Chapter 1. The Holomorphic Functions

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We begin with the description of complex numbers and their basic algebraic properties. We will assume that the reader had some previous encounters with the complex numbers and will be fairly brief, with the emphasis on some specifics that we will need later.

1 The Complex Plane

1.1 The complex numbers

We consider the set $\mathbb{C}$ of pairs of real numbers (x, y), or equivalently of points on the plane $\mathbb{R}^2$. Two vectors $z_1 = (x_1, x_2)$ and $z_2 = (x_2, y_2)$ are *equal* if and only if $x_1 = x_2$ and $y_1 = y_2$. Two vectors $z = (x, y)$ and $\bar{z} = (x, -y)$ that are symmetric to each other with respect to the x-axis are said to be *complex conjugate* to each other. We identify the vector $(x, 0)$ with a real number x. We denote by $\mathbb{R}$ the set of all real numbers (the x-axis).

Exercise 1.1 *Show that $z = \bar{z}$ if and only if z is a real number.*

We introduce now the operations of addition and multiplication on $\mathbb{C}$ that turn it into a field. The sum of two complex numbers and multiplication by a real number $\lambda \in \mathbb{R}$ are defined in the same way as in $\mathbb{R}^2$:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad \lambda(x, y) = (\lambda x, \lambda y).$$

Then we may write each complex number $z = (x, y)$ as

$$z = x \cdot \mathbf{1} + y \cdot i = x + iy, \tag{1.1}$$

where we denoted the two unit vectors in the directions of the x and y-axes by $\mathbf{1} = (1, 0)$ and $i = (0, 1)$.

You have previously encountered two ways of defining a product of two vectors: the inner product $(z_1 \cdot z_2) = x_1 x_2 + y_1 y_2$ and the skew product $[z_1, z_2] = x_1 y_2 - x_2 y_1$. However, none of them turn $\mathbb{C}$ into a field, and, actually $\mathbb{C}$ is not even closed under these operations: both the inner product and the skew product of two vectors is a number, not a vector. This leads us to introduce yet another product on $\mathbb{C}$. Namely, we postulate that $i \cdot i = i^2 = -1$ and define $z_1 z_2$ as a vector obtained by multiplication of $x_1 + iy_1$ and $x_2 + iy_2$ using the usual rules of algebra with the additional convention that $i^2 = -1$. That is, we define

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1). \tag{1.2}$$

More formally we may write

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

but we will not use this somewhat cumbersome notation.

Exercise 1.2 *Show that the product of two complex numbers may be written in terms of the inner product and the skew product as* $z_1 z_2 = (\bar{z}_1 \cdot z_2) + i[\bar{z}_1, z_2]$*, where* $\bar{z}_1 = x_1 - iy_1$ *is the complex conjugate of* z_1*.*

Exercise 1.3 *Check that the product (1.2) turns* $\mathbb{C}$ *into a field, that is, the distributive, commutative and associative laws hold, and for any* $z \neq 0$ *there exists a number* $z^{-1} \in \mathbb{C}$ *so that* $zz^{-1} = 1$*. Hint:* $z^{-1} = \dfrac{x}{x^2 + y^2} - \dfrac{iy}{x^2 + y^2}$.

Exercise 1.4 *Show that the following operations do not turn* $\mathbb{C}$ *into a field: (a)* $z_1 z_2 = x_1 x_2 + iy_1 y_2$*, and (b)* $z_1 z_2 = x_1 x_2 + y_1 y_2 + i(x_1 y_2 + x_2 y_1)$*.*

The product (1.2) turns $\mathbb{C}$ into a field (see Exercise 1.3) that is called the *field of complex numbers* and its elements, vectors of the form $z = x + iy$ are called *complex numbers*. The real numbers x and y are traditionally called the real and imaginary parts of z and are denoted by

$$x = \mathrm{Re} z, \quad y = \mathrm{Im} z. \tag{1.3}$$

A number $z = (0, y)$ that has the real part equal to zero, is called *purely imaginary*.

The Cartesian way (1.1) of representing a complex number is convenient for performing the operations of addition and subtraction, but one may see from (1.2) that multiplication and division in the Cartesian form are quite tedious. These operations, as well as raising a complex number to a power are much more convenient in the *polar representation* of a complex number:

$$z = r(\cos\phi + i \sin\phi), \tag{1.4}$$

that is obtained from (1.1) passing to the polar coordinates for (x, y). The polar coordinates of a complex number z are the polar radius $r = \sqrt{x^2 + y^2}$ and the polar angle ϕ, the angle between the vector z and the positive direction of the x-axis. They are called the *modulus* and *argument* of z are denoted by

$$r = |z|, \quad \phi = \mathrm{Arg} z. \tag{1.5}$$

The modulus is determined uniquely while the argument is determined up to addition of a multiple of 2π. We will use a shorthand notation

$$\cos\phi + i\sin\phi = e^{i\phi}. \tag{1.6}$$

Note that we have not yet defined the operation of raising a number to a complex power, so the right side of (1.6) should be understood at the moment just as a shorthand for the left side. We will define this operation later and will show that (1.6) indeed holds. With this convention the polar form (1.4) takes a short form

$$z = re^{i\phi}. \tag{1.7}$$

Using the basic trigonometric identities we observe that

$$\begin{aligned} r_1e^{i\phi_1}r_2e^{i\phi_2} &= r_1(\cos\phi_1 + i\sin\phi_1)r_2(\cos\phi_2 + i\sin\phi_2) \\ &= r_1r_2(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)) \\ &= r_1r_2(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)) = r_1r_2e^{i(\phi_1+\phi_2)}. \end{aligned} \tag{1.8}$$

This explains why notation (1.6) is quite natural. Relation (1.8) says that the modulus of the product is the product of the moduli, while the argument of the product is the sum of the arguments.

Exercise 1.5 *Show that if $z = re^{i\phi}$ then $z^{-1} = \frac{1}{r}e^{-i\phi}$, and more generally if $z_1 = r_1e^{i\phi_1}$, $z_2 = r_2e^{i\phi_2}$ with $r_2 \neq 0$, then*

$$\frac{z_1}{z_2} = \frac{r_1e^{i\phi_1}}{r_2e^{i\phi_2}} = \frac{r_1}{r_2}e^{i(\phi_1-\phi_2)}. \tag{1.9}$$

Sometimes it is convenient to consider a *compactification* of the set $\mathbb{C}$ of complex numbers. This is done by adding an ideal element that is call the point at infinity $z = \infty$. However, algebraic operations are not defined for $z = \infty$. We will call the compactified complex plane, that is, the plane $\mathbb{C}$ together with the point at infinity, the closed complex plane, denoted by $\overline{\mathbb{C}}$. Sometimes we will call $\mathbb{C}$ the open complex plane in order to stress the difference between $\mathbb{C}$ and $\overline{\mathbb{C}}$.

