Chapter 2. Properties of Holomorphic Functions

May 20, 2003

We will consider in this chapter some of the most important methods in the study of holomorphic functions. They are based on the representation of such functions as special integrals (the Cauchy integral) and as sums of power series (the Taylor and the Laurent series). We begin with the notion of the integral of a function of a complex variable.

1 The Integral

1.1 Definition of the integral

Definition 1.1 *Let $\gamma : I \to \mathbb{C}$ be a piecewise smooth path, where $I = [\alpha, \beta]$ is an interval on the real axis. Let a complex-valued function f be defined on $\gamma(I)$ so that the function $f \circ \gamma$ is a continuous function on I. The integral of f along the path γ is*

$$\int_\gamma f dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt. \tag{1.1}$$

The integral in the right side of (1.1) is understood to be $\int_\alpha^\beta g_1(t)dt + i\int_\alpha^\beta g_2(t)dt$, where g_1 and g_2 are the real and imaginary parts of the function $f(\gamma(t))\gamma'(t) = g_1(t) + ig_2(t)$.

Note that the functions g_1 and g_2 may have only finitely many discontinuities on I so that the integral (1.1) exists in the usual Riemann integral sense. If we set $f = u + iv$ and $dz = \gamma'(t)dt = dx + idy$ then (1.1) may be rewritten as

$$\int_\gamma f dz = \int_\gamma u dx - v dy + i\int_\gamma v dx + u dy. \tag{1.2}$$

One could also define the integral (1.1) as the limit of partial sums: divide the curve $\gamma(I)$ into finally many pieces $z_0 = \gamma(\alpha)$, $z_1 = \gamma(t_1)$, $\ldots$, $z_n = \gamma(\beta)$, $\alpha < t_1 < \cdots < \beta$, choose arbitrary points $\zeta_k = \gamma(\tau_k)$, $\tau_k \in [t_k, t_{k+1}]$ and define

$$\int_\gamma f dz = \lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\zeta_k)\Delta z_k, \tag{1.3}$$

where $\Delta z_k = z_{k+1} - z_k$, $k = 0, 1, \dots, n-1$ and $\delta = \max |\Delta z_k|$. Nevertheless we will use only the first definition and will not prove its equivalence to the other two.

If the path γ is just a rectifiable curve, then the Riemann integral is not defined even for continuous functions f because of the factor $\gamma'(t)$ in the right side of (1.1). One would have to use the Lebesgue integral in that case and assume that the function $f(\gamma(t))$ is Lebesgue integrable on I.

Example 1.2 Let γ be a circle $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, and $f(z) = (z-a)^n$, where $n = 0, \pm 1, \dots$ is an integer. Then we have $\gamma'(t) = re^{it}$, $f(\gamma(t)) = r^n e^{int}$ so that

$$\int_\gamma (z-a)^n dz = r^{n+1} i \int_0^{2\pi} e^{i(n+1)t} dt.$$

We have to consider two cases: when $n \neq 1$ we have

$$\int_\gamma (z-a)^n dz = r^{n+1} \frac{e^{2\pi i(n+1)} - 1}{n+1} = 0,$$

because of the periodicity of the exponential function, while when $n = -1$

$$\int_\gamma \frac{dz}{z-a} = i \int_0^{2\pi} dt = 2\pi i.$$

Therefore the integer powers of $z - a$ have the "orthogonality" property

$$\int_\gamma (z-a)^n = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n = -1 \end{cases} \tag{1.4}$$

that we will use frequently.

Example 1.3 Let $\gamma : \ I \to \mathbb{C}$ be an arbitrary piecewise smooth path and $n \neq 1$. We also assume that the path $\gamma(t)$ does not pass through the point $z = 0$ in the case $n < 0$. The chain rule implies that $\frac{d}{dt}\gamma^{n+1}(t) = (n+1)\gamma^n(t)\gamma'(t)$ so that

$$\int_\gamma z^n dz = \int_\alpha^\beta \gamma^n(t)\gamma'(t)dt = \frac{1}{n+1}[\gamma^{n+1}(\beta) - \gamma^{n+1}(\alpha)]. \tag{1.5}$$

We observe that the integrals of z^n, $n \neq -1$ depend not on the path but only on its endpoints. Their integrals over a closed path vanish.

We summarize the basic properties of the integral of a complex-valued function.

1. *Linearity.* If f and g are continuous on a piecewise smooth path γ then for any complex numbers α and β we have

$$\int_\gamma (\alpha f + \beta g) dz = \alpha \int_\gamma f dz + \beta \int_\gamma g dz. \tag{1.6}$$

This follows immediately from the definition.

2. *Additivity.* Let $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ and $\gamma_2 : [\beta_1, \beta_2] \to \mathbb{C}$ be two piecewise smooth paths so that $\gamma_1(\beta_1) = \gamma_2(\beta_1)$. The union $\gamma = \gamma_1 \cup \gamma_2$ is a path $\gamma : [\alpha_1, \beta_2] \to \mathbb{C}$ so that

$$\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [\alpha_1, \beta_1] \\ \gamma_2(t), & \text{if } t \in [\beta_1, \beta_2] \end{cases}.$$

We have then for any function f that is continuous on $\gamma_1 \cup \gamma_2$:

$$\int\limits_{\gamma_1 \cup \gamma_2} f dz = \int\limits_{\gamma_1} f dz + \int\limits_{\gamma_2} f dz. \tag{1.7}$$

One may drop the condition $\gamma_1(\beta_1) = \gamma_2(\beta_2)$ in the definition of the union $\gamma_1 \cup \gamma_2$. Then $\gamma_1 \cup \gamma_2$ will no longer be a continuous path but property (1.7) would still hold.

3. *Invariance.* Integral is invariant under a re-parameterization of the path.

Theorem 1.4 *Let a path $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ be obtained from a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$ by a legitimate re-parameterization, that is $\gamma = \gamma_1 \circ \tau$ where τ is an increasing piecewise smooth map $\tau : [\alpha, \beta] \to [\alpha_1, \beta_1]$. Then we have for any function f that is continuous on γ (and hence on γ_1):*

$$\int_{\gamma_1} f dz = \int_{\gamma} f dz. \tag{1.8}$$

Proof. The definition of the integral implies that

$$\int_{\gamma_1} f dz = \int_{\alpha_1}^{\beta_1} f(\gamma_1(s))\gamma_1'(s) ds.$$

Introducing the new variable t so that $\tau(t) = s$ and using the usual rules for the change of real variables in an integral we obtain

$$\begin{aligned} \int_{\gamma_1} f dz &= \int_{\alpha_1}^{\beta_1} f(\gamma_1(s))\gamma_1'(s) ds = \int_{\alpha}^{\beta} f(\gamma_1(\tau(t)))\gamma_1'(\tau(t))\tau'(t) dt \\ &= \int_{\alpha}^{\beta} f(\gamma(t))\gamma'(t) dt = \int_{\gamma} f dz. \ \square \end{aligned}$$

This theorem has an important corollary: the integral that we defined for a path makes sense also for a *curve* that is an equivalence class of paths. More precisely, the value of the integral along any path that defines a given curve is independent of the choice of path in the equivalence class of the curve.

As we have previously mentioned we will often identify the curve and the set of points on the complex plane that is the image of a path that defines this curve. Then we will talk about integral over this set understanding it as the integral along the corresponding set. For instance, expressions (1.4) may be written as

$$\int\limits_{\{|z-a|=r\}} \frac{dz}{z-a} = 2\pi i, \qquad \int\limits_{\{|z-a|=r\}} (z-a)^n dz = 0, \quad n \in \mathbb{Z}\backslash\{-1\}.$$

4. *Orientation.* Let γ^- be the path that is obtained out of a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$ by a change of variables $t \to \alpha + \beta - t$, that is, $\gamma^-(t) = \gamma(\alpha + \beta - t)$, and let f be a function continuous on γ. Then we have

$$\int_{\gamma^-} f dz = -\int_\gamma f dz. \tag{1.9}$$

This statement is proved exactly as Theorem 1.4.

We say that the path γ^- is obtained from γ by a change of orientation.

5. *A bound for the integral.*

Theorem 1.5 *Let f be a continuous function defined on a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$. Then the following inequality holds:*

$$\left| \int_\gamma f dz \right| \le \int_\gamma |f||d\gamma|, \tag{1.10}$$

where $|d\gamma| = |\gamma'(t)|dt$ is the differential of the arc length of γ and the integral on the right side is the real integral along a curve.

Proof. Let us denote $J = \int_\gamma f dz$ and let $J = |J|e^{i\theta}$, then we have

$$|J| = \int_\gamma e^{-i\theta} f dz = \int_\alpha^\beta e^{-i\theta} f(\gamma(t))\gamma'(t) dt.$$

The integral on the right side is a real number and hence

$$|J| = \int_\alpha^\beta \mathrm{Re}\left[e^{-i\theta} f(\gamma(t))\gamma'(t)\right] dt \le \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)| dt = \int_\gamma |f||d\gamma|. \ \square$$

Corollary 1.6 *Let assumptions of the previous theorem hold and assume that $|f(z)| \le M$ for a constant M, then*

$$\left| \int_\gamma f dz \right| \le M|\gamma|, \tag{1.11}$$

where $|\gamma|$ is the length of the path γ.

Inequality (1.11) is obtained from (1.10) if we estimate the integral on the right side of (1.10) and note that $\int_\gamma |d\gamma| = |\gamma|$.

Exercise 1.7 *Show that if a function f is $\mathbb{R}$-differentiable in a neighborhood of a point $a \in \mathbb{C}$ then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int\limits_{\{|z-a|=\varepsilon\}} f(z)dz = 2\pi i \frac{\partial f}{\partial \bar{z}}(a).$$

Hint: use the formula

$$f(z) = f(a) + \frac{\partial f}{\partial z}(a)(z-a) + \frac{\partial f}{\partial \bar{z}}(a)(\bar{z}-\bar{a}) + o(|z-a|)$$

and Example 1.3.

1.2 The anti-derivative

Definition 1.8 *An anti-derivative of a function f in a domain D is a holomorphic function F such that at every point $z \in D$ we have*

$$F'(z) = f(z). \tag{1.12}$$

If F is an anti-derivative of f in a domain D then any function of the form $F(z) + C$ where C is an arbitrary constant is also an anti-derivative of f in D. Conversely, let F_1 and F_2 be two anti-derivatives of f in D and let $\Phi = F_1 - F_2$. The function Φ is holomorphic in D and thus $\dfrac{\partial \Phi}{\partial \bar{z}} = 0$ in D. Moreover, $\dfrac{\partial \Phi}{\partial z} = F_1' - F_2' = 0$ in D and therefore $\dfrac{\partial \Phi}{\partial x} = \dfrac{\partial \Phi}{\partial y} = 0$ in D. The familiar result of the real analysis applied to the real-valued functions $\mathrm{Re}\Phi$ and $\mathrm{Im}\Phi$ implies that $\Phi = C$ is a constant in D. We have proved the following theorem.

Theorem 1.9 *If F is an anti-derivative of f in D then the collection of all anti-derivatives of f in D is described by*

$$F(z) + C, \tag{1.13}$$

where C is an arbitrary constant.

Therefore an anti-derivative of f in D if it exists is defined up to an arbitrary constant.

Let us now address the existence of anti-derivative. First we will look at the question of existence of a local anti-derivative that exists in a neighborhood of a point. We begin with a theorem that expresses in the simplest form the Cauchy theorem that lies at the core of the theory of integration of holomorphic functions.

Theorem 1.10 *(Cauchy) Let $f \in \mathcal{O}(D)$, that is, f is holomorphic in D. Then the integral of f along the oriented boundary*[1] *of any triangle Δ that is properly contained*[2] *in D is equal to zero:*

$$\int_{\partial \Delta} f dz = 0. \tag{1.14}$$

[1] We assume that the boundary $\partial\Delta$ (that we treat as a piecewise smooth curve) is oriented in such a way that the triangle Δ remains on one side of $\partial\Delta$ when one traces $\partial\Delta$.

[2] A set S is properly contained in a domain S' if S is contained in a compact subset of S'.

Proof. Let us assume that this is false, that is, there exists a triangle Δ properly contained in D so that

$$\left|\int_{\partial\Delta} f dz\right| = M > 0. \tag{1.15}$$

Let us divide Δ into four sub-triangles by connecting the midpoints of all sides and assume that the boundaries both of Δ and these triangles are oriented counter-clockwise. Then clearly the integral of f over $\partial\Delta$ is equal to the sum of the integrals over the boundaries of the small triangles since each side of a small triangle that is not part of the boundary $\partial\Delta$ belongs to two small triangles with two different orientations so that they do not contribute to the sum. Therefore there exists at least one small triangle that we denote Δ_1 so that

$$\left|\int_{\partial\Delta_1} f dz\right| \geq \frac{M}{4}.$$

We divide Δ_1 into four smaller sub-triangles and using the same considerations we find one of them denoted Δ_2 so that $\left|\int_{\partial\Delta_2} f dz\right| \geq \frac{M}{4^2}$.

Continuing this procedure we construct a sequence of nested triangles Δ_n so that

$$\left|\int_{\partial\Delta_n} f dz\right| \geq \frac{M}{4^n}. \tag{1.16}$$

The closed triangles Δ_n have a common point $z_0 \in \Delta \subset D$. The function f is holomorphic at z_0 and hence for any $\varepsilon > 0$ there exists $\delta > 0$ so that we may decompose

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \alpha(z)(z - z_0) \tag{1.17}$$

with $|\alpha(z)| < \varepsilon$ for all $z \in U = \{|z - z_0| < \delta\}$.

We may find a triangle Δ_n that is contained in U. Then (1.17) implies that

$$\int_{\partial\Delta_n} f dz = \int_{\partial\Delta_n} f(z_0)dz + \int_{\partial\Delta_n} f'(z_0)(z - z_0)dz + \int_{\partial\Delta_n} \alpha(z)(z - z_0)dz.$$

However, the first two integrals on the right side vanish since the factors $f(z_0)$ and $f'(z_0)$ may be pulled out of the integrals and the integrals of 1 and $z-z_0$ over a closed path $\partial\Delta_n$ are equal to zero (see Example 1.3). Therefore, we have $\int_{\partial\Delta_n} f dz = \int_{\partial\Delta_n} \alpha(z)(z-z_0)dz$, where $|\alpha(z)| < \varepsilon$ for all $z \in \partial\Delta_n$. Furthermore, we have $|z - z_0| \leq |\partial\Delta_n|$ for all $z \in \partial\Delta_n$ and hence we obtain using Theorem 1.5

$$\left|\int_{\partial\Delta_n} f dz\right| = \left|\int_{\partial\Delta_n} \alpha(z)(z - z_0)dz\right| < \varepsilon|\partial\Delta_n|^2.$$

However, by construction we have $|\partial\Delta_n| = |\partial\Delta|/2^n$, where $|\partial\Delta|$ is the perimeter of Δ, so that

$$\left|\int_{\partial\Delta_n} f dz\right| < \varepsilon|\partial\Delta|^2/4^n.$$

This together with (1.16) implies that $M < \varepsilon|\partial\Delta|^2$ which in turn implies $M = 0$ since ε is an arbitrary positive number. This contradicts assumption (1.15) and the conclusion of Theorem 1.10 follows. $\square$

We will consider the Cauchy theorem in its full generality in the next section. At the moment we will deduce the local existence of anti-derivative from the above Theorem.

Theorem 1.11 *Let $f \in \mathcal{O}(D)$ then it has an anti-derivative in any disk $U = \{|z-a| < r\} \subset D$:*

$$F(z) = \int\limits_{[a,z]} f(\zeta)d\zeta, \tag{1.18}$$

where the integral is taken along the straight segment $[a,z] \subset U$.

