{ij}=\partial Y_i/\partial x_j$ satisfies

$$\frac{dG_{ij}(t)}{dt}=-\frac{\partial u_i}{\partial y_k}\frac{\partial Y_k}{\partial x_j}, \tag{4.7}$$

so that, in the matrix form we have

$$\frac{dG}{dt}=-\nabla u(Y(t))G. \tag{4.8}$$

It follows that (as $G(0)=\mathrm{Id}$)

$$\|G(T,\cdot)\|_{L^\infty}\le\exp\left[C\int_0^T\|\nabla u(t,\cdot)\|_{L^\infty}dt\right], \tag{4.9}$$

and (4.5) implies that if $u(x)$ is mixing the set A on scale ε by time T, we need to have

$$\int_0^T\|\nabla u(t,\cdot)\|_{L^\infty}dt\ge C|\log\varepsilon|. \tag{4.10}$$

A more precise version of (4.10) is that

$$\|G(T,\cdot)\|_{L^\infty}\le\sup_x\exp\left[C\int_0^T|\nabla u(t,Y(t,x))|dt\right], \tag{4.11}$$

so that (4.5) implies

$$\sup_x \left[\int_0^T |\nabla u(t, Y(t,x))| dt\right] \geq C \log \varepsilon. \tag{4.12}$$

Unfortunately, this condition is essentially unverifiable as it involves an integral over trajectories, and they are not known a priori. However, if the flow is "sufficiently mixing" so that $Y(t,x)$ "visits all of $\mathbb{T}^2$" (which mixing on a small scale ε basically means), the flow is close to being ergodic, and we expect that for every x we have

$$\int_0^T |\nabla u(t, Y(t,x))| dt \approx \int_0^T \int_{\mathbb{T}^2} |\nabla u(t,x)| dx dt. \tag{4.13}$$

That is, "morally", the unknown time integral along the trajectories in (4.12) can be replaced by the (very much computable) spatial average of $|\nabla u(t,x)|$.

Alberto Bressan conjectured that, indeed, in order for an incompressible flow u to the set A on a scale ε by time T we need to have

$$M_1(T) := \int_0^T \int_{\mathbb{T}^2} |\nabla_x u(t, x_1, x_2)| dx_1 dx_2 dt \geq C |\log \varepsilon|, \tag{4.14}$$

for all $0 < \varepsilon < 1/4$. In particular, this limits from below the possible scale on which a given flow can mix. On the other hand, the smallest scale up to which u can mix the set A is bounded a priori from below by $\exp(-M_1/C)$, which is still a very small number! We will describe a result of Crippa and DeLellis that is only slightly weaker than Bressan's conjecture.

Theorem 4.1 *([20]) For any $p > 1$ there exists a constant $C > 0$ and $\varepsilon_0 > 0$ so that if the flow Φ defined by (4.2) mixes the set A on a scale ε by time T then*

$$M_p(T) := \int_0^T \left(\int_{\mathbb{T}^2} |\nabla_x u(t, x_1, x_2)|^p dx_1 dx_2\right)^{1/p} dt > C |\log \varepsilon| \text{ for every } 0 < \varepsilon < \varepsilon_0. \tag{4.15}$$

It is also proved in [20] that the constant ε_0 can be taken to be equal to $1/4$ for the particular choice of the set A in (4.4) – this constant, however, is not quite universal and does depend, in general on the choice of the set, so we refer the reader to that paper for this additional argument.

The basic ingredient in the proof of Theorem 4.1 is an estimate[6] on the Lipschitz constant of the trajectories $X(t,x)$ generated by an incompressible flow u on $\mathbb{T}^2$:

$$\frac{dX(t,x)}{dt} = u(t, X(t,x)), \quad X(0,x) = x \in \mathbb{T}^2, \tag{4.16}$$

with $\nabla \cdot u = 0$, in terms of $M_p(T)$.

Theorem 4.2 *Let $X(t,x)$ be solution of (4.16) on a time interval $0 \leq t \leq T$, and fix some $p > 1$. Then for every $\delta > 0$ there exists a set $K \subset \mathbb{T}^2$ such that $|\mathbb{T}^2 \setminus K| \leq \delta$, and for any $0 \leq t \leq T$ and any $x, x' \in K$ we have*

$$\frac{|X(t,x) - X(t,x')|}{|x - x'|} \leq \exp\left[\frac{C M_p(T)}{\delta^{1/p}}\right], \tag{4.17}$$

with the constant C that depends only on p.