One can make the compactification more visual if we represent the complex numbers as points not on the plane but on a two-dimensional sphere as follows. Let ξ, η and ζ be the Cartesian coordinates in the three-dimensional space so that the ξ and η-axes coincide with the x and y-axes on the complex plane. Consider the unit sphere

$$S: \quad \xi^2 + \eta^2 + \zeta^2 = 1 \tag{1.10}$$

in this space. Then for each point $z = (x, y) \in \mathbb{C}$ we may find a corresponding point $Z = (\xi, \eta, \zeta)$ on the sphere that is the intersection of S and the segment that connects the "North pole" $N = (0, 0, 1)$ and the point $z = (x, y, 0)$ on the complex plane.

The mapping $z \to Z$ is called *the stereographic projection*. The segment Nz may be parameterized as $\xi = tx$, $\eta = ty$, $\zeta = 1 - t$, $t \in [0, 1]$. Then the intersection point $Z = (t_0 x, t_0 y, 1 - t_0)$ with t_0 being the solution of

$$t_0^2 x^2 + t_0^2 y^2 + (1 - t_0)^2 = 1$$

so that $(1 + |z|^2)t_0 = 2$. Therefore the point Z has the coordinates

$$\xi = \frac{2x}{1 + |z|^2}, \quad \eta = \frac{2y}{1 + |z|^2}, \quad \zeta = \frac{|z|^2 - 1}{1 + |z|^2}. \tag{1.11}$$

The last equation above implies that $\dfrac{2}{1 + |z|^2} = 1 - \zeta$. We find from the first two equations the explicit formulae for the inverse map $Z \to z$:

$$x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\eta}{1 - \zeta}. \tag{1.12}$$

Expressions (1.11) and(1.12) show that the stereographic projection is a one-to-one map from $\mathbb{C}$ to $S \backslash N$ (clearly N does not correspond to any point z). We postulate that N corresponds to the point at infinity $z = \infty$. This makes the stereographic projection be a one-to-one map from $\bar{\mathbb{C}}$ to S. We will usually identify $\bar{\mathbb{C}}$ and the sphere S. The lattter is called *the sphere of complex numbers* or *the Riemann sphere*. The open plane $\mathbb{C}$ may be identified with $S \backslash N$, the sphere with the North pole deleted.

Exercise 1.6 *Let t and u be the longitude and the latitude of a point Z. Show that the corresponding point $z = se^{it}$, where $s = \tan(\pi/4 + u/2)$.*

We may introduce two metrics (distances) on $\mathbb{C}$ according to the two geometric descriptions presented above. The first is the usual Euclidean metric with the distance between the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in $\mathbb{C}$ given by

$$|z_2 - z_1| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.13}$$

The second is *the spherical metric* with the distance between z_1 and z_2 defined as the Euclidean distance in the three-dimensional space between the corresponding points Z_1 and Z_2 on the sphere. A straightforward calculation shows that

$$\rho(z_1, z_2) = \frac{2|z_2 - z_1|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}. \tag{1.14}$$

This formula may be extended to $\overline{\mathbb{C}}$ by setting

$$\rho(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}. \tag{1.15}$$

Note that (1.15) may be obtained from (1.14) if we let $z_1 = z$, divide the numerator and denominator by $|z_2|$ and let $|z_2| \to +\infty$.

Exercise 1.7 *Use the formula (1.11) for the stereographic projection to verify (1.14).*

Clearly we have $\rho(z_1, z_2) \leq 2$ for all $z_1, z_2 \in \overline{\mathbb{C}}$. It is straightforward to verify that both of the metrics introduced above turn $\mathbb{C}$ into a metric space, that is, all the usual axioms of a metric space are satisfied. In particular, the triangle inequality for the Euclidean metric (1.13) is equivalent to the usual triangle inequality for two-dimensional plane: $|z_1 + z_2| \leq |z_1| + |z_2|$.

Exercise 1.8 *Verify the triangle inequality for the metric $\rho(z_1, z_2)$ on $\overline{\mathbb{C}}$ defined by (1.14) and (1.15)*

We note that the Euclidean and spherical metrics are equivalent on bounded sets $M \subset \mathbb{C}$ that lie inside a fixed disk $\{|z| \leq R\}$, $R < \infty$. Indeed, if $M \subset \{|z| \leq R\}$ then (1.14) implies that for all $z_1, z_2 \in M$ we have

$$\frac{2}{1+R^2}|z_2 - z_1| \leq \rho(z_1, z_2) \leq 2|z_2 - z_1| \tag{1.16}$$

(this will be elaborated in the next section). Because of that the spherical metric is usually used only for unbounded sets. Typically, we will use the Euclidean metric for $\mathbb{C}$ and the spherical metric for $\overline{\mathbb{C}}$.

Now is the time for a little history. We find the first mention of the complex numbers as square rots of negative numbers in the book "Ars Magna" by Girolamo Cardano published in 1545. He thought that such numbers could be introduced in mathematics but opined that this would be useless: "Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, we obtain $25 - (-15)$. Therefore the product is 40. .... and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless." The baselessness of his verdict was realized fairly soon: Raphael Bombelli published his "Algebra" in 1572 where he introduced the algebraic operations over the complex numbers and explained how they may be used for solving the cubic equations. One may find in Bombelli's book the relation $(2 + \sqrt{-121})^{1/3} + (2 - \sqrt{-121})^{1/3} = 4$. Still, the complex numbers remained somewhat of a mystery for a long time. Leibnitz considered them to be "a beautiful and majestic refuge of the human spirit", but he also thought that it was impossible to factor $x^4 + 1$ into a product of two quadratic polynomials (though this is done in an elementary way with the help of complex numbers).

The active use of complex numbers in mathematics began with the works of Leonard Euler. He has also discovered the relation $e^{i\phi} = \cos\phi + i\sin\phi$. The geometric interpretation of complex numbers as planar vectors appeared first in the work of the Danish geographical surveyor Caspar Wessel in 1799 and later in the work of Jean Robert Argand in 1806. These papers were not widely known - even Cauchy who has obtained numerous fundamental results in complex analysis considered early in his career the complex numbers simply as symbols that were convenient for calculations, and equality of two complex numbers as a shorthand notation for equality of two real-valued variables.

The first systematic description of complex numbers, operations over them, and their geometric interpretation were given by Carl Friedreich Gauss in 1831 in his memoir "Theoria residuorum biquadraticorum". He has also introduced the name "complex numbers".

1.2 The topology of the complex plane

We have introduced distances on $\mathbb{C}$ and $\overline{\mathbb{C}}$ that turned them into metric spaces. We will now introduce the two topologies that correspond to these metrics.