Proof. We fix an arbitrary point $z \in U$ and assume that $|\Delta z|$ is so small that the point $z + \Delta z \in U$. Then the triangle Δ with vertices a, z and $z + \Delta z$ is properly contained in D so that Theorem 1.10 implies that

$$\int\limits_{[a,z]} f(\zeta)d\zeta + \int\limits_{[z,z+\Delta z]} f(\zeta)d\zeta + \int\limits_{[z+\Delta z,a]} f(\zeta)d\zeta = 0.$$

The first term above is equal to $F(z)$ and the third to $-F(z+\Delta z)$ so that

$$F(z+\Delta z) - F(z) = \int\limits_{[z,z+\Delta z]} f(\zeta)d\zeta. \tag{1.19}$$

On the other hand we have

$$f(z) = \frac{1}{\Delta z}\int\limits_{[z,z+\Delta z]} f(z)d\zeta$$

(we have pulled the constant factor $f(z)$ out of the integral sign above), which allows us to write

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int\limits_{[z,z+\Delta z]} [f(\zeta) - f(z)]d\zeta. \tag{1.20}$$

We use now continuity of the function f: for any $\varepsilon > 0$ we may find $\delta > 0$ so that if $|\Delta z| < \delta$ then we have $|f(\zeta) - f(z)| < \varepsilon$ for all $\zeta \in [z, z+\Delta z]$. We conclude from (1.20) that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon$$

provided that $|\Delta z| < \delta$. The above implies that $F'(z)$ exists and is equal to $f(z)$. $\square$

Remark 1.12 We have used only two properties of the function f in the proof of Theorem 1.11: f is continuous and its integral over any triangle Δ that is contained properly in D vanishes. Therefore we may claim that the function F defined by (1.18) is a local anti-derivative of any function f that has these two properties.

The problem of existence of a global anti-derivative in the whole domain D is somewhat more complicated. We will address it only in the next section, and now will just show how an anti-derivative that acts along a given path may be glued together out of local anti-derivatives.

Definition 1.13 *Let a function f be defined in a domain D and let $\gamma: I = [\alpha, \beta] \to D$ be an arbitrary continuous path. A function $\Phi: I \to \mathbb{C}$ is* an anti-derivative of f along the path γ *if (i) Φ is continuous on I, and (ii) for any $t_0 \in I$ there exists a neighborhood $U \subset D$ of the point $z_0 = \gamma(t_0)$ so that f has an anti-derivative F_U in U such that*

$$F_U(\gamma(t)) = \Phi(t) \tag{1.21}$$

for all t in a neighborhood $u_{t_0} \subset I$.

We note that if f has an anti-derivative F in the whole domain D then the function $F(\gamma(t))$ is an anti-derivative along the path γ. However, the above definition does not require the existence of a global anti-derivative in all of D – it is sufficient for it to exist *locally*, in a neighborhood of each point $z_0 \in \gamma$. Moreover, if $\gamma(t') = \gamma(t'')$ with $t' \neq t''$ then the two anti-derivatives of f that correspond to the neighborhoods $u_{t'}$ and $u_{t''}$ need not coincide: they may differ by a constant (observe that they are anti-derivatives of f in a neighborhood of the same point z' and hence Theorem 1.9 implies that their difference is a constant). Therefore anti-derivative along a path being a function of the parameter t might not be a function of the point z.

Theorem 1.14 *Let $f \in \mathcal{O}(D)$ and let $\gamma: I \to D$ be a continuous path. Then anti-derivative of f along γ exists and is defined up to a constant.*

Proof. Let us divide the interval $I = [\alpha, \beta]$ into n sub-intervals $I_k = [t_k, t'_k]$ so that each pair of adjacent sub-intervals overlap on an interval ($t_k < t'_{k-1} < t_{k+1} < t'_k$, $t_1 = \alpha$, $t'_n = \beta$). Using uniform continuity of the function $\gamma(t)$ we may choose I_k so small that the image $\gamma(I_k)$ is contained in a disk $U_k \subset D$. Theorem 1.10 implies that f has an anti-derivative F in each disk U_k. Let us choose arbitrarily an anti-derivative of f in U_1 and denote it F_1. Consider an anti-derivative of f defined in U_2. It may differ only by a constant from F_1 in the intersection $U_1 \cap U_2$. Therefore we may choose the anti-derivative F_2 of f in U_2 that coincides with F_1 in $U_1 \cap U_2$.

We may continue in this fashion choosing the anti-derivative F_k in each U_k so that $F_k = F_{k-1}$ in the intersection $U_{k-1} \cap U_k$, $k = 1, 2, \ldots, n$. The function

$$\Phi(t) = F_k \circ \gamma(t), \quad t \in I_k, \quad k = 1, 2, \ldots, n,$$

is an anti-derivative of f along γ. Indeed it is clearly continuous on γ and for each $t_0 \in I$ one may find a neighborhood u_{t_0} where $\Phi(t) = F_U \circ \gamma(t)$ where F_U is an anti-derivative of f in a neighborhood of the point $\gamma(t_0)$.

It remains to prove the second part of the theorem. Let Φ_1 and Φ_2 be two anti-derivatives of f along γ. We have $\Phi_1 = F^{(1)} \circ \gamma(t)$, $\Phi_2 = F^{(2)} \circ \gamma(t)$ in a neighborhood u_{t_0} of each point $t_0 \in I$. Here $F^{(1)}$ and $F^{(2)}$ are two anti-derivatives of f defined in a neighborhood of the point $\gamma(t_0)$. They may differ only by a constant so that $\phi(t) = \Phi_1(t) - \Phi_2(t)$ is constant in a neighborhood u_{t_0} of t_0. However, a locally constant function defined on a connected set is constant on the whole set [3]. Therefore $\Phi_1(t) - \Phi_2(t) = \text{const}$ for all $t \in I$. $\square$

If the anti-derivative of f along a path γ is known then the integral of f over γ is computed using the usual Newton-Leibnitz formula.

Theorem 1.15 *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise smooth path and let f be continuous on γ and have an anti-derivative $\Phi(t)$ along γ, then*

$$\int_\gamma f dz = \Phi(\beta) - \Phi(\alpha). \tag{1.22}$$

Proof. Let us assume first that γ is a smooth path and its image is contained in a domain D where f has an anti-derivative F. Then the function $F \circ \gamma$ is an anti-derivative of f along γ and hence differs from Φ only by a constant so that $\Phi(t) = F \circ \gamma(t) + C$. Since γ is a smooth path and $F'(z) = f(z)$ the derivative $\Phi'(t) = f(\gamma(t))\gamma'(t)$ exists and is continuous at all $t \in [\alpha, \beta]$. However, using the definition of the integral we have

$$\int_\gamma f dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t) dt = \int_\alpha^\beta \Phi'(t) dt = \Phi(\beta) - \Phi(\alpha)$$

and the theorem is proved in this particular case.

In the general case we may divide γ into a finite number of paths $\gamma_\nu : [\alpha_\nu, \alpha_{n+1}] \to \mathbb{C}$ $(\alpha_0 = \alpha < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta)$ so that each of them is smooth and is contained in a domain where f has an anti-derivative. As we have just shown,

$$\int_{\gamma_\nu} f dz = \Phi(\alpha_{\nu+1}) - \Phi(\alpha_\nu),$$

and summing over ν we obtain (1.22). $\square$

Remark 1.16 We may extend our definition of the integral to continuous paths (from piecewise smooth) by defining the integral of f over an arbitrary continuous path γ as the increment of its anti-derivative along the this path over the interval $[\alpha, \beta]$ of the parameter change. Clearly the right side of (1.22) does not change under a re-parameterization of the path. Therefore one may consider integrals of holomorphic functions over arbitrary continuous curves.

Remark 1.17 Theorem 1.15 allows us to verify that a holomorphic function might have no global anti-derivative in a domain that is not simply connected. Let $D = \{0 < |z| < 2\}$ be a punctured disk and consider the function $f(z) = 1/z$ that is holomorphic in D.

[3] Indeed, let $E = \{t \in I : \phi(t) = \phi(t_0)\}$. This set is not empty since it contains t_0. It is open since ϕ is locally constant so that if $t \in E$ and $\phi(t) = \phi(t_0)$ then $\phi(t') = \phi(t) = \phi(t_0)$ for all t' in a neighborhood u_t and thus $u_t \subset E$. However, it is also closed since ϕ is a continuous function (because it is locally constant) so that $\phi(t_n) = \phi(t_0)$ and $t_n \to t''$ implies $\phi(t'') = \phi(t_0)$. Therefore $E = I$.

This function may not have an anti-derivative in D. Indeed, were the anti-derivative F of f to exist in D, the function $F(\gamma(t))$ would be an anti-derivative along any path γ contained in D. Theorem 1.15 would imply that

$$\int_\gamma f dz = F(b) - F(a),$$

where $a = \gamma(\alpha)$, $b = \gamma(\beta)$ are the end-points of γ. In particular the integral of f along any closed path γ would vanish. However, we know that the integral of f over the unit circle is

$$\int\limits_{|z|=1} f dz = 2\pi i.$$

1.3 The Cauchy Theorem

We will prove now the Cauchy theorem in its general form - the basic theorem of the theory of integration of holomorphic functions (we have proved it in its simplest form in the last section). This theorem claims that the integral of a function holomorphic in some domain does not change if the path of integration is changed continuously inside the domain provided that its end-points remain fixed or a closed path remains closed. We have to define first what we mean by a continuous deformation of a path. We assume for simplicity that all our paths are parameterized so that $t \in I = [0, 1]$. This assumption may be made without any loss of generality since any path may be re-parameterized in this way without changing the equivalence class of the path and hence the value of the integral.

Definition 1.18 *Two paths $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ with common ends $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(1) = \gamma_1(1) = b$ are* homotopic *to each other in a domain D if there exists a continuous map $\gamma(s, t) : I \times I \to D$ so that*

$$\begin{aligned} \gamma(0, t) = \gamma_0(t), \quad \gamma(1, t) = \gamma_1(t), \quad t \in I \\ \gamma(s, 0) = a, \qquad \gamma(s, 1) = b, \qquad s \in I. \end{aligned} \tag{1.23}$$

The function $\gamma(s_0, t) : I \to D$ defines a path inside in the domain D for each fixed $s_0 \in I$. These paths vary continuously as s_0 varies and their family "connects" the paths γ_0 and γ_1 in D. Therefore the homotopy of two paths in D means that one path may be deformed continuously into the other inside D.

Similarly two closed paths $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ are homotopic in a domain D if there exists a continuous map $\gamma(s, t) : I \times I \to D$ so that

$$\begin{aligned} \gamma(0, t) = \gamma_0(t), \quad \gamma(1, t) = \gamma_1(t), \quad t \in I \\ \gamma(s, 0) = \gamma(s, 1), \qquad s \in I. \end{aligned} \tag{1.24}$$

Homotopy is usually denoted by the symbol $\sim$, we will write $\gamma_0 \sim \gamma_1$ if γ_0 is homotopic to γ_1.

It is quite clear that homotopy defines an equivalence relation. Therefore all paths with common end-points and all closed paths may be separated into equivalence classes. Each class contains all paths that are homotopic to each other.

A special homotopy class is that of paths homotopic to zero. We say that a closed path γ is homotopic to zero in a domain D if there exists a continuous mapping $\gamma(s,t) : \; I \times I \to D$ that satisfies conditions (1.24) and such that $\gamma_1(t) = \text{const.}$ That means that γ may be contracted to a point by a continuous transformation.

Any closed path is homotopic to zero in a simply connected domain, and thus any two paths with common ends are homotopic to each other. Therefore the homotopy classes in a simply connected domains are trivial.

Exercise 1.19 Show that the following two statements are equivalent: (i) any closed path in D is homotopic to zero, and (ii) any two paths in D that have common ends are homotopic to each other.

The notion of homotopy may be easily extended from paths to curves since homotopy is preserved under re-parameterizations of paths. Two curves (either with common ends or closed) are homotopic in D if the paths γ_1 and γ_2 that represent those curves are homotopic to each other.

We have introduced the notion of the integral first for a path and then verified that the value of the integral is determined not by a path but by a curve, that is, by an equivalence class of paths. The general Cauchy theorem claims that integral of a holomorphic function is determined not even by a curve but by the homotopy class of the curve. In other words, the following theorem holds.

Theorem 1.20 *(Cauchy) Let $f \in \mathcal{O}(D)$ and γ_0 and γ_1 be two paths homotopic to each other in D either as paths with common ends or as closed paths, then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz. \tag{1.25}$$

Proof. Let $\gamma : \; I \times I \to D$ be a function that defines the homotopy of the paths γ_0 and γ_1. We construct a system of squares K_{mn}, $m, n = 1, \ldots, N$ that covers the square $K = I \times I$ so that each K_{mn} overlaps each neighboring square. Uniform continuity of the function γ implies that the squares K_{mn} may be chosen so small that each K_{mn} is contained in a disk $U_{mn} \subset D$. The function f has an anti-derivative F_{mn} in each of those disks (we use the fact that a holomorphic function has an anti-derivative in any disk). We fix the subscript m and proceed as in the proof of theorem 1.14. We choose arbitrarily the anti-derivative F_{m1} defined in U_{m1} and pick the anti-derivative F_{m2} defined in U_{m2} so that $F_{m1} = F_{m2}$ in the intersection $U_{m1} \cap U_{m2}$. Similarly we may choose $F_{m3}, \ldots, F_{mN}$ so that $F_{m,n+1} = F_{mn}$ in the intersection $U_{m,n+1} \cap U_{mn}$ and define the function

$$\Phi_m(s,t) = F_{mn} \circ \gamma(s,t) \text{ for } (s,t) \in K_{mn},\; n = 1, \ldots, N. \tag{1.26}$$

The function Φ_{mn} is clearly continuous in the rectangle $K_m = \cup_{n=1}^{N} K_{mn}$ and is defined up to an arbitrary constant. We choose arbitrarily Φ_1 and pick Φ_2 so that $\Phi_1 = \Phi_2$ in the intersection $K_1 \cap K_2$ [4]. The functions $\Phi_3, \ldots, \Phi_N$ are chosen in exactly the same fashion so that $\Phi_{m+1} = \Phi_m$ in $K_{m+1} \cap K_m$. This allows us to define the function

$$\Phi(s,t) = \Phi_m(s,t) \text{ for } (s,t) \in K_m,\ m = 1, \ldots, N. \tag{1.27}$$

the function $\Phi(s,t)$ is clearly an anti-derivative along the path $\gamma_s(t) = \gamma(s,t) : I \to D$ for each fixed s. Therefore the Newton-Leibnitz formula implies that

$$\int_{\gamma_s} f dz = \Phi(s,1) - \Phi(s,0). \tag{1.28}$$

We consider now two cases separately.

(a) *The paths γ_0 and γ_1 have common ends.* Then according to the definition of homotopy we have $\gamma(s,0) = a$ and $\gamma(s,1) = b$ for all $s \in I$. Therefore the functions $\Phi(s,0)$ and $\Phi(s,1)$ are locally constant as functions of $s \in I$ at all s and hence they are constant on I. Therefore $\Phi(0,0) = \Phi(1,0)$ and $\Phi(1,0) = \Phi(1,1)$ so that (1.28) implies 1.25. $\Box$

(b) The paths γ_0 and γ_1 are closed. In this case we have $\gamma(s,0) = \gamma(s,1)$ so that the function $\Phi(s,0) - \Phi(s,1)$ is locally constant on I, and hence this function is a constant on I. Therefore once again (1.28) implies (1.25).

Exercise 1.21 Show that if f is holomorphic in an annulus $V = \{r < |z-a| < R\}$ then the integral $\displaystyle\int_{|z-a|=\rho} f dz$ has the same value for all ρ, $r < \rho < R$.