[6] We state this estimate here for periodic flows but it is proved in [20] for flows in $\mathbb{R}^n$, as well.

We postpone the proof of Theorem 4.2, and first use it to prove Theorem 4.1. Theorem 4.2 applies equally well to the mapping $\Phi:\ x \to X(T,x)$ and its inverse Φ^{-1}, as Φ^{-1} is simply the flow map of (4.16) with $u(t,x)$ replaced by $v(t,x) = -u(T-t,x)$. The idea is to to use the fact that Φ mixes the set A on scale ε to find two points z and z' such that, on one hand, the distance $|z-z'|$ is small but on the other, the distance between the corresponding pre-images $|\Phi^{-1}(z) - \Phi^{-1}(z')|$ is large. This gives a lower bound on the Lipschitz constant of Φ^{-1} which, in turn, gives a lower bound on $M_p(T)$ due to Theorem 4.2.

Here is how this idea is realized on the technical level. Theorem 4.2 implies that for every $\delta > 0$ there exists a set B_0 so that $|B_0| < \delta$, and outside B_0 we have the Lipschitz bound for Φ^{-1}:

$$|\Phi^{-1}(x) - \Phi^{-1}(x')| \le e^{\beta M_p(T)}|x-x'|, \quad \text{for all } x, x' \in \mathbb{T}^2 \setminus B_0, \tag{4.18}$$

with a constant β that depends only on δ and p. Let $A' = \mathbb{T}^2 \setminus A$ be the complement of A, then, as the set A is mixed on the scale ε by the mapping Φ, for every $x \in A$ we have

$$|B(\Phi(x);\varepsilon) \cap \Phi(A')| \ge \frac{1}{4}|B(\Phi(x);\varepsilon)|, \tag{4.19}$$

that is, in the ball of radius ε around the point $\Phi(x)$ there are always points that are images of points in A' by Φ. If we take a point x in the set

$$A_1 = \{(x_1,x_2) \in \mathbb{T}^2 :\ 1/6 \le x_2 \le 1/3\},$$

so that $\mathrm{dist}(x, A') > 1/6$ then we will have two points, $z = \Phi(x)$ and $z' = \Phi(x')$, with $x' \in A'$, so that $|z - z'| \le \varepsilon$ but $|x - x'| \ge 1/6$. This would give us exactly what we want provided we know that z, z' are not in the set B_0.

We now arrange for z and z' not to be in B_0. Consider the "dangerous set"

$$\tilde{A} = \{x \in A_1 :\ B(\Phi(x);\varepsilon) \cap [\Phi(A') \setminus B_0] = \emptyset\},$$

that is, the only points from $\Phi(A')$ inside the ball $B(\Phi(x);\varepsilon)$ are in B_0. Then (4.19) implies that for every $x \in \tilde{A}$ we have

$$|B(\Phi(x);\varepsilon) \cap B_0| \ge \frac{1}{4}|B(\Phi(x);\varepsilon)|. \tag{4.20}$$

The Besicovitch covering lemma implies that there exists a constant c_0 that depends only on the dimension $n = 2$ so that

$$|\Phi(\tilde{A})| \le 4c_0|B_0| \le 4c_0\delta. \tag{4.21}$$

As the mapping Φ is measure-preserving, we deduce that the set $\tilde{A}$ itself is small:

$$|\tilde{A}| \le 4c_0\delta. \tag{4.22}$$

Therefore, as, in addition, $|B_0| = |\Phi^{-1}(B_0)|$, we may choose δ so small that

$$|\tilde{A}| + |\Phi^{-1}(B_0)| \le \frac{1}{10} < |A_1|. \tag{4.23}$$

It follows that there exists a point $\bar{x} \in A_1$ such that $\bar{x} \notin \tilde{A} \cup \Phi^{-1}(B_0)$ and $\text{dist}(\bar{x}, A') > 1/6$. We set $z = \Phi(\bar{x})$. Since $\bar{x} \notin \tilde{A}$, there exists $z' \in B(\Phi(x), \varepsilon)$ such that $z' \notin B_0$ and $z' = \Phi(x')$ for some $x' \in A'$. As $z, z' \notin B_0$ by construction, we have, according to (4.18):

$$|\bar{x} - x'| = |\Phi^{-1}(\bar{z}) - \Phi^{-1}(z')| \leq |\bar{z} - z'| e^{\beta M_p(T)} \leq \varepsilon e^{\beta M_p(T)}. \tag{4.24}$$

Since $\text{dist}(\bar{x}, A') > 1/6$, we deduce that

$$\frac{1}{6} \leq \varepsilon e^{\beta M_p(T)}, \tag{4.25}$$

and the conclusion of Theorem 4.1 follows.