Let $\varepsilon > 0$ then an ε-neighborhood $U(z_0, \varepsilon)$ of $z_0 \in \mathbb{C}$ in the Euclidean metric is the disk of radius ε centered at z_0, that is, the set of points $z \in \mathbb{C}$ that satisfy the inequality

$$|z - z_0| < \varepsilon. \tag{1.17}$$

An ε-neighborhood of a point $z_0 \in \overline{\mathbb{C}}$ is the set of all points $z \in \overline{\mathbb{C}}$ such that

$$\rho(z, z_0) < \varepsilon. \tag{1.18}$$

Expression (1.15) shows that the inequality $\rho(z, \infty) < \varepsilon$ is equivalent to $|z| > \sqrt{\frac{4}{\varepsilon^2} - 1}$. Therefore an ε-neighborhood of the point at infinity is the outside of a disk centered at the origin complemented by $z = \infty$.

We say that a set Ω in $\mathbb{C}$ (or $\overline{\mathbb{C}}$) is *open* if for any point $z_0 \in \Omega$ there exists a neighborhood of z_0 that is contained in Ω. It is straightforward to verify that this notion of an open set turns $\mathbb{C}$ and $\overline{\mathbb{C}}$ into *topological spaces*, that is, the usual axioms of a topological space are satisfied.

Sometimes it will be convenient to make use of the so called *punctured neighborhoods*, that is, the sets of the points $z \in \mathbb{C}$ (or $z \in \overline{\mathbb{C}}$) that satisfy

$$0 < |z - z_0| < \varepsilon, \quad 0 < \rho(z, z_0) < \varepsilon. \tag{1.19}$$

We will introduce in this Section the basic topological notions that we will constantly use in the sequel.

Definition 1.9 *A point $z_0 \in \mathbb{C}$ (resp. in $\overline{\mathbb{C}}$) is a limit point of the set $M \subset \mathbb{C}$ (resp. $\overline{\mathbb{C}}$) if there is at least one point of M in any punctured neighborhood of z_0 in the topology of $\mathbb{C}$ (resp. $\overline{\mathbb{C}}$). A set M is said to be closed if it contains all of its limit points. The union of M and all its limit points is called the closure of M and is denoted $\overline{M}$.*

Example 1.10 The set $\mathbb{Z}$ of all integers $\{0, \pm 1, \pm 2, \dots\}$ has no limit points in $\mathbb{C}$ and is therefore closed in $\mathbb{C}$. It has one limit point $z = \infty$ in $\overline{\mathbb{C}}$ that does not belong to $\mathbb{Z}$. Therefore $\mathbb{Z}$ is not closed in $\overline{\mathbb{C}}$.

Exercise 1.11 *Show that any infinite set in $\overline{\mathbb{C}}$ has at least one limit point (compactness principle).*

This principle expresses the completeness (as a metric space) of the sphere of complex numbers and may be proved using the completeness of the real numbers. We leave the proof to the reader. However, as Example 1.10 shows, this principle fails in $\mathbb{C}$. Nevertheless it holds for infinite bounded subsets of $\mathbb{C}$, that is, sets that are contained in a disk $\{|z| < R\}$, $R < \infty$.

Inequality (1.16) shows that a point $z_0 \neq \infty$ is a limit point of a set M in the topology of $\mathbb{C}$ if and only if it is a limit point of M in the topology of $\overline{\mathbb{C}}$. In other words, when we talk about finite limit points we may use either the Euclidean or the spherical metric. That is what the equivalence of these two metrics on bounded sets, that we have mentioned before, means.

Definition 1.12 *A sequence $\{a_n\}$ is a mapping from the set $\mathbb{N}$ of non-negative integers into $\mathbb{C}$ (or $\overline{\mathbb{C}}$). A point $a \in \mathbb{C}$ (or $\overline{\mathbb{C}}$) is a limit point of the sequence $\{a_n\}$ if any neighborhood of a in the topology of $\mathbb{C}$ (or $\overline{\mathbb{C}}$) contains infinitely many elements of the sequence. A sequence $\{a_n\}$ converges to a if a is its only limit point. Then we write*

$$\lim_{n\to\infty} a_n = a. \tag{1.20}$$

Remark 1.13 *The notions of the limit point of a sequence $\{a_n\}$ and of the set of values $\{a_n\}$ are different. For instance, the sequence $\{1, 1, 1, \dots\}$ has a limit point $a = 1$, while the set of values consists of only one point $z = 1$ and has no limit points.*

Exercise 1.14 *Show that 1) A sequence $\{a_n\}$ converges to a if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $|a_n - a| < \varepsilon$ for all $n \geq N$ (if $a \neq \infty$), or $\rho(a_n, a) < \varepsilon$ (if $a = \infty$). 2) A point a is a limit point of a sequence $\{a_n\}$ if and only if there exists a subsequence $\{a_{n_k}\}$ that converges to a.*

The complex equation (1.20) is equivalent to two real equations. Indeed, (1.20) is equivalent to

$$\lim_{n\to\infty} |a_n - a| = 0, \tag{1.21}$$

where the limit above is understood in the usual sense of convergence of real-valued sequences. Let $a \neq \infty$, then without any loss of generality we may assume that $a_n \neq \infty$ (because if $a \neq \infty$ then there exists N so that $a_n \neq \infty$ for $n > N$ and we may restrict ourselves to $n > N$) and let $a_n = \alpha_n + i\beta_n$, $a = \alpha + i\beta$ (for $a = \infty$ the real and imaginary parts are not defined). Then we have

$$\max(|\alpha_n - \alpha|, |\beta_n - \beta|) \leq \sqrt{|\alpha_n - \alpha|^2 + |\beta_n - \beta|^2} \leq |\alpha_n - \alpha| + |\beta_n - \beta|$$

and hence (1.21) and the squeezing theorem imply that (1.20) is equivalent to a pair of equalities

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta. \tag{1.22}$$

In the case when $a \neq 0$ and $a \neq \infty$ we may assume that $a_n \neq 0$ and $a_n \neq \infty$ and write $a_n = r_n e^{i\phi_n}$, $a = re^{i\phi}$. Then

$$|a_n - a|^2 = r^2 + r_n^2 - 2rr_n \cos(\phi - \phi_n) = (r - r_n)^2 + 2rr_n(1 - \cos(\phi - \phi_n)) \tag{1.23}$$

and hence (1.20) holds if

$$\lim_{n\to\infty} r_n = r, \quad \lim_{n\to\infty} \phi_n = \phi. \tag{1.24}$$

Conversely, if (1.20) holds then (1.23) implies that the first equality in (1.24) holds and that $\lim_{n\to\infty} \cos(\phi - \phi_n) = 1$. Therefore if we choose $\phi_n \in [0, 2\pi)$ then (1.20) implies also the second equality in (1.24).