1.4 Some special cases

We consider in this section some special cases of the Cauchy theorem that are especially important and deserve to be stated separately.

Theorem 1.22 *Let $f \in \mathcal{O}(D)$ then its integral along any path that is contained in D and is homotopic to zero vanishes:*

$$\int_\gamma f dz = 0 \text{ if } \gamma \sim 0. \tag{1.29}$$

Proof. Since $\gamma \sim 0$ this path may be continuously deformed into a point $a \in D$ and thus into a circle $\gamma_\varepsilon = \{|z-a| = \varepsilon\}$ of an arbitrarily small radius $\varepsilon > 0$. The general Cauchy theorem implies that

$$\int_\gamma f dz = \int_{\gamma_\varepsilon} f dz.$$

The integral on the right side vanishes in the limit $\varepsilon \to 0$ since the function f is bounded in a neighborhood of the point a. However, the left side is independent of ε and thus it must be equal to zero. $\Box$

[4]This is possible since the function $\Phi_1 - \Phi_2$ is locally constant on a connected set $K_1 \cap K_2$ and is therefore constant on this set

Any closed path is homotopic to zero in a simply connected domain and thus the Cauchy theorem has a particularly simple form for such domains - this is its classical statement:

Theorem 1.23 *If a function f is holomorphic in a simply connected domain $D \subset \mathbb{C}$ then its integral over any closed path $\gamma: I \to D$ vanishes.*

Due to the importance of this theorem we also present its elementary proof under two additional assumptions: (1) the derivative f' is continuous[5], and (2) γ is a piecewise smooth Jordan path.

The second assumption implies that γ is the boundary of a domain G contained in D since the latter is simply connected. The first assumption allows to apply the Green's formula

$$\int\limits_{\partial G} Pdx + Qdy = \iint\limits_{G} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy. \tag{1.30}$$

Its derivation assumes the continuity of the partial derivatives of P and Q in $\bar{G}$ (here ∂G is the boundary of G traced counter-clockwise). Applying this formula to the real and imaginary parts of the integral

$$\int\limits_{\partial G} f dz = \int\limits_{\partial G} udx - vdy + i \int\limits_{\partial G} vdx + udy,$$

we obtain

$$\int\limits_{\partial G} f dz = \iint\limits_{G} \left\{ -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + i \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right\} dxdy.$$

The last equation may be re-written as

$$\int\limits_{\partial G} f dz = 2i \iint\limits_{G} \frac{\partial f}{\partial \bar{z}} dxdy, \tag{1.31}$$

which may be considered as the complex form of the Green's formula.

It is easy to deduce from the Cauchy theorem the global theorem of existence of an anti-derivative in a *simply connected* domain.

Theorem 1.24 *Any function f holomorphic in a simply connected domain D has an anti-derivative in this domain.*

Proof. We first show that the integral of f along a path in D is independent of the choice of the path and is completely determined by the end-points of the path. Indeed, let γ_1 and γ_2 be two paths that connect in D two points a and b. Without any loss of generality we may assume that the path γ_1 is parameterized on an interval $[\alpha, \beta_1]$ and γ_2 is parameterized on an interval $[\beta_1, \beta]$, $\alpha < \beta_1 < \beta$. Let us denote by γ the union of the paths γ_1 and γ_2^-, this is a closed path contained in γ, and, moreover,

$$\int_\gamma f dz = \int_{\gamma_1} f dz - \int_{\gamma_2} f dz.$$

[5] We will soon see that this assumption holds automatically.

However, Theorem 1.23 integral of f over any closed path vanishes and this implies our claim[6].

We fix now a point $a \in D$ and let z be a point in D. Integral of f over any path $\gamma = \widetilde{az}$ that connects a and z depends only on z and not on the choice of γ:

$$F(z) = \int\limits_{\widetilde{az}} f(\zeta) d\zeta. \tag{1.32}$$

Repeating verbatim the arguments in the proof of theorem 1.11 we verify that $F(z)$ is holomorphic in D and $F'(z) = f(z)$ for all $z \in D$ so that F is an anti-derivative of f in D. □

The example of the function $f = 1/z$ in an annulus $\{0 < |z| < 2\}$ (see Remark 1.17) shows that the assumption that D is simply connected is essential: the global existence theorem of anti-derivative does not hold in general for multiply connected domains.

The same example shows that the integral of a holomorphic function over a closed path in a multiply connected domain might not vanish, so that the Cauchy theorem in its classical form (Theorem 1.23) may not be extended to non-simply connected domains. However, one may present a reformulation of this theorem that allows such a generalization.

The boundary ∂D of a nice simply connected domain D is a closed curve that is homotopic to zero in the closer $\bar{D}$. One may not apply Theorem 1.22 to ∂D because f is defined only in D and it may be impossible to extend it to ∂D. If we require that $f \in \mathcal{O}(\bar{D})$, that is, that f may be extended into a domain G that contains D, then Theorem 1.29 may be applied. We obtain the following re-statement of the Cauchy theorem.

Theorem 1.25 *Let f be holomorphic in the closure $\bar{D}$ of a simply connected domain D that is bounded by a continuous curve, then the integral of f over the boundary of this domain vanishes.*

Exercise 1.26 Sometimes the assumptions of Theorem 1.25 may be weakened requiring only that f may be extended continuously to $\bar{D}$. For instance, let D be a star-shaped domain with respect to $z = 0$, that is, its boundary ∂D may be represented in polar coordinates as $r = r(\phi)$, $0 \leq \phi \leq 2\pi$ with $r(\phi)$ a single-valued function. Assume in addition that $r(\phi)$ is a piecewise smooth function. Show that the statement of theorem 1.25 holds for functions f that are holomorphic in D and continuous in $\bar{D}$.

Theorem 1.25 may be extended to multiply connected domains with the help of the following definition.

Definition 1.27 *Let the boundary of a compact domain D[7] consist of a finite number of closed curves γ_ν, $\nu = 0, \ldots, n$. We assume that the outer boundary γ_0, that is, the curve*

[6] One may also obtain this result directly from the general Cauchy theorem using the fact that any two paths with common ends are homotopic to each other in a simply connected domain.

[7] Recall that a domain D is compact if its closure does not contain the point at infinity.

that separates D from infinity, is oriented counterclockwise while the other boundary curves γ_ν, $\nu = 1, \dots, n$ are oriented clockwise. In other words, all the boundary curves are oriented in such a way that D remains on the left side as they are traced. The boundary of D with this orientation is called the oriented boundary *and denote by ∂D.*

We may now state the Cauchy theorem for multiply connected domains as follows.

Theorem 1.28 *Let a compact domain D be bounded by a finite number of continuous curves and let f be holomorphic in its closure $\bar{D}$. Then the integral of f over its oriented boundary ∂D is equal to zero:*

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \sum_{\nu=1}^{n} \int_{\gamma_\nu} f dz = 0. \tag{1.33}$$

Proof. Let us introduce a finite number of cuts $\lambda_\nu^\pm$ that connect the components of the boundary of this domain. It is clear that the closed curve Γ that consists of the oriented boundary ∂D and the unions $\Lambda^+ = \cup\lambda_\nu^+$ and $\Lambda^- = \cup\lambda_\nu^-$ is homotopic to zero in the domain G that contains $\bar{D}$, and such that f is holomorphic in D. Theorem 1.22 implies that the integral of f along Γ vanishes so that

$$\int_{\Gamma} f dz = \int_{\partial D} f dz + \int_{\Lambda^+} f dz + \int_{\Lambda^-} f dz = \int_{\partial D} f dz$$

since the integrals of f along Λ^+ and Λ^- cancel each other.$\square$

Example 1.29 Let $D = \{r < |z - a| < R\}$ be an annulus and $f \in \mathcal{O}(\bar{D})$ is a function holomorphic in a slightly larger annulus that contains $\bar{D}$. The oriented boundary of D consists of the circle $\gamma_0 = \{|z - a| = R\}$ oriented counterclockwise and the circle $\gamma_1^- = \{|z - a| = R\}$ oriented clockwise. According to Theorem 1.28

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \int_{\gamma_1^-} f dz = 0,$$

or

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz.$$

The last relation also follows from the Cauchy theorem for homotopic paths.

1.5 The Cauchy Integral Formula

We will obtain here a representation of functions holomorphic in a compact domain with the help of the integral over the boundary of the domain. This representation finds numerous applications both in theoretical and practical problems.

Theorem 1.30 *Let the function f be holomorphic in the closure of a compact domain D that is bounded by a finite number of continuous curves. Then the function f at any point $z \in D$ may be represented as*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1.34}$$

where ∂D is the oriented boundary of D.

The right side of (1.34) is called the Cauchy integral.

Proof. Let $\rho > 0$ be such that the disk $U_\rho = \{z' : \ |z - z'| < \rho\}$ is properly contained in D and let $D_\rho = \bar{D} \backslash \bar{U}_\rho$. The function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ is holomorphic in $\bar{D}_\rho$ as a ratio of two holomorphic functions with the numerator different from zero. The oriented boundary of D_ρ consists of the union of ∂D and the circle $\partial U_\rho = \{\zeta : \ |\zeta - z| = \rho\}$ oriented clockwise. Therefore we have

$$\frac{1}{2\pi i} \int_{\partial D_\rho} g(\zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

However, the function g is holomorphic in $\bar{D}_\rho$ (its singular point $\zeta = z$ lies outside this set) and hence the Cauchy theorem for multiply connected domains may be applied. We conclude that the integral of g over ∂D_ρ vanishes.

Therefore,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1.35}$$

where ρ may be taken arbitrarily small. Since the function f is continuous at the point z, for any $\varepsilon > 0$ we may choose $\delta > 0$ so that

$$|f(\zeta) - f(z)| < \varepsilon \text{ for all } \zeta \in \partial U_\rho$$

for all $\rho < \delta$. Therefore the difference

$$f(z) - \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(z) - f(\zeta)}{\zeta - z} d\zeta \tag{1.36}$$

does not exceed $\dfrac{1}{2\pi} \varepsilon \cdot 2\pi = \varepsilon$ and thus goes to zero as $\rho \to 0$. However, (1.35) shows that the left side in (1.36) is independent of ρ and hence is equal to zero for all sufficiently small ρ, so that

$$f(z) = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

This together with (1.35) implies (1.34). $\square$

Remark 1.31 If the point z lies outside $\bar{D}$ and conditions of Theorem 1.30 hold then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = 0. \tag{1.37}$$

This follows immediately from the Cauchy theorem since now the function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ is holomorphic in $\bar{D}$.

The integral Cauchy theorem expresses a very interesting fact: the values of a function f holomorphic in a domain $\bar{G}$ are completely determined by its values on the boundary ∂G. Indeed, if the values of f on ∂G are given then the right side of (1.34) is known and thus the value of f at any point $z \in D$ is also determined. This property is the main difference between holomorphic functions and differentiable functions in the real analysis sense.

Exercise 1.32 Let the function f be holomorphic in the closure of a domain D that contains the point at infinity and the boundary ∂D is oriented so that D remains on the left as the boundary is traced. Show that then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta + f(\infty).$$

An easy corollary of Theorem 1.30 is

Theorem 1.33 *The value of the function $f \in \mathcal{O}(D)$ at each point $z \in D$ is equal to the average of its values on any sufficiently small circle centered at z:*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + \rho e^{it})dt. \tag{1.38}$$

Proof. Consider the disk $U_\rho = \{z' : \ |z - z'| < \rho\}$ so that U_ρ is properly contained in D. The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i}\int_{\partial U_\rho}\frac{f(\zeta)}{\zeta - z}d\zeta. \tag{1.39}$$

Introducing the parameterization $\zeta = z + \rho e^{it}$, $t \in [0, 2\pi]$ of U_ρ and replacing $d\zeta = \rho i e^{it}dt$ we obtain (1.38) from (1.39). $\square$

The mean value theorem shows that holomorphic functions are built in a very regular fashion, so to speak, and their values are intricately related to the values at other points. This explains why these functions have specific properties that the real differentiable functions lack. We will consider many other such properties later.

Before we conclude we present an integral representation of $\mathbb{R}$-differentiable functions that generalizes the Cauchy integral formula.

Theorem 1.34 *Let $f \in C^1(\bar{D})$ be a continuously differentiable function in the real sense in the closure of a compact domain D bounded by a finite number of piecewise smooth curves. Then we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint_D \frac{\partial f}{\partial\bar{\zeta}}\frac{d\xi d\eta}{\zeta - z} \tag{1.40}$$

for all $z \in D$ (here $\zeta = \xi + i\eta$ inside the integral).

Proof. Let us delete a small disk $\bar{U}_\rho = \{\zeta : |\zeta - z| \le \rho\}$ out of D and apply the Green's formula in its complex form (1.31) to the function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ that is continuously differentiable in the domain $D_\rho = D\backslash\bar{U}_\rho$

$$\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \int_{\partial U_\rho}\frac{f(\zeta)}{\zeta - z}d\zeta = 2i\iint_{D_\rho}\frac{\partial f}{\partial\bar{\zeta}}\frac{d\xi d\eta}{\zeta - z}{}^{8}. \tag{1.41}$$

The function f is continuous at z so that $f(\zeta) = f(z) + O(\rho)$ for $\zeta \in U_\rho$, where $O(\rho) \to 0$ as $\rho \to 0$, and thus

$$\int_{\partial U_\rho}\frac{f(\zeta)}{\zeta - z}d\zeta = f(z)\int_{\partial U_\rho}\frac{1}{\zeta - z}d\zeta + \int_{\partial U_\rho}\frac{O(\rho)}{\zeta - z}d\zeta = 2\pi i f(z) + O(\rho).$$

Passing to the limit in (1.41) and using the fact that the double integrals in (1.40) and (1.41) are convergent[9] we obtain (1.40). □

Having described the basic facts of the theory of complex integration let us describe briefly its history. The main role in its development was played by the outstanding French mathematician A. Cauchy.

Augustin-Louis Cauchy was born in 1789 into an aristocratic family. He graduated from Ecole Polytechnique in Paris in 1807. This school was created in the time of the French revolution in order to prepare highly qualified engineers. Its graduates received fundamental training in mathematics, mechanics and technical drawing for two years and were afterward sent for two more years of engineering training to on one of the four specialized institutes. Cauchy was trained at Ecole des Ponts et Chaussées from which he graduated in 1810. At that time he started his work at Cherbourg on port facilities for Napoleon's English invasion fleet.

The work of Cauchy was quite diverse - he was occupied with elasticity theory, optics, celestial mechanics, geometry, algebra and number theory. But the basis of his interests was mathematical analysis, a branch of mathematics that underwent a serious transformation started by his work. Cauchy became a member of the Academy of Sciences in 1816 and a

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[8] We have $\dfrac{\partial g}{\partial\bar{\zeta}} = \dfrac{1}{\zeta - z}\dfrac{\partial f}{\partial\bar{\zeta}}$ since the function $1/(\zeta - z)$ is holomorphic in ζ so that its derivative with respect to $\bar{\zeta}$ vanishes.

[9] Our argument shows that the limit $\lim\limits_{\rho\to 0}\iint_{D_\rho}\dfrac{\partial f}{\partial\bar{\zeta}}\dfrac{d\xi d\eta}{\zeta - z}$ exists. Moreover, since $f \in C^1(D)$ the double integral in (1.40) exists as can be easily seen by passing to the polar coordinates and thus this limit coincides with it.

professor at College de France and Ecole Polytechnique in 1817. He presented there his famous course in analysis that were published in three volumes as Cours d'analyse (1821-1828).

Baron Cauchy was a devoted royalist. He followed the royal family and emigrated to Italy after the July revolution of 1830. His failure to return to Paris caused him to lose all his positions there. He returned to Paris in 1838 and regained his position at the Academy but not his teaching positions because he had refused to take an oath of allegiance. He taught at a Jesuit college and became a professor at Sorbonne when Louis Philippe was overthrown in 1848.