The proof of Theorem 4.2

We now prove Theorem 4.2. The main observation is that the Lipschitz constant is controlled by the following quantity:

$$A_p(T) = \left[\int_{\mathbb{T}^2} \left(\sup_{0 \leq t \leq T} \sup_{0 < r < 2} \frac{1}{|B(x;r)|} \int_{B(x;r)} \log \left(\frac{|X(t,x) - X(t,y)|}{r} + 1 \right) dy \right)^p dx \right]^{1/p}. \tag{4.26}$$

More precisely, we have the following two lemmas. The first gives an estimate of $A_p(T)$ in terms of $M_p(T)$.

Lemma 4.3 *There exists a constant C so that*

$$A_p(T) \leq C M_p(T). \tag{4.27}$$

The second estimates the Lipschitz constant on a large subset of the torus in terms of $A_p(T)$.

Lemma 4.4 *For every $\delta > 0$ we can find a set $K \subset \mathbb{T}^2$ so that $|\mathbb{T}^2 \setminus K| \leq \delta$, and for every $0 \leq t \leq T$ we have*

$$\frac{|X(t,x) - X(t,x')|}{|x - x'|} \leq \exp \frac{c_n A_p(T)}{\delta^{1/p}}, \tag{4.28}$$

for all $x, x' \in \mathbb{T}^2 \setminus K$.

Together, these lemmas imply the conclusion of Theorem 4.2. We first prove Lemma 4.4 simply because its proof is shorter.

The proof of Lemma 4.4

The definition of $A_p(T)$ and the Chebyshev inequality

$$|\{f(x) > \alpha\}| \leq \frac{\|f\|_{L^p}^p}{\alpha^p},$$

taken with $\alpha = A_p(T)/\delta^{1/p}$ imply that there exists a compact set $K \subset \mathbb{T}^2$ so that

$$|\mathbb{T}^2 \setminus K| \leq \delta,$$

and for all $x \in K$, $0 \le t \le T$ and all $0 < r < 2$ we have

$$\frac{1}{|B(x;r)|}\int_{B(x;r)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)dy \le \frac{A_p(T)}{\delta^{1/p}}. \tag{4.29}$$

Let us now take any $x, y \in K$ and set $r = |x-y|$. As we are on the two-dimensional torus, we have $r < 2$, thus

$$\begin{aligned}
\log\left[\frac{|X(t,x)-X(t,y)|}{r}+1\right] &= \frac{1}{|B(x;r)\cap B(y;r)|}\int_{B(x;r)\cap B(y;r)} \log\left[\frac{|X(t,x)-X(t,y)|}{r}+1\right]dz \\
&\le \frac{1}{|B(x;r)\cap B(y;r)|}\int_{B(x;r)\cap B(y;r)} \log\left[\frac{|X(t,x)-X(t,z)|}{r}+1\right]\,dz \\
&+\frac{1}{|B(x;r)\cap B(y;r)|}\int_{B(x;r)\cap B(y;r)} \log\left[\frac{|X(t,z)-X(t,y)|}{r}+1\right]\,dz \\
&\le \frac{C}{|B(x;r)|}\int_{B(x;r)} \log\left[\frac{|X(t,x)-X(t,z)|}{r}+1\right]\,dz \\
&+\frac{C}{|B(y;r)|}\int_{B(y;r)} \log\left[\frac{|X(t,z)-X(t,y)|}{r}+1\right]\,dz \le 2C\frac{A_p(R,T)}{\delta^{1/p}}.
\end{aligned} \tag{4.30}$$

We used above the inequality

$$\log(1+a+b) \le \log(1+a)+\log(1+b), \quad a \ge 0, \quad b \ge 0,$$

and that the intersection $|B(x,r)\cap B(y,r)|$ has a volume comparable to r^2 when $r = |x-y|$, in addition to the assumption that $x, y \in K$ and (4.29). It follows that

$$\frac{|X(t,x)-X(t,y)|}{|x-y|} \le \exp\left[2C\frac{A_p(T)}{\delta^{1/p}}\right], \tag{4.31}$$

for all $x, y \in \mathbb{T}^2 \setminus K$, and all $0 \le t \le T$, and the conclusion of Lemma 4.4 follows.