Exercise 1.15 *Show that 1) the sequence $a_n = e^{in}$ diverges, and 2) if a series $\sum_{n=1}^{\infty} a_n$ converges and $|\arg a_n| \le \alpha < \pi/2$, then the series converges absolutely. Here $\arg a_n$ is the value of Arg a_n that satisfies $-\pi < \arg a_n \le \pi$.*

We will sometimes use the notion of the distance between two sets M and N, which is equal to the least upper bound of all distances between pairs of points from M and N:

$$\rho(M, N) = \inf_{z \in M, z' \in N} \rho(z, z'). \tag{1.25}$$

One may use the Euclidean metric to define the distance between sets as well, of course.

Theorem 1.16 *Let M and N be two non-overlapping closed sets: $M \cap N = \emptyset$, then the distance between M and N is positive.*

Proof. Let us assume that $\rho(M, N) = 0$. Then there exist two sequences of points $z_n \in M$ and $z'_n \in N$ so that $\lim_{n\to\infty} \rho(z_n, z'_n) = 0$. According to the compactness principle the sequences z_n and z'_n have limit points z and z', respectively. Moreover, since both M and N are closed, we have $z \in M$ and $z' \in N$. Then there exist a subsequence $n_k \to \infty$ so that *both* $z_{n_k} \to z$ and $z'_{n_k} \to z'$. The triangle inequality for the spherical metric implies that

$$\rho(z, z') \le \rho(z, z_{n_k}) + \rho(z_{n_k}, z'_{n_k}) + \rho(z'_{n_k}, z').$$

The right side tends to zero as $k \to \infty$ while the left side does not depend on k. Therefore, passing to the limit $k \to \infty$ we obtain $\rho(z, z') = 0$ and thus $z = z'$. However, $z \in M$ and $z' \in N$, which contradicts the assumption that $M \cap N = \emptyset$. □

1.3 Paths and curves

Definition 1.17 *A path γ is a continuous map of an interval $[\alpha, \beta]$ of the real axis into the complex plane $\mathbb{C}$ (or $\overline{\mathbb{C}}$). In other words, a path is a complex valued function $z = \gamma(t)$ of a real argument t, that is continuous at every point $t_0 \in [\alpha, \beta]$ in the following sense: for any $\varepsilon > 0$ there exists $\delta > 0$ so that $|\gamma(t) - \gamma(t_0)| < \varepsilon$ (or $\rho(\gamma(t), \gamma(t_0)) < \varepsilon$ if $\gamma(t_0) = \infty$) provided that $|t - t_0| < \delta$. The points $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ are called the endpoints of the path γ. The path is closed if $\gamma(\alpha) = \gamma(\beta)$. We say that a path γ lies in a set M if $\gamma(t) \in M$ for all $t \in [\alpha, \beta]$.*

Sometimes it is convenient to distinguish between a path and a curve. In order to introduce the latter we say that two paths

$$\gamma_1 : [\alpha_1, \beta_1] \to \overline{\mathbb{C}} \text{ and } \gamma_2 : [\alpha_2, \beta_2] \to \overline{\mathbb{C}}$$

are *equivalent* ($\gamma_1 \sim \gamma_2$) if there exists an increasing continuous function

$$\tau : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2] \tag{1.26}$$

such that $\tau(\alpha_1) = \alpha_2$, $\tau(\beta_1) = \beta_2$ and so that $\gamma_1(t) = \gamma_2(\tau(t))$ for all $t \in [\alpha_1, \beta_1]$.

Exercise 1.18 *Verify that relation $\sim$ is reflexive: $\gamma \sim \gamma$, symmetric: if $\gamma_1 \sim \gamma_2$, then $\gamma_2 \sim \gamma_1$ and transitive: if $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$ then $\gamma_1 \sim \gamma_3$.*

Example 1.19 Let us consider the paths $\gamma_1(t) = t$, $t \in [0, 1]$; $\gamma_2(t) = \sin t$, $t \in [0, \pi/2]$; $\gamma_3(t) = \cos t$, $t \in [0, \pi/2]$ and $\gamma_4(t) = \sin t$, $t \in [0, \pi]$. The set of values of $\gamma_j(t)$ is always the same: the interval $[0, 1]$. However, we only have $\gamma_1 \sim \gamma_2$. These two paths trace $[0, 1]$ from left to right once. The paths γ_3 and γ_4 are neither equivalent to these two, nor to each other: the interval $[0, 1]$ is traced in a different way by those paths: γ_3 traces it from right to left, while γ_4 traces $[0, 1]$ twice.

Exercise 1.20 Which of the following paths: a) $e^{2\pi it}$, $t \in [0, 1]$; b) $e^{4\pi it}$, $t \in [0, 1]$; c) $e^{-2\pi it}$, $t \in [0, 1]$; d) $e^{4\pi i \sin t}$, $t \in [0, \pi/6]$ are equivalent to each other?

Definition 1.21 *A curve is an equivalence class of paths. Sometimes, when this will cause no confusion, we will use the word 'curve' to describe a set $\gamma \in \overline{\mathbb{C}}$ that may be represented as an image of an interval $[\alpha, \beta]$ under a continuous map $z = \gamma(t)$.*

Below we will introduce some restrictions on the curves and paths that we will consider. We say that $\gamma : [\alpha, \beta] \to \overline{\mathbb{C}}$ is a *Jordan* path if the map γ is continuous and *one-to-one*. The definition of a closed Jordan path is left to the reader as an exercise.

A path $\gamma : [\alpha, \beta] \to \mathbb{C}$ ($\gamma(t) = x(t) + iy(t)$) is *continuously differentiable* if derivative $\gamma'(t) := x'(t) + iy'(t)$ exists for all $t \in [\alpha, \beta]$. A continuously differentiable path is said to be *smooth* if $\gamma'(t) \neq 0$ for all $t \in [\alpha, \beta]$. This condition is introduced in order to avoid singularities. A path is called *piecewise smooth* if $\gamma(t)$ is continuous on $[\alpha, \beta]$, and $[\alpha, \beta]$ may be divided into a finite number of closed sub-intervals so that the restriction of $\gamma(t)$ on each of them is a smooth path.

We will also use the standard notation to describe smoothness of functions and paths: the class of continuous functions is denoted C, or C^0, the class of continuously differentiable functions is denoted C^1, etc. A function that has n continuous derivatives is said to be a C^n-function.