The first results on complex integration by Cauchy were presented in his memoir on the theory of definite integrals presented to the Academy in 1814 and published only in 1825. Similarly to Euler Cauchy came to these problems from hydrodynamics. He starts with the relation

$$\int_{x_0}^{X}\int_{y_0}^{Y} f(x,y)dxdy = \int_{y_0}^{Y} dy \int_{x_0}^{X} f(x,y)dx \tag{1.42}$$

and considers two real valued functions S and V put together in one complex function $F = S + iV$. Inserting $f = \dfrac{\partial V}{\partial y} = \dfrac{\partial S}{\partial x}$ into (1.42) Cauchy obtains the formula that relates the integrals of these functions:

$$\int_{x_0}^{X} [V(x,Y) - V(x,y_0)]dx = \int_{y_0}^{Y} [S(X,y) - S(x_0,y)]dy.$$

He obtained a similar formula using $f = \dfrac{\partial V}{\partial x} = -\dfrac{\partial S}{\partial y}$ but only in 1822 he arrived at the idea of putting together in the complex form that he put as a footnote in his memoir of 1825. This is the Cauchy theorem for a rectangular contour though the geometric meaning of that identity is missing here.

We note that his work differs little from the work of Euler presented in 1777 at the Saint Petersburg Academy of Sciences that contains the formula

$$\int (u+iv)(dx+idy) = \int udx - vdy + i\int vdx + udy$$

and describes some of its applications. However, in the same year 1825 Cauchy published separately his memoir on definite integrals with imaginary limits, where he considered the complex integral as the limit of partial sums and observed that to make its meaning precise one should define the continuous monotone functions $x = \phi(t)$, $y = \chi(t)$ on an interval $t_0 \le t \le T$ such that $\phi(t_0) = x_0$, $\chi(t_0) = y_0$, $\phi(T) = X$, $\chi(T) = Y$. It seems that Cauchy was not yet aware of the geometric interpretation of the integral as a path in the complex plane as well as of the geometric interpretation of complex numbers in general at that time.

He has formulated his main theorem as follows: "if $F(x + y\sqrt{-1})$ is finite and continuous for $x_0 \le x \le X$ and $y_0 \le y \le Y$ then the value of the integral does not depend on the nature of the functions $\phi(t)$ and $\chi(t)$." He proves it varying the functions ϕ and χ and verifying that the variation of the integral is equal to zero. We should note that the clear notion of the integral of a function of a complex variable as integral along a path in the complex plane and the formulation of the independence of the integral from the path appeared first in the letter by Gauss to Bessel in 1831.

The Cauchy integral formula was first proved by him in 1831 in a memoir on celestial mechanics. Cauchy proved it for a disk which is quite sufficient for developing functions in power series (see the next section). We will describe other results by Cauchy as they are presented in the course.

2 The Taylor series

We will obtain the representation of holomorphic functions as sums of power series (the Taylor series) in this section.

Let us recall the simplest results regarding series familiar from the real analysis. A series (of complex numbers) $\sum_{n=0}^{\infty} a_n$ is convergent if the sequence of its partial sums $s_k = \sum_{n=0}^{k} a_n$ has a finite limit s. This limit is called the sum of the series.

A functional series $\sum_{n=0}^{\infty} f_n(z)$ with the functions f_n defined on a set $M \subset \bar{\mathbb{C}}$ converges uniformly on M if it converges at all $z \in M$, and, moreover, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geq N$ the remainder of the series satisfies $|\sum\limits_{k=n+1}^{\infty} f_k(z)| < \varepsilon$ for all $z \in M$.

The series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on M if the series $\sum_{n=0}^{\infty} \|f_n\|$ converges. Here $\|f_n\| = \sup\limits_{z \in M} |f_n(z)|$, and the proof is identical to that in the real analysis. This condition implies that the functional series is majorized by a convergent series of numbers. We also recall that the sum of a uniformly convergent series of continuous functions $f_n(z)$ on M is also continuous on M, and that one may integrate term-wise a uniformly convergent series along a smooth curve. The proofs are once again identical to those in the real analysis.

2.1 The Taylor series

One of the main theorems of the theory of functions of a complex variable is

Theorem 2.1 *Let $f \in \mathcal{O}(D)$ and let $z_0 \in D$ be an arbitrary point in D. Then the function f may be represented as a sum of a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (2.1)$$

inside any disk $U = \{|z - z_0| < R\} \subset D$.

Proof. Let $z \in U$ be an arbitrary point. Choose $r > 0$ so that $|z - z_0| < r < R$ and denote by $\gamma_r = \{\zeta : \ |\zeta - z_0| = r\}$ The integral Cauchy formula implies that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

In order to represent f as a power series let us represent the kernel of this integral as the sum of a geometric series:

$$\frac{1}{\zeta - z} = \left[(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)\right]^{-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{2.2}$$

We multiply both sides by $\frac{1}{2\pi i} f(\zeta)$ and integrate the series term-wise along γ_r. The series (2.2) converges uniformly on γ_r since

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{r} = q < 1$$

for all $\zeta \in \gamma_r$. Uniform convergence is preserved under multiplication by a continuous and hence bounded function $\frac{1}{2\pi i} f(\zeta)$. Therefore our term-wise integration is legitimate and we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{n=0}^{\infty} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where[10]

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 0, 1, \ldots . \square \tag{2.3}$$

Definition 2.2 *The power series (2.1) with coefficients given by (2.3) is* the Taylor series *of the function f at the point z_0 (or centered at z_0).*

The Cauchy theorem 1.20 implies that the coefficients c_n of the Taylor series defined by (2.3) do not depend on the radius r of the circle γ_r, $0 < r < R$.

Exercise 2.3 Find the radius of the largest disk where the function $z/\sin z$ may be represented by a Taylor series centered at $z_0 = 0$.

Exercise 2.4 Let f be holomorphic in $\mathbb{C}$. Show that (a) f is even if and only if its Taylor series at $z = 0$ contains only even powers; (b) f is real on the real axis if and only if $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

We present some simple corollaries of Theorem 2.1.

The Cauchy inequalities. *Let the function f be holomorphic in a closed disk $\bar{U} = \{|z - z_0| \le r\}$ and let its absolute value on the circle $\gamma_r = \partial U$ be bounded by a constant M. Then the coefficients of the Taylor series of f at z_0 satisfy the inequalities*

$$|c_n| \le M/r^n, \quad (n = 0, 1, \ldots). \tag{2.4}$$

[10]This theorem was presented by Cauchy in 1831 in Turin. Its proof was first published in Italy, and it appeared in France in 1841. However, Cauchy did not justify the term-wise integration of the series. This caused a remark by Chebyshev in his paper from 1844 that such integration is possible only in some "particular cases".

Proof. We deduce from (2.3) using the fact that $|f(\zeta)| \le M$ for all $\zeta \in \gamma_r$:

$$|c_n| \le \frac{1}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{M}{r^n}. \square$$

Exercise 2.5 Let $P(z)$ be a polynomial in z of degree n. Show that if $|P(z)| \le M$ for $|z| = 1$ then $|P(z)| \le M|z|^n$ for all $|z| \ge 1$.

The Cauchy inequalities imply the interesting

Theorem 2.6 *(Liouville[11]) If the function f is holomorphic in the whole complex plane and bounded then it is equal identically to a constant.*

Proof. According to Theorem 2.1 the function f may be represented by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in any closed disk $\bar{U} = \{|z| \le R\}$, $R < \infty$ with the coefficients that do not depend on R. Since f is bounded in $\mathbb{C}$, say, $|f(z)| \le M$ then the Cauchy inequalities imply that for any $n = 0, 1, \ldots$ we have $|c_n| \le M/R^n$. We may take R to be arbitrarily large and hence the right side tends to zero as $R \to +\infty$ while the left side is independent of R. Therefore $c_n = 0$ for $n \ge 1$ and hence $f(z) = c_0$ for all $z \in \mathbb{C}$. $\square$

Therefore the two properties of a function – to be holomorphic and bounded are realized simultaneously only for the trivial functions that are equal identically to a constant.

Exercise 2.7 Prove the following properties of functions f holomorphic in the whole plane $\mathbb{C}$:

(1) Let $M(r) = \sup\limits_{|z|=r} |f(z)|$, then if $M(r) = Ar^N + B$ where r is an arbitrary positive real number and A, B and N are constants, then f is a polynomial of degree not higher than N.

(2) If all values of f belong to the right half-plane then $f = \text{const}$.

(3) If $\lim_{z\to\infty} f(z) = \infty$ then the set $\{z \in \mathbb{C} : \; f(z) = 0\}$ is not empty.

The Liouville theorem may be reformulated:

Theorem 2.8 *If a function f is holomorphic in the closed complex plane $\overline{\mathbb{C}}$ then it is equal identically to a constant.*

Proof. if the function f is holomorphic at infinity the limit $\lim_{z\to\infty} f(z)$ exists and is finite. Therefore f is bounded in a neighborhood $U = \{|z| > R\}$ of this point. However, f is also bounded in the complement $\bar{U}^c = \{|z| \le R\}$ since it is continuous there and the set $\bar{U}^c$ is compact. Therefore f is holomorphic and bounded in $\mathbb{C}$ and thus Theorem 2.6 implies that is equal to a constant. $\square$

[11] Actually this theorem was proved by Cauchy in 1844 while Liouville has proved only a partial result in the same year. The wrong attribution was started by a student of Liouville who has learned the theorem at one of his lectures.

Exercise 2.9 Show that a function $f(z)$ that is holomorphic at $z = 0$ and satisfies $f(z) = f(2z)$, is equal identically to a constant.

Theorem 2.1 claims that any function holomorphic in a disk may be represented as a sum of a convergent power series inside this disk. We would like to show now that, conversely, the sum of a convergent power series is a holomorphic function. Let us first recall some properties of power series that are familiar from the real analysis.

Lemma 2.10 *If the terms of a power series*

$$\sum_{n=0}^{\infty} c_n(z-a)^n \tag{2.5}$$

are bounded at some point $z_0 \in \mathbb{C}$, that is,

$$|c_n(z_0-a)^n| \le M, \quad (n = 0, 1, \dots), \tag{2.6}$$

then the series converges in the disk $U = \{z : \ |z-a| < |z_0-a|\}$. Moreover, it converges absolutely and uniformly on any set K that is properly contained in U.

Proof. We may assume that $z_0 \neq a$, so that $|z_0 - a| = \rho > 0$, otherwise the set U is empty. Let K be properly contained in U, then there exists $q < 1$ so that $|z-a|/\rho \le q < 1$ for all $z \in K$. Therefore for any $z \in K$ and any $n \in \mathbb{N}$ we have $|c_n(z-a)^n| \le |c_n|\rho^n q^n$. However, assumption (2.6) implies that $|c_n|\rho^n \le M$ so that the series (2.5) is majorized by a convergent series $M\sum_{n=0}^{\infty} q^n$ for all $z \in K$. Therefore the series (2.5) converges uniformly and absolutely on K. This proves the second statement of this lemma. The first one follows from the second since any point $z \in U$ belongs to a disk $\{|z-a| < \rho'$, with $\rho' < \rho$, that is properly contained in U. $\Box$

Theorem 2.11 *(Abel[12]) Let the power series (2.5) converge at a point $z_0 \in \mathbb{C}$. Then this series converges in the disk $U = \{z : \ |z-a| < |z_0-a|\}$ and, moreover, it converges uniformly and absolutely on every compact subset of U.*

Proof. Since the series (2.5) converges at the point z_0 the terms $c_n(z_0-a)^n$ converge to zero as $n \to \infty$. However, every converging sequence is bounded, and hence the assumptions of the previous lemma are satisfied and both claims of the present theorem follow from this lemma. $\Box$

The Cauchy-Hadamard formula. *Let the coefficients of the power series (2.5) satisfy*

$$\limsup_{n\to\infty} |c_n|^{1/n} = \frac{1}{R}, \tag{2.7}$$

with $0 \le R \le \infty$ (we set $1/0 = \infty$ and $1/\infty = 0$). Then the series (2.5) converges at all z such that $|z-a| < R$ and diverges at all z such that $|z-a| > R$.

Proof. Recall that $A = \limsup_{n\to\infty} \alpha_n$ if (1) there exists a subsequence $\alpha_{n_k} \to A$ as $k \to \infty$, and (2) for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $\alpha_n < A + \varepsilon$ for all $n \ge N$. This

[12] This theorem was published in 1826 by a Norwegian mathematician Niels Abel (1802-1829).

includes the cases $A = \pm\infty$. However, if $A = +\infty$ then condition (2) is not necessary, and if $A = -\infty$ then the number $A + \varepsilon$ in condition (2) is replaced by an arbitrary number (in the latter case condition (1) holds automatically and $\lim_{n\to\infty} \alpha_n = -\infty$). It is shown in real analysis that $\limsup \alpha_n$ exists for any sequence $\alpha_n \in \mathbb{R}$ (either finite or infinite).

Let $0 < R < \infty$, then for any $\varepsilon > 0$ we may find N such that for all $n \geq N$ we have $|c_n|^{1/n} \leq \dfrac{1}{R} + \varepsilon$. Therefore, we have

$$|c_n(z-a)^n| < \left\{ \left(\frac{1}{R} + \varepsilon \right) |z-a| \right\}^n . \tag{2.8}$$

Furthermore, given $z \in \mathbb{C}$ such that $|z-a| < R$ we may choose ε so small that we have $\left(\dfrac{1}{R} + \varepsilon \right) |z-a| = q < 1$. Then (2.8) shows that the terms of the series (2.5) are majorized by a convergent geometric series q^n for $n \geq N$, and hence the series (2.5) converges when $|z-a| < R$.

Condition (1) in the definition of lim sup implies that for any $\varepsilon > 0$ one may find a subsequence c_{n_k} so that $|c_{n_k}|^{1/n_k} > \dfrac{1}{R} - \varepsilon$ and hence

$$|c_{n_k}(z-a)^{n_k}| > \left\{ \left(\frac{1}{R} - \varepsilon \right) |z-a| \right\}^{n_k} . \tag{2.9}$$

Then, given $z \in \mathbb{C}$ such that $|z-a| > R$ we may choose ε so small that we have $\left(\dfrac{1}{R} - \varepsilon \right) |z-a| > 1$. then (2.9) implies that $|c_{n_k}(z-a)^{n_k}| > 1$ for all k and hence the n-th term of the power series (2.5) does not vanish as $n \to \infty$ so that the series diverges if $|z-a| > R$.

We leave the proof in the special case $R = 0$ and $R = \infty$ as an exercise for the reader. □

Definition 2.12 *The domain of convergence of a power series (2.5) is the interior of the set E of the points $z \in \mathbb{C}$ where the series converges.*

Theorem 2.13 *The domain of convergence of the power series (2.5) is the open disk $\{|z-a| < R\}$, where R is the number determined by the Cauchy-Hadamard formula.*

Proof. The previous proposition shows that the set E where the series (2.5) converges consists of the disk $U = \{|z-a| < R\}$ and possibly some other set of points on the boundary $\{|z-a| = R\}$ of U. Therefore the interior of E is the open disk $\{|z-a| < R\}$. □

The open disk in Theorem 2.13 is called the disk of convergence of the power series (2.5), and the number R is its radius of convergence.

Example 2.14 1. The series

$$(a)\ \sum_{n=1}^{\infty}(z/n)^n, \quad (b)\sum_{n=1}^{\infty} z^n, \quad (c)\sum_{n=1}^{\infty}(nz)^n \tag{2.10}$$

have the radii of convergence $R = \infty$, $R = 1$ and $R = 0$, respectively. Therefore the domain of convergence of the first is $\mathbb{C}$, of the second – the unit disk $\{|z| < 1\}$ and of the third – an empty set.