The proof of Lemma 4.3

Let us define

$$Q(t,x,r) = \frac{1}{|B(x;r)|}\int_{B(x;r)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)dy, \tag{4.32}$$

for $0 \le t \le T$ and $x \in \mathbb{T}^2$, and $0 < r < 2$. The time derivative of this function satisfies

$$\frac{dQ(t,x,r)}{dt} = \frac{1}{|B(x;r)|}\int_{B(x;r)} \frac{1}{|X(t,x)-X(t,y)|+r}\frac{d}{dt}[|X(t,x)-X(t,y)|]dy. \tag{4.33}$$

For a vector-valued function $v(t)$ we have

$$2|v|\frac{d}{dt}(|v|) = \frac{d}{dt}|v|^2 = 2(v\cdot\frac{dv}{dt}) \le 2|v|\cdot|\frac{dv}{dt}|,$$

hence

$$\frac{d}{dt}(|v|) \le |\frac{dv}{dt}|.$$

Using this inequality in (4.33) gives

$$\begin{aligned}\frac{dQ(t,x,r)}{dt} &\le \frac{1}{|B(x;r)|}\int_{B(x;r)} \frac{1}{|X(t,x)-X(t,y)|+r}\left|\frac{dX(t,x)}{dt} - \frac{dX(t,y)}{dt}\right|dy \\ &= \frac{1}{|B(x;r)|}\int_{B(x;r)} \frac{|u(X(t,x))-u(X(t,y))|}{|X(t,x)-X(t,y)|+r}dy. \end{aligned} \tag{4.34}$$

Let us recall the definition of the maximal function

$$M_\lambda f(x) = \sup_{0<r<\lambda}\frac{1}{|B(x;r)|}\int_{B(x;r)}|f(y)|dy.$$

We will make use of the following lemma.

Lemma 4.5 *There exists a constant $C_n > 0$ (that depends only on the dimension n) so that for all $|x-y| \le \lambda$ we have*

$$\frac{|u(x)-u(y)|}{|x-y|} \le C_n(M_\lambda \nabla u(x) + M_\lambda \nabla u(y)). \tag{4.35}$$

Proof of Lemma 4.5. Let us assume that $|x-y| = 2R$. Without loss of generality we may assume that $x = (-R,0)$ and $y = (R,0)$. Consider the family of paths Γ_β, parametrized by $\beta \in [-1,1]$, that consist of two straight lines: the first one connects x and the point $z_\beta = (0,\beta R)$, and the second line connects z_β and y. For each $\beta \in [-1,1]$ we have

$$|u(x)-u(y)| \le \int_{\Gamma_\beta}|\nabla u|dl,$$

hence

$$\begin{aligned}|u(x)-u(y)| &\le \sqrt{2}\int_{-1}^{1}\left(\int_{-R}^{R}|\nabla u(z_1,\beta(R-|z_1|))dz_1\right)d\beta \\ &= \sqrt{2}\int_{-R}^{R}dz_1\int_{|z_1|-R}^{R-|z_1|}\frac{|\nabla u(z_1,z_2)|}{R-|z_1|}dz_2 \le 2\sqrt{2}\left(\int_{B_R(x)}\frac{|\nabla u(z)|}{|z-x|}dz + \int_{B_R(y)}\frac{|\nabla u(z)|}{|z-y|}dz\right).\end{aligned} \tag{4.36}$$