Example 1.22 The paths γ_1, γ_2 and γ_3 of the previous example are Jordan, while γ_4 is not Jordan. The circle $z = e^{it}$, $t \in [0, 2\pi]$ is a closed smooth Jordan path; the four-petal rose $z = e^{it} \cos 2t$, $t \in [0, 2\pi]$ is a smooth non-Jordan path; the semi-cubic parabola $z = t^2(t + i)$, $t \in [-1, 1]$ is a Jordan continuously differentiable piecewise smooth path. The path $z = t\left(1 + i \sin\left(\frac{1}{t}\right)\right)$, $t \in [-1/\pi, 1/\pi]$ is a Jordan non-piecewise smooth path.

One may introduce similar notions for curves. A *Jordan curve* is a class of paths that are equivalent to some Jordan path (observe that since the change of variables (1.26) is one-to-one, all paths equivalent to a Jordan path are also Jordan).

The definition of a smooth curve is slightly more delicate: this notion has to be invariant with respect to a replacement of a path that represents a given curve by an equivalent one. However, a continuous monotone change of variables (1.26) may map a smooth path onto a non-smooth one unless we impose some additional conditions on the functions τ allowed in (1.26).

More precisely, a smooth curve is a class of paths that may be obtained out of a smooth path by all possible re-parameterizations (1.26) with $\tau(s)$ being a continuously differentiable function with a positive derivative. One may define a piecewise smooth curve in a similar fashion: the change of variables has to be continuous everywhere, and in addition have a continuous positive derivative except possibly at a finite set of points.

Sometimes we will use a more geometric interpretation of a curve, and say that a Jordan, or smooth, or piecewise smooth curve is a set of points $\gamma \subset \mathbb{C}$ that may be represented as the image of an interval $[\alpha, \beta]$ under a map $z = \gamma(t)$ that defines a Jordan, smooth or piecewise smooth path.

1.4 Domains

We say that a set D is *pathwise-connected* if for any two points $a, b \in D$ there exists a path that lies in D and has endpoints a and b.

Definition 1.23 *A domain D is a subset of $\mathbb{C}$ (or $\overline{\mathbb{C}}$) that is both open and pathwise-connected.*

The limit points of a domain D that do not belong to D are called the *boundary points* of D. These are the points z so that any neighborhood of z contains some points in D and at least one point not in D. Indeed, if $z_0 \in \partial D$ then any neighborhood of z contains a point from D since z_0 is a limit point of D, and it also contains z_0 itself that does not lie in D. Conversely, if any neighborhood of z_0 contains some points in D and at least one point not in D then $z_0 \notin D$ since D is open, and z_0 is a limit point of D, so that $z_0 \in \partial D$. The collection of all boundary points of D is called the *boundary* of D and is denoted by ∂D. The *closure* of D is the set $\bar{D} = D \cup \partial D$. The *complement* of D is the set $D^c = \mathbb{C} \backslash \bar{D}$, the points z that lie in D^c are called the *outer points* of D.

Exercise 1.24 Show that the set D^c is open.

Theorem 1.25 *The boundary ∂D of any domain D is a closed set.*

Proof. Let ζ_0 be a limit point of ∂D. We have to show that $\zeta_0 \in \partial D$. Let U be a punctured neighborhood of ζ_0. Then U contains a point $\zeta \in \partial D$. Furthermore, there exists a neighborhood V of ζ so that $V \subset U$. However, since ζ is a boundary point of D, the set V must contain points both from D and not from D. Therefore U also contains both points from D and not in D and hence $\zeta_0 \in \partial D$. □

We will sometimes need some additional restrictions on the boundary of domains. The following definition is useful for these purposes.

Definition 1.26 *The set M is connected if it is impossible to split it as $M = M_1 \cup M_2$ so that both M_1 and M_2 are not empty while the intersections $\bar{M}_1 \cap M_2$ and $M_1 \cap \bar{M}_2$ are empty.*

Exercise 1.27 Show that a closed set is connected if and only if it cannot be represented as a union of two non-overlapping non-empty closed sets.

One may show that a pathwise connected set is connected. The converse, however, is not true.

Let M be a non-connected set. A subset $N \subset M$ is called a *connected component* if M if N is connected and is not contained in any other connected subset of M. One may show that any set is the union of its connected components (though, it may have infinitely many connected components).

A domain $D \subset \overline{\mathbb{C}}$ is *simply connected* if its boundary ∂D is a connected set.

Example 1.28 (a) The interior of figure eight is not a domain since it is not pathwise-connected. (b) The set of points between two circles tangent to each other is a simply connected domain.

Sometimes we will impose further conditions. A domain D is *Jordan* if its boundary is a union of closed Jordan curves. A domain D is *bounded* if it lies inside a bounded disk $\{|z| < R,\ R < \infty\}$. A set M is properly embedded in a domain D if its closure $\bar{M}$ in $\overline{\mathbb{C}}$ is contained in D. We will then write $M \subset\subset D$.

We will often make use of the following theorem. A neighborhood of a point z in the relative topology of a set M is the intersection of a usual neighborhood of z and M.

Theorem 1.29 *Let $M \subset \overline{\mathbb{C}}$ be a connected set and let N be its non-empty subset. If N is both open and closed in the relative topology of M then $M = N$.*

Proof. Let the set $N' = M \backslash N$ be non-empty. The closure $\bar{N}$ of N in the usual topology of $\overline{\mathbb{C}}$ is the union of its closure $(\bar{N})_M$ of N in the relative topology of M, and some other set (possibly empty) that does not intersect M. Therefore we have $\bar{N} \cap N' = (\bar{N})_M \cap N'$. However, N is closed in the relative topology of M so that $(\bar{N})_M = N$ and hence $(\bar{N})_M \cap N' = N \cap N' = \emptyset$.

Furthermore, since N is also open in the relative topology of M, its complement N' in the same topology is closed (the limit points of N' may not belong to N since the latter is open, hence they belong to N' itself). Therefore we may apply the previous argument to N' and conclude that $\bar{N}' \cap N$ is empty. This contradicts the assumption that M is connected. $\square$

2 Functions of a complex variable

2.1 Functions

A complex valued function $f : M \to \mathbb{C}$, where $M \subset \overline{\mathbb{C}}$ is one-to-one, if for any two points $z_1 \neq z_2$ in M the images $w_1 = f(z_1)$, $w_2 = f(z_2)$ are different: $w_1 \neq w_2$. Later we will need the notion of a multi-valued function that will be introduced in Chapter 3.

Defining a function $f : M \to \mathbb{C}$ is equivalent to defining two real-valued functions

$$u = u(z), \quad v = v(z). \tag{2.1}$$

Here $u : M \to \mathbb{R}$ and $v : M \to \mathbb{R}$ are the real and imaginary parts of f: $f(x+iy) = u(x+iy) + iv(x+iy)$. Furthermore, if $f \neq 0, \neq \infty$ (this notation means that $f(z) \neq 0$ and $f(z) \neq \infty$ for all $z \in M$) we may write $f = \rho e^{i\psi}$ with

$$\rho = \rho(z), \quad \psi = \psi(z) + 2k\pi, \quad (k = 0, \pm 1, \dots). \tag{2.2}$$

At the points where $f = 0$, or $f = \infty$, the function $\rho = 0$ or $\rho = \infty$ while ψ is not defined.