2. The same formula shows that the domain of convergence of all three series

$$(a)\ \sum_{n=1}^{\infty} z^n, \quad (b)\sum_{n=1}^{\infty} z^n/n, \quad (c)\sum_{n=1}^{\infty} z^n/n^2 \tag{2.11}$$

is the unit disk $\{|z| < 1\}$. However, the sets where the three series converge are different. The series (a) diverges at all points on the circle $\{|z| = 1\}$ since its n-th term does not vanish as $n \to +\infty$. The series (b) converges at some points of the circle $\{|z| = 1\}$ (for example, at $z = -1$) and diverges at others (for example, at $z = 1$). The series (c) converges at all points on this circle since it is majorized by the converging series $sum_{n=1}^{\infty} 1/n^2$ at all z such that $|z| = 1$.

We pass now to the proof that the sum of a power series is holomorphic.

Theorem 2.15 *The sum of a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \tag{2.12}$$

is holomorphic in its domain of convergence.

Proof. We assume that the radius of convergence $R > 0$, otherwise there is nothing to prove. Let us define the formal series of derivatives

$$\sum_{n=1}^{\infty} nc_n(z-a)^{n-1} = \phi(z). \tag{2.13}$$

Its convergence is equivalent to that of the series $\sum_{n=1}^{\infty} nc_n(z-a)^n$. However, since $\limsup_{n\to\infty} |nc_n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n}$ the radius of convergence of the series (2.13) is also equal to R. Therefore this series converges uniformly on compact subsets of the disk $U = \{|z-a| < R\}$ and hence the function $\phi(z)$ is continuous in this disk.

Moreover, for the same reason the series (2.13) may be integrated term-wise along the boundary of any triangle Δ that is properly contained in U:

$$\int_{\partial\Delta} \phi dz = \sum_{n=1}^{\infty} nc_n \int_{\partial\Delta} (z-a)^{n-1} dz = 0.$$

The integrals on the right side vanish by the Cauchy theorem. Therefore we may apply Theorem 1.11 and Remark 1.12 which imply that the function

$$\int_{[a,z]} \phi(\zeta)d\zeta = \sum_{n=1}^{\infty} nc_n \int_{[a,z]} (\zeta - a)^{n-1} d\zeta = \sum_{n=1}^{\infty} c_n (z-a)^n$$

has a derivative at all $z \in U$ that is equal to $\phi(z)$. Once again we used uniform convergence to justify the term-wise integration above. However, then the function

$$f(z) = c_0 + \int_{[a,z]} \phi(\zeta)d\zeta$$

has a derivative at all $z \in U$ that is also equal to $\phi(z)$. $\square$

2.2 Properties of holomorphic functions

We discuss some corollaries of Theorem 2.15.

Theorem 2.16 *Derivative of a function $f \in \mathcal{O}(D)$ is holomorphic in the domain D.*

Proof. Given a point $z_0 \in D$ we construct a disk $U = \{|z - z_0| < R\}$ that is contained in D. Theorem 2.1 implies that f may be represented as a sum of a converging power series in this disk. Theorem 2.15 implies that its derivative $f' = \phi$ may also be represented as a sum of a power series converging in the same disk. Therefore one may apply Theorem 2.15 also to the function ϕ and hence ϕ is holomorphic in the disk U. $\square$

This theorem also implies directly the necessary condition for the existence of anti-derivative that we have mentioned in Section 1.2:

Corollary 2.17 *If a continuous function f has an anti-derivative F in a domain D then f is holomorphic in D.*

Using Theorem 2.16 once again we obtain

Theorem 2.18 *Any function $f \in \mathcal{O}(D)$ has derivatives of all orders in D that are also holomorphic in D.*

The next theorem establishes uniqueness of the power series representation of a function relative to a given point.

Theorem 2.19 *Let a function f have a representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{2.14}$$

in a disk $\{|z - z_0| < R\}$. Then the coefficients c_n are determined uniquely as

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, \dots \tag{2.15}$$

Proof. Inserting $z = z_0$ in (2.14) we find $c_0 = f(z_0)$. Differentiating (2.14) termwise we obtain

$$f'(z) = c_1 + 2c_2(z - z_0) + 3c_3(z - z_0)^2 + \dots$$

Inserting $z = z_0$ above we obtain $c_1 = f'(z_0)$. Differentiating (2.14) n times we obtain (we do not write out the formulas for $\tilde{c}_j$ below)

$$f^{(n)}(z) = n!c_n + \tilde{c}_1(z - z_0) + \tilde{c}_1(z - z_0)^2 + \dots$$

and once again using $z = z_0$ we obtain $c_n = f^{(n)}(z_0)/n!$. $\square$

Sometimes Theorem 2.19 is formulated as follows: "Every converging power series is the Taylor series for its sum."

Exercise 2.20 Show that a differential equation $dw/dz = P(w, z)$ where P is a polynomial both in z and w has no more than solution $w(z)$ holomorphic near a given point $z = a$ such that $w(a) = b$ with a given $b \in \mathbb{C}$.

Expression (2.14) allows to calculate the Taylor series of elementary functions. For example, we have

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \tag{2.16}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots, \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \tag{2.17}$$

with all three expansions valid at all $z \in \mathbb{C}$ (they have infinite radius of convergence $R = \infty$).

Comparing expressions (2.15) for the coefficients c_n with their values given by (2.3) we obtain the formulas for the derivatives of holomorphic functions:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 1, 2\dots \tag{2.18}$$

If the function f is holomorphic in a domain D and G is a sub-domain of D that is bounded by finitely many continuous curves and such that $z_0 \in G$ then we may replace the contour γ_r in (2.18) by the oriented boundary ∂G, using the invariance of the integral under homotopy of paths. Then we obtain *the Cauchy integral formula for derivatives* of holomorphic functions:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial G}\frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}, \quad n = 1, 2\dots \tag{2.19}$$

These formulas may be also obtained from the Cauchy integral formula

$$f(z)\frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)d\zeta}{(\zeta - z)},$$

by differentiating with respect to the parameter z inside the integral. Our indirect argument allowed us to bypass the justification of this operation.

Theorem 2.21 *(Morera[13]) If a function f is continuous in a domain D and its integral over the boundary $\partial\Delta$ of any triangle Δ vanishes then f is holomorphic in D.*

Proof. Given $a \in D$ we construct a disk $U = \{|z - a| < r\} \subset D$. The function $F(z) = \int_{[a,z]} f(\zeta)d\zeta$ is holomorphic in U (see remark after Theorem 1.11). Theorem 2.16 implies then that f is also holomorphic in D. This proves that f is holomorphic at all $a \in D$. $\square$

Remark 2.22 The Morera Theorem states the converse to the Cauchy theorem as formulated in Theorem 1.10, that is, that integral of a holomorphic function over the boundary of any triangle vanishes. However, the Morera theorem also requires that f is continuous in D. This assumption is essential: for instance, the function f that is equal to zero everywhere in $\mathbb{C}$ except at $z = 0$, where it is equal to one, is not even continuous at $z = 0$ but its integral over any triangle vanishes.

However, the Morera theorem does not require any differentiability of f: from the modern point of view we may say that a function satisfying the assumptions of this theorem is a generalized solution of the Cauchy-Riemann equations. The theorem asserts that any generalized solution is a classical solution, that is, it has partial derivatives that satisfy the Cauchy-Riemann equations.

Exercise 2.23 Let f be continuous in a disk $U = \{|z| < 1\}$ and holomorphic everywhere in U except possibly on the diameter $[-1, 1]$. Show that f is holomorphic in all of U.

Finally, we present the list of equivalent definitions of a holomorphic function.

Theorem 2.24 *The following are equivalent:*
(R) The function f is $\mathbb{C}$-differentiable in a neighborhood U of the point a.
(C) The function f is continuous in a neighborhood U of the point a and its integral over the boundary of any triangle in $\Delta \subset U$ vanishes.
(W) the function f may be represented as the sum of a converging power series in a neighborhood U of the point a.

These three statements reflect three concepts in the development of the theory of functions of a complex variable. Usually a function f that satisfies (R) is called holomorphic in the sense of Riemann, those that satisfy (C) - holomorphic in the sense of Cauchy, and (W) - holomorphic in the sense of Weierstrass[14] The implication (R)→(C) was proved in the Cauchy theorem 1.11, (C)→(W) in the Taylor theorem 2.1, and (W)→(R) in Theorem 2.15.

Remark 2.25 We have seen that the representation as a power series in a disk $\{|z-a| < R\}$ is a necessary an sufficient condition for f to be holomorphic in this disk. However, convergence of the power series on the boundary of the disk is not related to it being

[13]The theorem was proved by an Italian mathematician Giacinto Morera in 1889.
[14]These names approximately correspond to the true order of the events.

holomorphic at those points. This may be sen on simplest examples. Indeed, the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{2.20}$$

converges in the open disk $\{|z| < 1\}$. The series (2.20) diverges at all points on $\{|z| = 1\}$ since its n-th term does not vanish in the limit $n \to \infty$. On the other hand, the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2} \tag{2.21}$$

converges at all points of $\{|z| = 1\}$ since it is majorized by the convergent number series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. However, its sum may not be holomorphic at $z = 1$ since its derivative $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$ is unbounded as z tends to one along the real axis.

2.3 The Uniqueness theorem

Definition 2.26 *A zero of the function f is a point $a \in \overline{\mathbb{C}}$ where f vanishes, that is, solution of $f(z) = 0$.*

Zeroes of differentiable functions in the real analysis may have limit points where the function f remains differentiable, for example, $f(x) = x^2 \sin(1/x)$ behaves in this manner at $x = 0$. The situation is different in the complex analysis: zeroes of a holomorphic function must be isolated, they may have limit points only on the boundary of the domain where the function is holomorphic.

Theorem 2.27 *Let the point $a \in \mathbb{C}$ be a zero of the function f that is holomorphic at this point, and f is not equal identically to zero in a neighborhood of a. Then there exists a number $n \in \mathbb{N}$ so that*

$$f(z) = (z-a)^n \phi(z), \tag{2.22}$$

where the function ϕ is holomorphic at a and is different from zero in a neighborhood of a.

Proof. Indeed, f may be represented by a power series in a neighborhood of a: $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$. The first coefficient $c_0 = 0$ but not all c_n are zero, otherwise f would vanish identically in a neighborhood of a. Therefore there exists the smallest n so that $c_n \neq 0$ and the power series has the form

$$f(z) = c_n (z-a)^n + c_{n+1}(z-a)^{n+1} + \ldots, \quad c_n \neq 0. \tag{2.23}$$

Let us denote by

$$\phi(z) = c_n + c_{n+1}(z-a) + \ldots \tag{2.24}$$

so that $f(z) = (z - a)^n \phi(z)$. The series (2.24) converges in a neighborhood of a (it has the same radius of convergence as f) and thus ϕ is holomorphic in this neighborhood. Moreover, since $\phi(a) = c_n \neq 0$ and ϕ is continuous at a, $\phi(z) \neq 0$ in a neighborhood of a. $\square$

Theorem 2.28 *(Uniqueness) Let $f_1, f_2 \in \mathcal{O}(D)$, then if $f_1 = f_2$ on a set E that has a limit point in D then $f_1(z) = f_2(z)$ for all $z \in D$.*

Proof. The function $f = f_1 - f_2$ is holomorphic in D. We should prove that $f \equiv 0$ in D, that is, that the set $F = \{z \in D : f(z) = 0\}$, that contains in particular the set E, coincides with D. The limit point a of E belongs to E (and hence to F) since f is continuous. Theorem 2.23 implies that $f \equiv 0$ in a neighborhood of a, otherwise it would be impossible for a to e a limit point of the set of zeroes of f.

Therefore the interior F^o of F is not empty - it contains a. Moreover, F^o is an open set as the interior of a set. However, it is also closed in the relative topology of D. Indeed, let $b \in D$ be a limit point of F^o, then the same Theorem 2.27 implies that $f \equiv 0$ in a neighborhood of b so that $b \in F^o$. Finally, the set D being a domain is connected, and hence $F^o = D$ by Theorem 1.29 of Chapter 1. $\square$

This theorem shows another important difference of a holomorphic function from a real differentiable function in the sense of real analysis. Indeed, even two infinitely differentiable functions may coincide on an open set without being identically equal to each other everywhere else. However, according to the previous theorem tow holomorphic functions that coincide on a set that has a limit point in the domain where they are holomorphic (for instance on a small disk, or an arc inside the domain) have to be equal identically in the whole domain.

Exercise 2.29 Show that if f is holomorphic at $z = 0$ then there exists $n \in \mathbb{N}$ so that $f(1/n) \neq (-1)^n/n^3$.

We note that one may simplify the formulation of Theorem 2.27 using the Uniqueness theorem. That is, the assumption that f is not equal identically to zero in any neighborhood of the point a may be replaced by the assumption that f is not equal identically to zero everywhere (these two assumptions coincide by the Uniqueness theorem).

Theorem 2.27 shows that holomorphic functions vanish as an integer power of $(z-a)$.

Definition 2.30 *The order, or multiplicity, of a zero $a \in \mathbb{C}$ of a function f holomorphic at this point, is the order of the first non-zero derivative $f^{(k)}(a)$. in other words, a point a is a zero of f of order n if*

$$f(a) = \cdots = f^{(k-1)}(a) = 0, \quad f^{(n)}(a) \neq 0, \quad n \geq 1. \tag{2.25}$$

Expressions $c_k = f^{(k)}(a)/k!$ for the coefficients of the Taylor series show that the order of zero is the index of the first non-zero Taylor coefficient of the function f at the point a, or, alternatively, the number n in Theorem 2.27. The Uniqueness theorem shows that holomorphic functions that are not equal identically to zero may not have zeroes of infinite order.

Similar to what is done for polynomials, one may define the order of zeroes using division.

Theorem 2.31 *The order of zero $a \in \mathbb{C}$ of a holomorphic function f coincides with the order of the highest degree $(z-a)^k$ that is a divisor of f in the sense that the ratio $\dfrac{f(z)}{(z-a)^k}$ (extended by continuity to $z = a$) is a holomorphic function at a.*

Proof. Let us denote by n the order of zero a and by N the highest degree of $(z-a)$ that is a divisor of f. Expression (2.22) shows that f is divisible by any power $k \le n$:

$$\frac{f(z)}{(z-a)^k} = (z-a)^{n-k}\phi(z),$$

and thus $N \ge n$. Let f be divisible by $(z-a)^N$ so that the ratio

$$\psi(z) = \frac{f(z)}{(z-a)^N}$$

is a holomorphic function at a. Developing ψ as a power series in $(z-a)$ we find that the Taylor expansion of f at a starts with a power not smaller than N. Therefore $n \ge N$ and since we have already shown that $n \le N$ we conclude that $n = N$. $\square$

Example 2.32 The function $f(z) = \sin z - z$ has a third order zero at $z = 0$. Indeed, we have $f(0) = f'(0) = f''(0)$ but $f'''(0) \ne 0$. This may also be seen from the representation

$$f(z) = -\frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

Remark 2.33 Let f be holomorphic at infinity and equal to zero there. It is natural to define the order of zero at this point as the order of zero the order of zero at $z = 0$ of the function $\phi(z) = f(1/z)$. The theorem we just proved remains true also for $a = \infty$ if instead of dividing by $(z-a)^k$ we consider multiplication by z^k. For example, the function $f(z) = \dfrac{1}{z^3} + \dfrac{1}{z^2}$ has order 3 at infinity.

2.4 The Weierstrass theorem

Recall that termwise differentiation of a series in real analysis requires uniform convergence of the series in a neighborhood of a point as well as uniform convergence of the series of derivatives. The situation is simplified in the complex analysis. The following theorem holds.