The last inequality above follows from elementary geometry. On the other hand, for any $0 < \rho \le R$ we have

$$M_{2R}\nabla u(x) \ge \frac{C}{\rho^2}\int_{B_\rho(x)}|\nabla u(z)|dz,$$

so that

$$\begin{aligned}M_{2R}\nabla u(x) &\ge \frac{C}{R}\int_0^R\frac{1}{\rho^2}\left(\int_{B_\rho(x)}|\nabla u(z)|dz\right)d\rho \\ &= \frac{C}{R}\int_0^R d\rho\int_0^\rho dr\int_0^{2\pi}d\phi\frac{r}{\rho^2}|\nabla u(z)| = \frac{C}{R}\int_0^R dr\int_r^R d\rho\int_0^{2\pi}d\phi\frac{r}{\rho^2}|\nabla u(z)| \\ &= \frac{C}{R}\int_{B_R(x)}\left[1-\frac{|z-x|}{R}\right]\frac{|\nabla u(z)|}{|z-x|}dz \ge \frac{C}{2R}\int_{B_{R/2}(x)}\frac{|\nabla u(z)|}{|z-x|}dz.\end{aligned} \tag{4.37}$$

Finally, for the integral over the annulus $\{R/2 \le |z| \le R\}$ we have

$$\frac{1}{R}\int_{B_R(x)\setminus B_{R/2}(x)} \frac{|\nabla u(z)|}{|z-x|}dz \le \frac{C}{R^2}\int_{B_R(x)} |\nabla u(z)|dz \le CM_{2R}\nabla u(x). \tag{4.38}$$

This inequality, combined with (4.36) and (4.37) finishes the proof of Lemma 4.5.

The end of the proof of Lemma 4.3. We now conclude the proof of Lemma 4.3. On a torus, we have

$$|X(t,x) - X(t,y)| \le \bar{R} = 2,$$

hence Lemma 4.5 and (4.34) imply that

$$\begin{aligned}\frac{dQ(t,x,r)}{dt} &\le \frac{C}{|B(x;r)|}\int_{B(x;r)} (M_{\bar{R}}\nabla u(X(t,x)) + M_{\bar{R}}\nabla u(X(t,y)))dy \\ &\le CM_{\bar{R}}\nabla u(X(t,x)) + \frac{C}{|B(x;r)|}\int_{B(x;r)} M_{\bar{R}}\nabla u(X(t,y))dy. \end{aligned}\tag{4.39}$$

Integrating in t and taking supremum over t and r gives:

$$\begin{aligned}\sup_{0\le t\le T}\sup_{0<r<2} Q(t,x,r) &\le C_0 + C\int_0^T M_{\bar{R}}\nabla u(X(t,x))dt \\ &+ C\int_0^T \sup_{0<r<2}\left[\frac{1}{|B(x;r)|}\int_{B(x;r)} M_{\bar{R}}\nabla u(X(t,y))dy\right]dt.\end{aligned}\tag{4.40}$$

Taking the L^p-norm over the torus we obtain

$$\begin{aligned}A_p(T) &\le C + C\left\|\int_0^T M_{\bar{R}}\nabla u(X(t,x))dt\right\|_{L^p_x(\mathbb{T}^2)} \\ &+C\left\|\int_0^T \sup_{0<r<2}\left[\frac{1}{|B(x;r)|}\int_{B(x;r)} M_{\bar{R}}\nabla u(X(t,y))dy\right]dt\right\|_{L^p_x(\mathbb{T}^2)}.\end{aligned}\tag{4.41}$$

Recall that the maximal function satisfies the following L^p-estimate for $p > 1$:

$$\int_{B(0,r)} |M_\lambda f(y)|^p dy \le C(p)\int_{B(0,r+\lambda)} |f(y)|^p dp. \tag{4.42}$$

It follows that, using incompressibility of u, we have

$$\begin{aligned}\left\|\int_0^T M_{\bar{R}}\nabla u(X(t,x))dt\right\|_{L^p_x(\mathbb{T}^2)} &\le \int_0^T \|(M_{\bar{R}}\nabla u\circ X(t,\cdot))(x)\|_{L^p_x(\mathbb{T}^2)}dt \\ &= \int_0^T \|M_{\bar{R}}\nabla u(t,\cdot)\|_{L^p(\mathbb{T}^2)}dt \le C(p)\int_0^T \|\nabla u(t,\cdot)\|_{L^p(\mathbb{T}^2)}dt.\end{aligned}\tag{4.43}$$

The last term in (4.41) can be estimated very similarly, except that the L^p-bound (4.42) has to be applied to the iterated maximal function $M_{2R}(M_{\bar{R}}\nabla u\circ X(t,\cdot)(x))$. This finishes the proof of Lemma 4.3.

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