We will constantly use the geometric interpretation of a complex valued function. The form (2.1) suggests representing f as two surfaces $u = u(x, y)$, $v = v(x, y)$ in the three-dimensional space. However, this is not convenient since it does not represent (u, v) as one complex number. Therefore we will represent a function $f : M \to \overline{\mathbb{C}}$ as a map of M into a sphere $\overline{\mathbb{C}}$.

We now turn to the basic notion of the limit of a function.

Definition 2.1 *Let the function f be defined in a punctured neighborhood of a point $a \in \overline{\mathbb{C}}$. We say that the number $A \in \overline{\mathbb{C}}$ is its limit as z goes to a and write*

$$\lim_{z \to a} f(z) = A, \tag{2.3}$$

if for any neighborhood U_A of A their exists a punctured neighborhood U'_a of a so that for all $z \in U'_a$ we have $f(z) \in U_A$. Equivalently, for any $\varepsilon > 0$ there exists $\delta > 0$ so that the inequality

$$0 < \rho(z, a) < \delta \tag{2.4}$$

implies

$$\rho(f(z), A) < \varepsilon. \tag{2.5}$$

If $a, A \neq \infty$ then (2.4) and (2.5) may be replaced by the inequalities $0 < |z - a| < \delta$ and $|f(z) - A| < \varepsilon$. If $a = \infty$ and $A \neq \infty$ then they may written as $\delta < |z| < \infty$, $|f(z) - A| < \varepsilon$. You may easily write them in the remaining cases $a \neq \infty$, $A = \infty$ and $a = A = \infty$.

We set $f = u + iv$. It is easy to check that for $A \neq \infty$, $A = A_1 + iA_2$, (2.3) is equivalent to two equalities

$$\lim_{z \to a} u(z) = A_1, \quad \lim_{z \to a} v(z) = A_2. \tag{2.6}$$

If we assume in addition that $A \neq 0$ and choose $\arg f$ appropriately then (2.3) may be written in polar coordinates as

$$\lim_{z \to a} |f(z)| = |A|, \quad \lim_{z \to a} \arg f(z) = \arg A. \tag{2.7}$$

The elementary theorems regarding the limits of functions in real analysis, such as on the limit of sums, products and ratios may be restated verbatim for the complex case and we do not dwell on their formulation and proof.

Sometimes we will talk about the limit of a function along a set. Let M be a set, a be its limit point and f a function defined on M. We say that f tends to A as z tends to a along M and write

$$\lim_{z\to a, z\in M} f(z) = A \tag{2.8}$$

if for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $z \in M$ and $0 < \rho(z,a) < \delta$ we have $\rho(f(z), A) < \varepsilon$.

Definition 2.2 *Let f be defined in a neighborhood of $a \in \overline{\mathbb{C}}$. We say that f is continuous at a if*

$$\lim_{z\to a} f(z) = f(a). \tag{2.9}$$

For the reasons we have just discussed the elementary theorems about the sum, product and ratio of continuous functions in real analysis translate immediately to the complex case.

One may also define continuity of f at a along a set M, for which a is a limit point, if the limit in (2.9) is understood along M. A function that is continuous at every point of M (along M) is said to be continuous on M. In particular if f is continuous at every point of a domain D it is continuous in the domain.

We recall some properties of continuous functions on closed sets $K \subset \overline{\mathbb{C}}$:

1. Any function f that is continuous on K is bounded on K, that is, there exists $A \geq 0$ so that $|f(z)| \leq A$ for all $z \in K$.
2. Any function f that is continuous on K attains its maximum and minimum, that is, there exist $z_1, z_2 \in K$ so that $|f(z_1)| \leq |f(z)| \leq |f(z_2)|$ for all $z \in K$.
3. Any function f that is continuous on K is uniformly continuous, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(z_1) - f(z_2)| < \varepsilon$ provided that $\rho(z_1, z_2) < \delta$.

The proofs of these properties are the same as in the real case and we do not present them here.

2.2 Differentiability

The notion of differentiability is intricately connected to linear approximations so we start with the discussion of linear functions of complex variables.

Definition 2.3 *A function $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, or $\mathbb{R}$-linear, respectively, if*
(a) $l(z_1 + z_2) = l(z_1) + l(z_2)$ for all $z_1, z_2 \in \mathbb{C}$,
(b) $l(\lambda z) = \lambda l(z)$ for all $\lambda \in \mathbb{C}$, or, respectively, $\lambda \in \mathbb{R}$.

Thus $\mathbb{R}$-linear functions are linear over the field of real numbers while $\mathbb{C}$-linear are linear over the field of complex numbers. The latter form a subset of the former.

Let us find the general form of an $\mathbb{R}$-linear function. We let $z = x + iy$, and use properties (a) and (b) to write $l(z) = xl(1) + yl(i)$. Let us denote $\alpha = l(1)$ and $\beta = l(i)$, and replace $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$. We obtain the following theorem.

Theorem 2.4 *Any $\mathbb{R}$-linear function has the form*

$$l(z) = az + b\bar{z}, \tag{2.10}$$

where $a = (\alpha - i\beta)/2$ and $b = (\alpha + i\beta)/2$ are complex valued constants.

Similarly writing $z = 1 \cdot z$ we obtain

Theorem 2.5 *Any $\mathbb{C}$-linear function has the form*

$$l(z) = az, \tag{2.11}$$

where $a = l(1)$ is a complex valued constant.

Theorem 2.6 *An $\mathbb{R}$-linear function is $\mathbb{C}$-linear if and only if*

$$l(iz) = il(z). \tag{2.12}$$

Proof. The necessity of (2.12) follows immediately from the definition of a $\mathbb{C}$-linear function. Theorem 2.4 implies that $l(z) = az + b\bar{z}$, so $l(iz) = i(az - b\bar{z})$. Therefore, $l(iz) = il(z)$ if and only if

$$iaz - b\bar{z} = iaz + ib\bar{z}.$$

Therefore if $l(iz) = il(z)$ for all $z \in \mathbb{C}$ then $b = 0$ and hence l is $\mathbb{C}$-linear.