Theorem 2.34 *(Weierstrass) If the series*

$$f(z) = \sum_{n=0}^{\infty} f_n(z) \tag{2.26}$$

of functions holomorphic in a domain D converges uniformly on any compact subset of this domain then
(i) the sum of this series is holomorphic in D;
(ii) the series may be differentiated termwise arbitrarily many times at any point in D.

Proof. Let a be arbitrary point in D and consider the disk $U = \{|z-a| < r\}$ that is properly contained in D. The series (2.26) converges uniformly in U by assumption and thus its sum is continuous in U. Let $\Delta \subset U$ be a triangle contained in U and let $\gamma = \partial\Delta$. Since the series (2.26) converges uniformly in U we may integrate it termwise along γ:

$$\int_\gamma f(z)dz = \sum_{n=0}^{\infty} \int_\gamma f_n(z)dz.$$

However, the Cauchy theorem implies that all integrals on the right side vanish since the functions f_n are holomorphic. Hence the Morera theorem implies that the function f is holomorphic and part (i) is proved.

In order to prove part (ii) we once again take an arbitrary point $a \in D$, consider the same disk U as in the proof of part (i) and denote by $\gamma_r = \partial U = \{|z-a| = r\}$. The Cauchy formulas for derivatives imply that

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{k+1}} d\zeta. \tag{2.27}$$

The series

$$\frac{f(\zeta)}{(\zeta-a)^{k+1}} = \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{(\zeta-a)^{k+1}} \tag{2.28}$$

differs from (2.26) by a factor that has constant absolute value $\dfrac{1}{r^{k+1}}$ for all $\zeta \in \gamma_r$. Therefore it converges uniformly on γ_r and may be integrated termwise in (2.27). Using expressions (2.27) in (2.28) we obtain

$$f^{(k)}(a) = \frac{k!}{2\pi i} \sum_{n=0}^{\infty} \int_{\gamma_r} \frac{f_n(\zeta)}{(\zeta-a)^{k+1}} d\zeta = \sum_{n=0}^{\infty} f_n^{(k)}(a),$$

and part (ii) is proved. □

Exercise 2.35 Explain why the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^3 z)}{n^2}$ may not be differentiated termwise.

3 The Laurent series and singular points

The Taylor series are well suited to represent holomorphic functions in a disk. We will consider here more general power series with both positive and negative powers of $(z-a)$. Such series represent functions holomorphic in annuli

$$V = \{z \in \mathbb{C} : \; r < |z-a| < R\}, \quad r \geq 0, \; R \leq \infty.$$

Such representations are especially important when the inner radius is zero, that is, in punctured neighborhoods. They allow to study functions near the singular points where they are not holomorphic.

3.1 The Laurent series

Theorem 3.1 *(Laurent[15]) Any function f holomorphic in an annulus*

$$V = \{z \in \mathbb{C} : \ r < |z-a| < R\}$$

may be represented in this annulus as a sum of a converging power series

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n. \tag{3.1}$$

Its coefficients are determined by the formulas

$$c_n = \frac{1}{2\pi i} \int\limits_{|z-a|=\rho} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{3.2}$$

where $r < \rho < R$.

Proof. We fix an arbitrary point $z \in V$ and consider the annulus $V' = \{\zeta : \ r' < |z-a| < R'\}$ such that $z \in V' \subset V$. The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i} \int_{\partial V'} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{3.3}$$

The circles $\Gamma' = \{|z-a| = R'\}$ and $\gamma' = \{|z-a| = r'\}$ are both oriented counterclockwise. We have $\left|\frac{z-a}{\zeta - a}\right| = q < 1$ for all $\zeta \in \Gamma'$. Therefore the geometric series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a)\left(1 - \frac{z-a}{\zeta - a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}$$

converges uniformly and absolutely for $\zeta \in \Gamma'$. We multiply this series by a bounded function $\frac{1}{2\pi i} f(\zeta)$ (this does not violate uniform convergence) and integrating termwise along Γ' we obtain

$$\frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{0}^{\infty} c_n (z-a)^n \tag{3.4}$$

with

$$c_n = \frac{1}{2\pi i} \int\limits_{\Gamma'} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}, \quad n = 0, 1, 2, \dots \tag{3.5}$$

[15] This theorem was proved by Weierstrass in his Münster notebooks in 1841, but they were not published until 1894. A French engineer and mathematician Pierre Alphonse Laurent has proved this theorem in his memoir submitted in 1842 for the Grand Prize after the deadline has passed. It was not approved for the award.

The second integral in (3.3) has to be treated differently. We have $\left|\dfrac{\zeta-a}{z-a}\right| = q_1 < 1$ for all $\zeta \in \gamma'$. Therefore we obtain an absolutely and uniformly converging on γ' geometric series as

$$-\frac{1}{\zeta - z} = \frac{1}{(z-a)\left(1-\frac{\zeta-a}{z-a}\right)} = \sum_{n=1}^{\infty} \frac{(\zeta-a)^{n-1}}{(z-a)^n}.$$

Once again multiplying this series by $\dfrac{1}{2\pi i} f(\zeta)$ and integrating termwise along γ' we get

$$-\frac{1}{2\pi i}\int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{1}^{\infty} \frac{d_n}{(z-a)^n} \tag{3.6}$$

with

$$d_n = \frac{1}{2\pi i}\int\limits_{\Gamma'} f(\zeta)(\zeta-a)^{n-1}d\zeta, \quad n = 1, 2, \dots \tag{3.7}$$

We replace now the index n in (3.6) and (3.7) that takes values $1, 2, \dots$ by index $-n$ that takes values $-1, -2, \dots$ (this does not change anything) and denote[16]

$$c_n = -d_n = \frac{1}{2\pi i}\int\limits_{\Gamma'} f(\zeta)(\zeta-a)^{-n-1}d\zeta. \quad n = 1, 2, \dots \tag{3.8}$$

Now decomposition (3.6) takes the form

$$-\frac{1}{2\pi i}\int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{n=-1}^{\infty} c_n(z-a)^n. \tag{3.9}$$

We now insert (3.4) and (3.9) into (3.1) and obtain the decomposition (3.1): $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n(z-a)^n$, where the infinite series is understood as the sum of the series (3.4) and (3.9). It remains to observe that the Cauchy theorem 1.20 implies that the circles γ' and Γ' in (3.5) and (3.8) may be replaced by any circle $\{|\zeta - a| = \rho\}$ with any $r < \rho < R$. Then these expressions becomes (3.2). $\Box$

Definition 3.2 *The series (3.1) with the coefficients determined by (3.2) is called the Laurent series of the function f in the annulus V. The terms with non-negative powers constitute its regular part, while the terms with the negative powers constitute the principal part (we will see in the next section that these names are natural).*

Let us consider the basic properties of the power series in integer powers of $(z-a)$. As before we define such a series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \tag{3.10}$$

[16]Note that we have so far we used only c_n with positive indices so we do not interfere with previously defined c_n's.

as the sum of two series

$$(\Sigma_1): \sum_{n=0}^{\infty} c_n(z-a)^n \text{ and } (\Sigma_2): \sum_{n=-1}^{-\infty} c_n(z-a)^n. \tag{3.11}$$

The series (Σ_1) is a usual power series, its domain of converges is the disk $\{|z-a| < R\}$ where the radius R is determined by the Cauchy-Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}. \tag{3.12}$$

The series (Σ_2) is a power series in the variable $Z = 1/(z-a)$:

$$(\Sigma_2): \sum_{n=1}^{\infty} c_{-n} Z^n. \tag{3.13}$$

Therefore its domain of convergence is the outside of the disk $\{|z-a| > r\}$ where

$$r = \limsup_{n\to\infty} |c_{-n}|^{1/n} \tag{3.14}$$

as follows from the Cauchy-Hadamard formula applied to the series (3.13). The number R is not necessarily larger than r therefore the domain of convergence of the series (3.10) may be empty. However, if $r < R$ then the domain of convergence of the series (3.10) is the annulus $V = \{r < |z-a| < R\}$. We note that the set of points where (3.10) converges may differ from V by a subset of the boundary ∂V.

The series (3.10) converges uniformly on any compact subset of V according to the Abel theorem. Therefore the Weierstrass theorem implies that its sum is holomorphic in V.

These remarks imply immediately the uniqueness of the representation of a function as a power series in both negative and positive powers in a given annulus.

Theorem 3.3 *If a function f may be represented by a series of type (3.1) in an annulus $V = \{r < |z-a| < R\}$ then the coefficients of this series are determined by formulas (3.2).*

Proof. Consider a circle $\gamma = \{|z-a| = \rho\}$, $r < \rho < R$. The series

$$\sum_{k=-\infty}^{\infty} c_k(z-a)^k = f(z)$$

converges uniformly on γ. This is still true if we multiply both sides by an arbitrary power $(z-a)^{-n-1}$, $n = 0, \pm 1, \pm 2, \ldots$:

$$\sum_{n=-\infty}^{\infty} c_k(z-a)^{k-n-1} = \frac{f(z)}{(z-a)^{n+1}}.$$

Integrating this series term-wise along γ we obtain

$$\sum_{n=-\infty}^{\infty} c_k \int_\gamma (z-a)^{k-n-1} dz = \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}.$$

The orthogonality (1.4) implies that all integrals on the left side vanish except the one with $k = n$ that is equal to $2\pi i$. We get

$$2\pi c_n = \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}$$

which is nothing but (3.2). $\square$

Theorem 3.3 may be reformulated as follows: any converging series in negative and positive powers is the Laurent series of its sum.

Expression (3.2) for the coefficients of the Laurent series are rarely used in practice since they require computation of integrals. The uniqueness theorem that we have just proved implies that any legitimate way of getting the Laurent series may be used: they all lead to the same result.

Example 3.4 The function $f(z) = \frac{1}{(z-1)(z-2)}$ is holomorphic in the annuli $V_1 = \{0 < |z| < 1\}$, $V_2 = \{1 < |z| < 2\}$, $V_3 = \{2 < |z| < \infty\}$. In order to obtain its Laurent series we represent f as $f = \frac{1}{z-2} - \frac{1}{z-1}$. The two terms may be represented by the following geometric series in the annulus V_1:

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{(converges for } |z| < 2) \qquad (3.15) \\ -\frac{1}{z-1} &= \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{(converges for } |z| < 1). \end{aligned}$$

Therefore the function f is given in V_1 by the series

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n,$$

that contains only positive powers (the Taylor series). The first series in (3.15) still converges in V_2 but the second ones needs to be replaced by the decomposition

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\sum_{n=-1}^{-\infty} z^n \quad \text{(converges for } |z| > 1). \qquad (3.16)$$

The function f is represented by the Laurent series in this annulus:

$$f(z) = -\sum_{n=-1}^{-\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$

Finally, the series (3.16) converges in V_3 while the first expansion in (3.15) should be replaced by

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{2}\sum_{n=-1}^{-\infty}\left(\frac{z}{2}\right)^n \quad \text{(converges for } |z|>2).$$

Therefore we have in V_3:

$$f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}}-1\right)z^n.$$

We observe that the coefficients of the Laurent series are determined by formulas (3.2) that coincide with the integral formulas for the coefficients of the Taylor series[17] Repeating the arguments in the derivation of the Cauchy inequalities for the coefficients of the Taylor series we obtain

Theorem 3.5 *The Cauchy inequalities (for the coefficients of the Laurent series). Let the function f be holomorphic in the annulus $V = \{r < |z-a| < R\}$ and let its absolute value be bounded by M on a circle $\gamma_\rho = \{|z-a| = \rho\}$ then the coefficients of the Laurent series of the function f in V satisfy the inequalities*

$$|c_n| \le M/\rho^n, \quad n = 0, \pm 1, \pm 2 \ldots \tag{3.17}$$

We now comment on the relation between the Laurent and Fourier series. The Fourier series of a function ϕ that is integrable on $[0, 2\pi]$ is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt, \tag{3.18}$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi}\phi(t)\cos ntdt, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi}\phi(t)\sin ntdt, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.19}$$

with $b_0 = 0$. Such a series may be re-written in the complex form using the Euler formulas $\cos nt = \dfrac{e^{int}+e^{-int}}{2}$, $\sin nt = \dfrac{e^{int}-e^{-int}}{2i}$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

[17]However, the coefficients of the Laurent series may not be written as $c_n = f^{(n)}(a)/n!$ – for the simple reason that f might be not defined for $z = a$.

where we set

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_0^{2\pi} \phi(t) e^{-int} dt, \quad n = 0, 1, \dots,$$

and

$$c_n = \frac{a_{-n} + ib_{-n}}{2} = \frac{1}{2\pi}\int_0^{2\pi} \phi(t) e^{-int} dt, \quad n = -1, -2, \dots$$

The series

$$\sum_{n=-\infty}^{\infty} c_n e^{int} \tag{3.20}$$

with the coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(t) e^{-int} dt \tag{3.21}$$

is the Fourier series of the function ϕ written in the complex form.

Let us now set $e^{it} = z$ and $\phi(t) = f(e^{it}) = f(z)$, then the series (3.20) takes the form

$$\sum_{n=-\infty}^{\infty} c_n z^n \tag{3.22}$$

and its coefficients are

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}) e^{-int} dt = \frac{1}{2\pi i}\int_{|z|=1} f(z) \frac{dz}{z^{n+1}}. \tag{3.23}$$

Therefore the Fourier series of a function $\phi(t)$, $t \in [0, 2\pi]$ written in the complex form is the Laurent series of the function $f(z) = \phi(t)$ with $z = e^{it}$, on the unit circle $|z| = 1$.

Clearly, conversely, the Laurent series of a function $f(z)$ on the unit circle is the Fourier series of the function $f(e^{it}) = \phi(t)$ on the interval $[0, 2\pi]$.

We note that in general even if the Fourier series converges to the function ϕ at all points $[0, 2\pi]$ the corresponding Laurent series may have $R = r = 1$ so that its domain of convergence is empty. Domain of convergence is not empty only under fairly restrictive assumptions on the function ϕ.

Example 3.6 Let $\phi(t) = \dfrac{a \sin t}{a^2 - 2a\cos t + 1}$, then we set $z = e^{it}$ and find

$$f(z) = \frac{a(z - \frac{1}{z})}{2i\left\{a^2 - a(z + \frac{1}{z}) + 1\right\}} = \frac{1}{2i} \cdot \frac{1 - z^2}{z^2 - (a + \frac{1}{a})z + 1} = \frac{1}{2i}\left(\frac{1}{1 - az} - \frac{1}{1 - \frac{a}{z}}\right)$$

. This function is holomorphic in the annulus $\{|a| < |z| < 1/|a|\}$. As in the previous example we obtain its Laurent series in this annulus:

$$f(z) = \frac{1}{2i}\sum_{n=1}^{\infty} a^n \left(z^n - \frac{1}{z^n}\right).$$

Replacing again $z = e^{it}$ we obtain the Fourier series of the function ϕ:

$$\phi(t) = \sum_{n=1}^{\infty} a^n \sin nt.$$

3.2 Isolated singular points

We begin to study the points where analyticity of a function is violated. We first consider the simplest type of such points.

Definition 3.7 *A point $a \in \bar{\mathbb{C}}$ is an isolated singular point of a function f if there exists a punctured neighborhood of this point (that is, a set of the form $0 < |z - a| < r$ if $a \neq \infty$, or of the form $R < |z| < \infty$ if $a = \infty$), where f is holomorphic.*

We distinguish three types of singular points depending on the behavior of f near such point.

Definition 3.8 *An isolated singular point a of a function f is said to be*
(I) removable if the limit $\lim_{z\to a} f(z)$ exists and is finite;
(II) a pole if the limit $\lim_{z\to a} f(z)$ exists and is equal to ∞.
(III) an essential singularity if f has neither a finite nor infinite limit as $z \to a$.

Example 3.9 1. All three types of singular points may be realized. For example, the function $z/\sin z$ has a removable singularity at $z = 0$ as may be seen from the Taylor expansion

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \dots$$

that implies that the limit $\lim_{z=0} \frac{\sin z}{z} = 1$ exists and thus so does $\lim_{z=0} \frac{z}{\sin z} = 1$. The functions $1/z^n$, where n is a positive integer have a pole at $z = 0$. The function e^z has an essential singularity at $z = 0$, since, for instance, its limits as $z = x$ tends to zero from the left and right are different (the limit on the left is equal to zero, and the limit on the right is infinite), while it has no limit as z goes to zero along the imaginary axis: $e^{iy} = \cos(1/y) + i \sin(1/y)$ has no limit as $y \to 0$.

Non-isolated singular points may exist as well. For instance, the function $\dfrac{1}{\sin(\pi z)}$ has poles at the points $z = 1/n$, $n \in \mathbb{Z}$ and hence $z = 0$ is non-isolated singular point of this function - a limit point of poles.

2. A more complicated set of singular points is exhibited by the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = 1 + z^2 + z^4 + z^8 + \dots \tag{3.24}$$

According to the Cauchy-Hadamard formula the series (3.24) converges in the open disk $\{|z|, 1\}$ and hence f is holomorphic in this disk. Furthermore, $f(z)$ tends to infinity as $z \to 1$ along the real axis and hence $z = 1$ is a singular point of this function. However, we have

$$f(z^2) = 1 + z^4 + z^8 + \dots = f(z) - z^2$$

and hence $f(z)$ tends to infinity also when $z \to -1$ along the radial direction. Similarly $f(z) = z^2 + z^4 + f(z^4)$ and hence $f \to \infty$ as $z \to \pm i$ along the radius of the disk. In general,

$$f(z) = z^2 + \dots + z^{2^n} + f(z^{2^n})$$

for any $n \in \mathbb{N}$. Therefore $f \to \infty$ as z tends to any "dyadic" point $z = e^{ik\cdot 2\pi/2^n}$, $k = 0, 1, \ldots, 2^n - 1$ on the circle along the radial direction. Since the set of "dyadic" points is dense on the unit circle each point on this circle is a singular point of f. Therefore f is singular along a whole curve that consists of non-isolated singular points.

The type of an isolated singular point $z = a$ is closely related to the Laurent expansion of f in a punctured neighborhood of a. This relation is expressed by the following three theorems for finite singular points.

Theorem 3.10 *An isolated singular point $a \in \mathbb{C}$ of a function f is a removable singularity if and only if its Laurent expansion around a contains no principal part:*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \tag{3.25}$$

Proof. Let a be a removable singularity of f, then the limit $\lim_{z\to a} f(z) = A$ exists and is finite. Therefore f is bounded in a punctured neighborhood $\{0 < |z-a| < R\}$ of f, say, $|f| \le M$. Let ρ be such that $0 < \rho < R$ and use the Cauchy inequalities:

$$|c_n| \le M/\rho^n, \quad n = 0, \pm 1, \pm 2, \ldots$$

If $n < 0$ then the right side vanishes in the limit $\rho \to 0$ while the left side is independent of ρ. Therefore $c_n = 0$ when $n < 0$ and the Laurent series has no principal part.

Conversely, let $f(z)$ has a Laurent expansion around a that has no principal part. This is a Taylor expansion and hence the limit

$$\lim_{z\to a} f(z) = c_0$$

exists and is finite. Therefore a is a removable singularity of f. □

Remark 3.11 The same argument proves the following.

Theorem 3.12 *An isolated singular point a of a function f is removable if and only if f is bounded in a neighborhood of the point a.*

Extending f to a removable singular point a by continuity we set $f(a) = \lim_{z\to a} f(z)$ and obtain a function holomorphic at this point – this removes the singularity. That explains the name "removable singularity". In the future we will consider such points as regular and not singular points.

Exercise 3.13 *Show that if f is holomorphic in a punctured neighborhood of a point a and we have $Ref > 0$ at all points in this neighborhood, then a is a removable singularity of f.*

Theorem 3.14 *An isolated singular point $a \in \mathbb{C}$ is a pole if and only if the principal part of the Laurent expansion near a contains only finite (and positive) number of non-zero terms:*

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n, \quad N > 0. \tag{3.26}$$

Proof. Let a be a pole of f. There exists a punctured neighborhood of a where f is holomorphic and different from zero since $\lim_{z\to a} f(z) = \infty$. The function $\phi(z) = 1/f(z)$ is holomorphic in this neighborhood and the limit $\lim_{z\to a} \phi(z) = 0$ exists. Therefore a is a removable singularity of ϕ (and its zero) and the Taylor expansion holds:

$$\phi(z) = b_N (z-a)^N + b_{N+1}(z-a)^{N+1} + \ldots, \quad b_N \neq 0.$$

Therefore we have in the same neighborhood

$$f(z) = \frac{1}{\phi(z)} = \frac{1}{(z-a)^N} \cdot \frac{1}{b_N + b_{N+1}(z-a) + \ldots}. \tag{3.27}$$

The second factor above is a holomorphic function at a and thus admits the Taylor expansion

$$\frac{1}{b_N + b_{N+1}(z-a) + \ldots} = c_{-N} + c_{-N+1}(z-a) + \ldots, \quad c_{-N} = \frac{1}{b_N} \neq 0.$$

Using this expansion in (3.27) we find

$$f(z) = \frac{c_{-N}}{(z-a)^N} + \frac{c_{-N+1}}{(z-a)^{N-1}} + \cdots + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

This is the Laurent expansion of f near a and we see that its principal part contains finitely many terms.

Let f be represented by a Laurent expansion (3.26) in a punctured neighborhood of a with the principal part that contains finitely many terms, and $c_{-N} \neq 0$. Then both f and $\phi(z) = (z-a)^N f(z)$ are holomorphic in this neighborhood. The latter has the expansion

$$\phi(z) = c_{-N} + c_{-N+1}(z-a) + \ldots$$

that shows that a is a removable singularity of ϕ and the limit $\lim_{z\to a} \phi(z) = c_{-N}$ exists. Then the function $f(z) = \phi(z)/(z-a)^N$ tends to infinity as $z \to a$ and hence a is a pole of f. $\square$

We note another simple fact that relates poles and zeros.

Theorem 3.15 *A point a is a pole of the function f if and only if the function $\phi = 1/f$ is holomorphic in a neighborhood of a and $\phi(a) = 0$.*

Proof. The necessity of this condition has been proved in the course of the proof of Theorem 3.14. Let us show it is also sufficient. If ϕ is holomorphic at a and $\phi(a) = 0$ but ϕ is not equal identically to a constant then the uniqueness theorem implies that there exists a punctured neighborhood of this point where $\phi \neq 0$. the function $f = 1/\phi$ is holomorphic in this neighborhood and hence a is an isolated singular point of f. However, $\lim_{z\to a} f(z) = \infty$ and thus a is a pole of f. $\square$

This relation allows to introduce the following definition.

Definition 3.16 *The order of the pole a of a function f is the order of this point as a zero of $\phi = 1/f$.*

The proof of Theorem 3.14 shows that the order of a pole coincides with the index N of the leading term in the Laurent expansion of the function around the pole.

Theorem 3.17 *An isolated singular point of a is an essential singularity if and only if the principal part of the Laurent expansion of f near a contains infinitely many non-zero terms.*

Proof. This theorem is essentially contained in Theorems 3.10 and 3.14 (if the principal part contains infinitely many terms then a may be neither removable singularity nor a pole; if a is an essential singularity then the principal part may neither be absent nor contain finitely many terms). $\square$

Exercise 3.18 Show that if a is an essential singularity of a function f then

$$\rho^k \sup_{|z-a|=\rho} |f(z)| \to \infty$$

as $\rho \to 0$ for any natural k.

Behavior of a function near an essential singularity is characterized by the following interesting

Theorem 3.19 *If a is an essential singularity of a function f then for any $A \in \overline{\mathbb{C}}$ we may find a sequence $z_n \to a$ so that*

$$\lim_{n\to\infty} f(z) = A. \tag{3.28}$$

Proof. Let $A = \infty$. Since f may not be bounded in a punctured neighborhood $\{0 < |z-a| < r\}$ there exists a point z_1 so that $|f(z_1)| > 1$. Similarly there exists a point z_2 in $\{0 < |z-a| < |z_1 - a|/2\}$ such that $|f(z_2)| > 2$ etc.: there exists a point z_n in the neighborhood $\{0 < |z-a| < |z_{n-1} - a|/2\}$ so that $|f(z_n)| > n$. Clearly we have both $z_n \to a$ and $f(z_n) \to \infty$.

Let us consider now the case $A \neq \infty$. Then either there exists a sequence of points $\zeta_k \to a$ so that $f(\zeta_k) = A$ or there exists a neighborhood $\{0 < |z-a| < r\}$ so that $f(z) \neq A$ in this neighborhood. The function $\phi(z) = 1/(f(z) - A)$ is holomorphic in this neighborhood. Moreover, a is an essential singularity of ϕ (otherwise $f(z) = A + \dfrac{1}{\phi(z)}$

would have a limit as $z \to a$). The first part of this proof implies that there exists a sequence $z_k \to a$ so that $\phi(z_k) \to \infty$ which in turn implies that

$$\lim_{n\to\infty} f(z_n) = A + \lim_{n\to\infty} \frac{1}{\phi(z)} = A. \ \square$$

The collection of all possible limits of $f(z_k)$ for all sequences $z_k \to a$ is called the indeterminacy set of f at the point a. If a is a removable singularity or a pole of f the indeterminacy set of f at a consists of one point (either finite or infinite). Theorem 3.19 claims that the other extreme is realized at an essential singularity: the indeterminacy set fills the whole closed complex plane $\overline{\mathbb{C}}$.

Exercise 3.20 (i) Show that the conclusion of Theorem 3.19 holds also for a singular point that is a limit point of poles.
(ii) Let a be an essential singularity of f: which type of singularity may the function $1/f$ have at a? (Hint: it is either an essential singularity or a limit point of poles.)

We briefly comment now on the isolated singularities at infinity. The classification and Theorems 3.12, 3.15 and 3.19 are applicable in this case without any modifications. However, Theorems 3.10, 3.14 and 3.17 related to the Laurent expansion require changes. The reason is that the type of singularity at a finite singular point is determined by the principal part of the Laurent expansion that contains the negative powers of $(z-a)$ that are singular at those points. However, the negative powers are regular at infinity and the type of singularity is determined by the positive powers of z. Therefore it is natural to define the principal part of the Laurent expansion at infinity as the collection of the positive powers of z of this expansion. Theorems 3.10, 3.14 and 3.17 hold after that modification also for $a = \infty$.

This result may be obtained immediately with the change of variables $z = 1/w$: if we denote $f(z) = f(1/w) = \phi(w)$ then clearly

$$\lim_{z\to\infty} f(z) = \lim_{w\to 0} \phi(w)$$

and hence ϕ has the same type of singularity at $w = 0$ as f at the point $z = \infty$. For example, in the case of a pole ϕ has an expansion in $\{0 < |w| < r\}$

$$\phi(w) = \frac{b_{-N}}{w^N} + \cdots + \frac{b_{-1}}{w} + \sum_{n=0}^{\infty} b_n w^n, \quad b_{-N} \neq 0.$$

Replacing w by $1/z$ we get the expansion for f in the annulus $\{R < |z| < \infty\}$ with $R = 1/r$:

$$f(z) = \sum_{n=-1}^{-\infty} c_n z^n + c_0 + c_1 z + \cdots + c_N z^N$$

with $c_n = b_{-n}$. Its principal part contains finitely many terms. We may consider the case of a removable or an essential singularity in a similar fashion.

We describe now the classification of the simplest holomorphic functions according to their singular points. According to the Liouville theorem the functions that have no singularities in $\overline{\mathbb{C}}$ are constants. The next level of complexity is exhibited by the entire functions.

Definition 3.21 *A function $f(z)$ is called entire if it is holomorphic in $\mathbb{C}$, that is, if it has no finite singular points.*

The point $a = \infty$ is therefore an isolated singularity of an entire function f. If it is a removable singularity then $f = \text{const}$. If it is a pole then the principal part of the Laurent expansion at infinity is a polynomial $g(z) = c_1 z + \cdots + c_N z^N$. Subtracting the principal part from f we observe that the function $f - g$ is entire and has a removable singularity at infinity. Therefore it is a constant and hence f is a polynomial. Therefore an entire function with a pole at infinity must be a polynomial.

Entire functions with an essential singularity at infinity are called entire transcendental functions, such as e^z, $\sin z$ or $\cos z$.

Exercise 3.22 (i) Show that an entire function such that $|f(z)| \geq |z|^N$ for sufficiently large $|z|$ is a polynomial.
(ii) Deduce Theorem 3.19 for entire functions and $a = \infty$ from the Liouville theorem.

Definition 3.23 *A function f is meromorphic if it has no singularities in $\mathbb{C}$ except poles.*

Entire functions form a sub-class of meromorphic functions that have no singularities in $\mathbb{C}$. Since each pole is an isolated singular point a meromorphic function may have no more than countably many poles in $\mathbb{C}$. Indeed, every disk $\{|z| < n\}$ contains finitely many poles (otherwise the set of poles would have a limit point that would be a non-isolated singular point and not a pole) and hence all poles may be enumerated. Examples of meromorphic functions with infinitely many poles are given by functions $\tan z$ and $\text{cotan}\, z$.

Theorem 3.24 *If a meromorphic function f has a pole or a removable singularity at infinity (that is, if all its singularities in $\overline{\mathbb{C}}$ are poles) then f is a rational function.*

Proof. The number of poles of f is finite - otherwise a limit point of poles would exist in $\overline{\mathbb{C}}$ since the latter is compact, and it would be a non-isolated singular point and not a pole. Let us denote by a_ν, $\nu = 1, \ldots, n$ the finite poles of f and let

$$g_\nu(z) = \frac{c_{-N_\nu}^{(\nu)}}{(z - a_\nu)^{N_\nu}} + \cdots + \frac{c_{-1}^{(\nu)}}{z - a_\nu} \tag{3.29}$$

be the principal part of f near the pole a_ν. We also let

$$g(z) = c_1 z + \cdots + c_N z^N \tag{3.30}$$

be the principal part of f at infinity. If $a = \infty$ is a removable singularity of f we set $g = 0$.

Consider the function

$$\phi(z) = f(z) - g(z) - \sum_{\nu=1}^{n} g_\nu(z).$$

It has no singularities in $\overline{\mathbb{C}}$ and hence $\phi(z) = c_0$. Therefore

$$f(z) = c_0 + g(z) + \sum_{\nu=1}^{n} g_\nu(z) \tag{3.31}$$

is a rational function. $\square$

Remark 3.25 Expression (3.31) is the decomposition of f into an entire part and simple fractions. Our argument gives a simple existence proof for such a decomposition.

Sometimes we will use the term "meromorphic function" in a more general sense. We say that f is meromorphic in a domain D if it has no singularities in D other than poles. Such function may also have no more than countably many poles. Indeed we may construct a sequence of compact sets $K_1 \subset K_2 \dots \subset K_n \subset \dots$ so that $D = \cup_{n=1}^{\infty} K_n$: it suffices to take $K_n = \{z : \ |z| \leq n, \ \mathrm{dist}(z, \partial D) \geq 1/n\}$. Then f may have only finitely many poles in each K_n and hence it has no more than countably many poles in all of D. If the set of poles of f in D is infinite then the limit points of this set belong to the boundary ∂D.