We set $a = a_1 + ia_2$, $b = b_1 + ib_2$, and also $z = x + iy$, $w = u + iv$. We may represent an $\mathbb{R}$-linear function $w = az + b\bar{z}$ as two real equations

$$u = (a_1 + b_1)x - (a_2 - b_2)y, \quad v = (a_2 + b_2)x + (a_1 - b_1)y.$$

Therefore geometrically an $\mathbb{R}$-linear function is an affine transform of a plane $\mathbf{y} = A\mathbf{x}$ with the matrix

$$A = \begin{pmatrix} a_1 + b_1 & -(a_2 - b_2) \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}. \tag{2.13}$$

Its Jacobian is

$$J = a_1^2 - b_1^2 + a_2^2 - b_2^2 = |a|^2 - |b|^2. \tag{2.14}$$

This transformation is non-singular when $|a| \neq |b|$. It transforms lines into lines, parallel lines into parallel lines and squares into parallelograms. It preserves the orientation when $|a| > |b|$ and changes it if $|a| < |b|$.

However, a $\mathbb{C}$-linear transformation $w = az$ may not change orientation since its jacobian $J = |a|^2 \geq 0$. They are not singular unless $a = 0$. Letting $a = |a|e^{i\alpha}$ and recalling the geometric interpretation of multiplication of complex numbers we find that a non-degenerate $\mathbb{C}$-linear transformation

$$w = |a|e^{i\alpha}z \tag{2.15}$$

is the composition of dilation by $|a|$ and rotation by the angle α. Such transformations preserve angles and map squares onto squares.

Exercise 2.7 Let $b = 0$ in (2.13) and decompose A as a product of two matrices, one corresponding to dilation by $|a|$, another to rotation by α.

We note that preservation of angles characterizes $\mathbb{C}$-linear transformations. Moreover, the following theorem holds.

Theorem 2.8 *If an $\mathbb{R}$-linear transformation $w = az + b\bar{z}$ preserves orientation and angles between three non-parallel vectors $e^{i\alpha_1}$, $e^{i\alpha_2}$, $e^{i\alpha_3}$, $\alpha_j \in \mathbb{R}$, $j = 1, 2, 3$, then w is $\mathbb{C}$-linear.*

Proof. Let us assume that $w(e^{i\alpha_1}) = \rho e^{i\beta_1}$ and define $w'(z) = e^{-i\beta_1} w(ze^{i\alpha_1})$. Then $w'(z) = a'z + b'\bar{z}$ with

$$a' = ae^{i(\alpha_1 - \beta_1)}a, \quad b' = be^{-i(\alpha_1+\beta_1)},$$

and, moreover $w'(1) = e^{-i\beta_1}\rho e^{i\beta_1} = \rho > 0$. Therefore we have $a' + b' > 0$. Furthermore, w' preserves the orientation and angles between vectors $v_1 = 1$, $v_2 = e^{i(\alpha_2-\alpha_1)}$ and $v_3 = e^{i(\alpha_3-\alpha_1)}$. Since both v_1 and its image lie on the positive semi-axis and the angles between v_1 and v_2 and their images are the same, we have $w'(v_2) = h_2 v_2$ with $h_2 > 0$. This means that

$$a'e^{i\beta_2} + b'e^{-i\beta_2} = h_2 e^{i\beta_2}, \quad \beta_2 = \alpha_2 - \alpha_1,$$

and similarly

$$a'e^{i\beta_3} + b'e^{-i\beta_3} = h_3 e^{i\beta_3}, \quad \beta_3 = \alpha_3 - \alpha_1,$$

with $h_3 > 0$. Hence we have

$$a' + b' > 0, \quad a' + b'e^{-2i\beta_2} > 0, \quad a' + b'e^{-2i\beta_3} > 0.$$

This means that unless $b' = 0$ there exist three different vectors that connect the vector a' to the real axis, all having the same length $|b'|$. This is impossible, and hence $b' = 0$ and w is $\mathbb{C}$-linear.

Exercise 2.9 (a) Give an example of an $\mathbb{R}$-linear transformation that is not $\mathbb{C}$-linear but preserves angles between two vectors.
(b) Show that if an $\mathbb{R}$-linear transformation preserves orientation and maps some square onto a square it is $\mathbb{C}$-linear.

Now we may turn to the notion of differentiability of complex functions. Intuitively, a function is differentiable if it is well approximated by linear functions. Two different definitions of linear functions that we have introduced lead to different notions of differentiability.

Definition 2.10 *Let $z \in \mathbb{C}$ and let U be a neighborhood of z. A function $f : U \to \mathbb{C}$ is $\mathbb{R}$-differentiable (respectively, $\mathbb{C}$-differentiable) at the point z if we have for sufficiently small $|\Delta z|$:*

$$\Delta f = f(z + \Delta z) - f(z) = l(\Delta z) + o(\Delta z), \tag{2.16}$$

where $l(\Delta z)$ (with z fixed) is an $\mathbb{R}$-linear (respectively, $\mathbb{C}$-linear) function of Δz, and $o(\Delta z)$ satisfies $o(\Delta z)/\Delta z \to 0$ as $\Delta z \to 0$. The function l is called the differential of f at z and is denoted df.

The increment of an $\mathbb{R}$-differentiable function has, therefore, the form

$$\Delta f = a\Delta z + b\overline{\Delta z} + o(\Delta z). \tag{2.17}$$

Taking the increment $\Delta z = \Delta x$ along the x-axis, so that $\overline{\Delta z} = \Delta x$ and passing to the limit $\Delta x \to 0$ we obtain

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{\partial f}{\partial x} = a + b.$$

Similarly, taking $\Delta z = i\Delta y$ (the increment is long the y-axis) so that $\overline{\Delta z} = -i\Delta y$ we obtain

$$\lim_{\Delta y \to 0} \frac{\Delta f}{i\Delta y} = \frac{1}{i}\frac{\partial f}{\partial y} = a - b.$$

The two relations above imply that

$$a = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad b = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

These coefficients are denoted as

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{2.18}$$

and are sometimes called the formal derivatives of f at the point z. They were first introduced by Riemann in 1851.

Exercise 2.11 Show that (a) $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$; (b) $\frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}$, $\frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}}$.

Using the obvious relations $dz = \Delta z$, $d\bar{z} = \Delta\bar{z}$ we arrive at the formula for the differential of $\mathbb{R}$-differentiable functions

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}. \tag{2.19}$$

Therefore, all the functions $f = u + iv$ such that u and v have usual differentials as functions of two real variables x and y turn out to be $\mathbb{R}$-differentiable. This notion does not bring any essential new ideas to analysis. The complex analysis really starts with the notion of $\mathbb{C}$-differentiability.