Theorem 3.24 may now be formulated as follows: any function meromorphic in the closed complex plane $\overline{\mathbb{C}}$ is rational.

3.3 The Residues

Somewhat paradoxically the most interesting points in the study of holomorphic functions are those where functions cease being holomorphic – the singular points. We will encounter many observations in the sequel that demonstrate that the singular points and the Laurent expansions around them contain the basic information about the holomorphic functions.

We illustrate this point on the problem of computing integrals of holomorphic functions. Let f be holomorphic in a domain D everywhere except possibly at a countable set of isolated singular points. Let G be properly contained in D, and let the boundary ∂G consist of finitely many continuous curves and not contain any singular points of f. There is a finite number of singular points contained inside G that we denote by $a_1, a_2, \dots, a_n$. Let us consider the circles $\gamma_\nu = \{|z - a_\nu| = t\}$ oriented counterclockwise, and of so small a radius that the disks $\bar{U}_\nu$ bounded by them do not overlap and are all contained in G. Let us also denote the domain $G_r = G \backslash (\cup_{\nu=1}^{n} \bar{U}_\nu)$. The function f is holomorphic in $\bar{G}_r$ and hence the Cauchy theorem implies that

$$\int_{\partial G_r} f dz = 0. \tag{3.32}$$

However, the oriented boundary ∂G_r consists of ∂G and the circles γ_ν^- oriented clockwise so that

$$\int_{\partial G} f dz = \sum_{\nu=1}^{n} \int_{\gamma_\nu} f dz. \tag{3.33}$$

Therefore the computation of the integral of a function along the boundary of a domain is reduced to the computation of the integrals over arbitrarily small circles around its singular points.

Definition 3.26 *The integral of a function f over a sufficiently small circle centered at an isolated singular point $a \in \mathbb{C}$ of this function, divided by $2\pi i$ is called the residue of f at a and is denoted by*

$$res_a f = \frac{1}{2\pi i} \int_{\gamma_r} f dz. \tag{3.34}$$

The Cauchy theorem on invariance of the integral under homotopic variations of the contour implies that the residue does not depend on the choice of r provided that r is sufficiently small and is completely determined by the local behavior of f near a.

Relation (3.33) above expresses the Cauchy theorem on residues[18]:

Theorem 3.27 *Let the function f be holomorphic everywhere in a domain D except at an isolated set of singular points. Let the domain G be properly contained in D and let its boundary ∂G contain no singular points of f. Then we have*

$$\int_{\partial G} f dz = 2\pi i \sum_{(G)} res_{a_\nu} f, \tag{3.35}$$

where summation is over all singular points of f contained in G.

This theorem is of paramount importance as it allows to reduce the computation of a global quantity such as integral over a curve to a computation of local quantities – residues of the function at its singular points.

As we will now see the residues of a function at its singular points are determined completely by the principal part of its Laurent expansion near the singular points. This will show that it suffices to have the information about the singular points of a function and the principal parts of the corresponding Laurent expansions in order to compute its integrals.

Theorem 3.28 *The residue of a function f at an isolated singular point $a \in \mathbb{C}$ is equal to the coefficient in front of the term $(z-a)^{-1}$ in its Laurent expansion around a:*

$$res_a f = c_{-1}. \tag{3.36}$$

[18]Cauchy first considered residues in his memoirs of 1814 and 1825 where he studied the difference of integrals with common ends that contain a pole of the function between them. This explains the term "residue" that first appeared in a Cauchy memoir of 1826. Following this work Cauchy has published numerous papers on the applications of residues to calculations of integrals, series expansions, solution of differential equations etc.

Proof. The function f has the Laurent expansion around a:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

The series converges uniformly on a sufficiently small circle $\gamma_r = \{|z-a| = r\}$. Integrating the series termwise over γ_r and using (1.4) we find $\int_{\gamma_r} f dz = 2\pi i c_{-1}$. The definition (3.34) of the residue implies (3.36). □

Corollary 3.29 *The residue at a removable singularity $a \in \mathbb{C}$ vanishes.*

We present now some formulas for the computation of the residue at a pole. First we let a be a pole of order one. The Laurent expansion of the function near a has the form

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

This immediately leads to the formula for the residue at a pole of order one:

$$c_{-1} = \lim_{z \to a} (z-a) f(z). \tag{3.37}$$

A simple modification of this formula is especially convenient. Let

$$f(z) = \frac{\phi(z)}{\psi(z)}$$

with the functions ϕ and ψ holomorphic at a so that $\psi(a) = 0$, $\psi'(a) \neq 0$, and $\phi(a) \neq 0$. This implies that a is a pole of order one of the function f. Then (3.37) implies that

$$c_{-1} = \lim_{z \to a} \frac{(z-a)\phi(z)}{\psi(z)} = \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z)-\psi(a)}{z-a}}$$

so that

$$c_{-1} = \frac{\phi(a)}{\psi'(a)}. \tag{3.38}$$

Let f now have a pole of order n at a, then its Laurent expansion near this point has the form

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

We multiply both sides by $(z-a)^n$ in order to get rid of the negative powers in the Laurent expansion and then differentiate $n-1$ times in order to single out c_{-1} and pass to the limit $z \to a$. We obtain the expression for the residue at a pole of order n:

$$c_{-1} = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]. \tag{3.39}$$

There no analogous formulas for the calculation of residues at an essential singularity: one has to compute the principal part of the Laurent expansion.

A couple of remarks on residue at infinity.

Definition 3.30 *Let infinity be an isolated singularity of the function f. The residue of f at infinity is*

$$res_\infty f = \frac{1}{2\pi i}\int_{\gamma_R^-} f dz, \tag{3.40}$$

where γ_R^- is the circle $\{|z| = R\}$ of a sufficiently large radius R oriented clockwise.

The orientation of γ_R^- is chosen so that the neighborhood $\{R < |z| < \infty\}$ remains on the left as the circle is traversed. The Laurent expansion of f at infinity has the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Integrating the series termwise along γ_R^- and using (1.4) we obtain

$$\mathrm{res}_\infty f = c_{-1}. \tag{3.41}$$

The terms with the negative powers constitute the regular part of the Laurent expansion at infinity. Therefore unlike at finite singular points the residue at infinity may be non-zero even if $z = \infty$ is not a singular point of the function.

We present a simple theorem on the total sum of residues.

Theorem 3.31 *Let the function f be holomorphic everywhere in the complex plane $\mathbb{C}$ except at a finite number of points a_ν, $\nu = 1, \ldots, n$. Then the sum of its residues at all of its finite singular points and the residue at infinity vanishes:*

$$\sum_{\nu=1}^{n} res_{a_\nu} f + res_\infty f = 0. \tag{3.42}$$

Proof. We consider the circle $\gamma_R = \{|z| = R\}$ of such a large radius that it contains all finite singular points a_ν of f. Let γ_R be oriented counterclockwise. The Cauchy theorem on residues implies that

$$\frac{1}{2\pi}\int_{\gamma_R} f dz = \sum_{\nu=1}^{n} \mathrm{res}_{a_\nu} f,$$

while the Cauchy theorem 1.20 implies that the left side does not change if R is increased further. Therefore it is equal to the negative of the residue of f at infinity. Thus the last equality is equivalent to (3.42).

Example 3.32 One needs not compute the residues at all the eight poles of the second order in order to compute the integral $I = \int\limits_{|z|=2} \frac{dz}{(z^8+1)^2}$. It suffices to apply the theorem on the sum of residues that implies that

$$\sum_{\nu=1}^{n} \mathrm{res}_{a_\nu} \frac{1}{(z^8+1)^2} + \mathrm{res}_\infty \frac{1}{(z^8+1)^2} = 0.$$

However, the function f has a zero of order sixteen at infinity. Thus its Laurent expansion at infinity has negative powers starting at z^{-16}. Hence its residue at infinity is equal to zero, and hence the sum of residues at finite singular points vanishes so that $I = 0$.

We present several examples of the application of the Cauchy theorem on residues to the computation of definite integrals of functions of a real variable. Let us compute the integral along the real axis

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx, \tag{3.43}$$

where t is a real number. The integral converges absolutely since it is majorized by the converging integral of $1/(1+x^2)$.

The residues are used as follows. We extend the integrand to the whole complex plane

$$f(z) = \frac{e^{itz}}{1+z^2}$$

and choose a closed contour so that it contains the interval $[-R, R]$ of the real axis and an arc that connects the end-points of this segment. The Cauchy theorem on residues is applied to this closed contour and then the limit $R \to \infty$ is taken. If the limit of the integral along the arc may be found then the problem is solved.

Let $z = x + iy$, given that $|e^{izt}| = e^{-yt}$ we consider separately two cases: $t \geq 0$ and $t < 0$. In the former case we close the contour by using the upper semi-circle $\gamma_R' = \{|z| = R,\ \mathrm{Im} z > 0\}$ that is traversed counterclockwise. When $R > 1$ the resulting contour contains on pole $z = i$ of f of the first order. The residue at this point is easily found using (3.38):

$$\mathrm{res}_i \frac{e^{izt}}{1+z^2} = \frac{e^{-t}}{2i}.$$

The Cauchy theorem on residues implies then that

$$\int_{-R}^{R} f(x)dx + \int_{\gamma_R'} f dz = \pi e^{-t}. \tag{3.44}$$

The integral over γ_R' is bounded as follows. We have $|e^{itz}| = e^{-ty} \leq 1$, $|1+z^2| \geq R^2 - 1$ when $t \geq 0$ and $z \in \gamma_R'$. Therefore we have an upper bound

$$\left| \int_{\gamma_R'} \frac{e^{itz}}{1+z^2} dz \right| \leq \frac{\pi R}{R^2 - 1} \tag{3.45}$$

that shows that this integral vanishes in the limit $R \to \infty$. Therefore passing to the limit $R \to \infty$ in (3.44) we obtain for $t \geq 0$:

$$\int_{-\infty}^{\infty} f(x)dx = \pi e^{-t}. \tag{3.46}$$

The estimate (3.45) fails when $t < 0$ since $|e^{izt}| = e^{-yt}$ grows as $y \to +\infty$. Therefore we replace the semi-circle γ_R' by the lower semi-circle $\gamma_R'' = \{|z| = R,\ \text{Im}z < 0\}$ that is traversed clockwise. Then the Cauchy theorem on residues implies for $R > 1$:

$$\int_{-R}^{R} f(x)dx + \int_{\gamma_R''} f dz = -2\pi \text{res}_{-i} f = \pi e^t. \tag{3.47}$$

We have $|e^{itz}| = e^{ty} \leq 1$, $|1+z^2| \geq R^2 - 1$ when $t < 0$ and $z \in \gamma_R''$. Therefore the integral over γ_R'' also vanishes in the limit $R \to \infty$ and (3.47) becomes in the limit $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)dx = \pi e^t. \tag{3.48}$$

Putting (3.46) and (3.48) together we obtain the final answer

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx = \pi e^{-|t|}. \tag{3.49}$$

We will often use residues to compute various integrals. We present a lemma useful in such calculations.

Lemma 3.33 *(Jordan[19]) Let the function f be holomorphic everywhere in $\{Imz \geq 0\}$ except possibly at an isolated set of singular points and $M(R) = \sup_{\gamma_R} |f(z)|$ over the semi-circle $\gamma_R = \{|z| = R,\ Imz \geq 0\}$ tends to zero as $R \to \infty$ (or along a sequence $R_n \to \infty$ such that γ_{R_n} do not contain singular points of f). Then the integral*

$$\int_{\gamma_R} f(z) e^{i\lambda z} dz \tag{3.50}$$

tends to zero as $R \to \infty$ (or along the corresponding sequence $R_n \to \infty$) for all $\lambda > 0$.

The main point of this lemma is that $M(R)$ may tend to zero arbitrary slowly so that the integral of f over γ_R needs not vanish as $R \to \infty$. Multiplication by the exponential $e^{i\lambda z}$ with $\lambda > 0$ improves convergence to zero.

Proof. Let us denote by $\gamma_R' = \{z = Re^{i\phi},\ 0 \leq \phi \leq \pi/2\}$ the right half of γ_R. We have $\sin\phi \geq \frac{2}{\pi}\phi$ for $\phi \in [0, \pi/2]$ because $\sin\phi$ is a concave function on the interval. Therefore the bound $\left|e^{i\lambda z}\right| = e^{-\lambda R \sin\phi} \leq e^{-2\lambda R\phi/\pi}$ holds and thus

$$\left|\int_{\gamma_R} f(z) e^{i\lambda z} dz\right| \leq M(R) \int_0^{\pi/2} e^{-2\lambda R\phi/\pi} R d\phi = M(R) \frac{\pi}{2\lambda}(1 - e^{-\lambda R}) \to 0$$

as $R \to \infty$. The bound for $\gamma_R'' = \gamma_R \backslash \gamma_R'$ is obtained similarly. □

As the proof of this lemma shows the assumption that f is holomorphic is not essential in this lemma.

[19] This lemma appeared first in 1894 in the textbook on analysis written by Camille Jordan (1838-1922).

4 Exercises for Chapter 2

1. An integral of the Cauchy type is an integral of the form

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

where γ is a smooth curve in $\mathbb{C}$ and f is a continuous function on γ. Show that F is a holomorphic function in $\overline{\mathbb{C}}\backslash\gamma$ that vanishes at infinity.

2. Let γ be a smooth closed Jordan curve that bounds a domain D: $\gamma = \partial D$, and let $f \in C^1(\gamma)$. Show that the value of the integral of the Cauchy type jumps by the value of f at the crossing point when we cross γ. More precisely, if $\zeta_0 \in \gamma$ and $z \to \zeta_0$ from one side of γ then F has two limiting values $F^+(\zeta_0)$ and $F^-(\zeta_0)$ so that

$$F^+(\zeta_0) - F^-(\zeta_0) = f(\zeta_0).$$

Here $+$ corresponds to inside of D and $-$ to the outside. Hint: write F as

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{(f(\zeta) - f(\zeta_0))d\zeta}{\zeta - z} + \frac{f(\zeta_0)}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}.$$

3. Under the assumptions of the previous problem show that each of the following conditions is necessary and sufficient for the integral of the Cauchy type to be the Cauchy integral:

$$(a) \int_\gamma \frac{f(\zeta)}{\zeta - z} = 0 \ \text{ for all } z \in \overline{\mathbb{C}}\backslash\bar{D}$$

and

$$(b) \int_\gamma \zeta^n f(\zeta)d\zeta = 0 \ \text{ for all } n = 0, 1, 2\ldots$$

4. Let f be holomorphic in the disk $\{|z| < R\}$, $R > 1$. Show that the average of the square of its absolute value on the unit circle $\{|z| = 1\}$ is equal to $\sum_{n=0}^{\infty} |c_n|^2$, where c_n are the Taylor coefficients of f at $z = 0$.

5. The series $\displaystyle\sum_{n=0}^{\infty} \frac{x^2}{n^2x^2 + 1}$ converges for all real x but its sum may not be expanded in the Taylor series at $z = 0$. Explain.

6. Show that any entire function that satisfies the conditions $f(z + i) = f(z)$ and $f(z + 1) = f(z)$ is equal to a constant.

7. Show that the function $f(z) = \displaystyle\int_0^1 \frac{\sin tz}{t}dt$ is entire.

8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in a closed disk $\bar{U} = \{|z| \le R\}$ and $a_0 \neq 0$. Show that f is different from zero in the disk $\left\{|z| < \dfrac{|a_0|R}{|a_0| + M}\right\}$ where $M = \sup_{z\partial U} |f(z)|$.

9. Show that a power series may not converge absolutely at any boundary point of the disk of convergence if the boundary contains at least one pole of the function.

10. Show that a function holomorphic outside two non-intersecting compact sets may be represented as a sum of two functions, one of which is holomorphic outside of one compact set and the other outside the other compact set.