The increment of a $\mathbb{C}$-differentiable function has the form

$$\Delta f = a\Delta z + o(\Delta z) \tag{2.20}$$

and its differential is a $\mathbb{C}$-linear function of Δz (with z fixed). Expression (2.19) shows that $\mathbb{C}$-differentiable functions are distinguished from $\mathbb{R}$-differentiable ones by an additional condition

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{2.21}$$

If $f = u + iv$ then (2.18) shows that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

so that the complex equation (2.21) may be written as a pair of real equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.22}$$

The notion of complex differentiability is clearly very restrictive: while it is fairly difficult to construct an example of a continuous but nowhere real differentiable function, most trivial functions turn out to be non-differentiable in the complex sense. For example, the function $f(z) = x + 2iy$ is nowhere $\mathbb{C}$-differentiable: $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 2$ and conditions (2.22) fail everywhere.

Exercise 2.12 1. Show that $\mathbb{C}$-differentiable functions of the form $u(x) + iv(y)$ are necessarily $\mathbb{C}$-linear.
2. Let $f = u + iv$ be $\mathbb{C}$-differentiable in the whole plane $\mathbb{C}$ and $u = v^2$ everywhere. Show that $f = \text{const}$.

Let us consider the notion of a derivative starting with that of the directional derivative. We fix a point $z \in \mathbb{C}$, its neighborhood U and a function $f : U \to \mathbb{C}$. Setting $\Delta z = |\Delta z| e^{i\theta}$ we obtain from (2.17) and (2.19):

$$\Delta f = \frac{\partial f}{\partial z}|\Delta z| e^{i\theta} + \frac{\partial f}{\partial \bar{z}}|\Delta z| e^{-i\theta} + o(\Delta z).$$

We divide both sides by Δz, pass to the limit $|\Delta z| \to 0$ with θ fixed and obtain the derivative of f at the point z in direction θ:

$$\frac{\partial f}{\partial z_\theta} = \lim_{|\Delta z| \to 0, \arg z = \theta} \frac{\Delta f}{\Delta z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} e^{-2i\theta}. \tag{2.23}$$

This expression shows that when z is fixed and θ changes between 0 and 2π the point $\frac{\partial f}{\partial z_\theta}$ traverses twice a circle centered at $\frac{\partial f}{\partial z}$ with the radius $\left|\frac{\partial f}{\partial \bar{z}}\right|$.

Hence if $\frac{\partial f}{\partial \bar{z}} \neq 0$ then the directional derivative depends on direction θ, and only if $\frac{\partial f}{\partial \bar{z}} = 0$, that is, if f is $\mathbb{C}$-differentiable, all directional derivatives at z are the same.

Clearly, the derivative of f at z exists if and only if the latter condition holds. It is defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}. \tag{2.24}$$

The limit is understood in the topology of $\mathbb{C}$. It is also clear that if $f'(z)$ exists then it is equal to $\frac{\partial f}{\partial z}$. This proposition is so important despite its simplicity that we formulate it as a separate theorem.

Theorem 2.13 *Complex differentiability of f at z is equivalent to the existence of the derivative $f'(z)$ at z.*

Proof. If f is $\mathbb{C}$-differentiable at z then (2.20) with $a = \dfrac{\partial f}{\partial z}$ implies that

$$\Delta f = \frac{\partial f}{\partial z}\Delta z + o(\Delta z).$$

Then, since $\lim\limits_{\Delta z\to 0}\dfrac{o(\Delta z)}{\Delta z} = 0$, we obtain that the limit $f'(z) = \lim\limits_{\Delta z\to 0}\dfrac{\Delta f}{\Delta z}$ exists and is equal to $\dfrac{\partial f}{\partial z}$. Conversely, if $f'(z)$ exists then by the definition of the limit we have

$$\frac{\Delta f}{\Delta z} = f'(z) + \alpha(\Delta z),$$

where $\alpha(\Delta z) \to 0$ as $\Delta z \to 0$. Therefore the increment $\Delta f = f'(z)\Delta z + \alpha(\Delta z)\Delta z$ may be split into two parts so that the first is linear in Δz and the second is $o(\Delta z)$, which is equivalent to $\mathbb{C}$-differentiability of f at z.□

The definition of the derivative of a function of a complex variable is exactly the same as in the real analysis, and all the arithmetic rules of dealing with derivatives translate into the complex realm without any changes. Thus the elementary theorems regarding derivatives of a sum, product, ratio, composition and inverse function apply verbatim in the complex case. We skip their formulation and proofs.

Let us mention a remark useful in computations. The derivative of a function $f = u + iv$ does not depend on direction (if it exists), so it may be computed in particular in the direction of the x-axis:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{2.25}$$

We should have convinced ourselves that the notion of $\mathbb{C}$-differentiability is very natural. However, as we will see later, $\mathbb{C}$-differentiability at one point is not sufficient to build an interesting theory. Therefore we will require $\mathbb{C}$-differentiability not at one point but in a whole neighborhood.

Definition 2.14 *A function f is holomorphic (or analytic) at a point $z \in \mathbb{C}$ if it is $\mathbb{C}$-differentiable in a neighborhood of z.*

Example 2.15 The function $f(z) = |z|^2 = z\bar{z}$ is clearly $\mathbb{R}$-differentiable everywhere in $\mathbb{C}$. However, $\dfrac{\partial f}{\partial \bar{z}} = 0$ only at $z = 0$, so f is only $\mathbb{C}$-differentiable at $z = 0$ but is not holomorphic at this point.

The set of functions holomorphic at a point z is denoted by $\mathcal{O}_z$. Sums and products of functions in $\mathcal{O}_z$ also belong to $\mathcal{O}_z$, so this set is a ring. We note that the ratio f/g of two functions in $\mathcal{O}_z$ might not belong to $\mathcal{O}_z$ if $g(z) = 0$.

Functions that are $\mathbb{C}$-differentiable at all points of an open set $D \subset \mathbb{C}$ are clearly also holomorphic at all points $z \in D$. We say that such functions are holomorphic in D and denote their collection by $\mathcal{O}(D)$. The set $\mathcal{O}(D)$ is also a ring. In general a function is holomorphic on a set $M \subset \mathbb{C}$ if it may extended to a function that is holomorphic on an open set D that contains M.

Finally we say that f is holomorphic at infinity if the function $g(z) = f(1/z)$ is holomorphic at $z = 0$. This definition allows to consider functions holomorphic in $\overline{\mathbb{C}}$. However, the notion of derivative at $z = \infty$ is not defined.

The notion of complex differentiability lies at the heart of complex analysis. A special role among the founders of complex analysis was played by Leonard Euler, "the teacher of all mathematicians of the second half of the XVIIIth century" according to Laplace. Let us describe briefly his life and work.

Euler was born in 1707 into a family of a Swiss pastor and obtained his Master's diploma at Basel in 1724. He studied theology for some time but then focused solely on mathematics and its applications. Nineteen-year old Euler moved to Saint Petersburg in 1727 and took the vacant position in physiology at the Russian Academy of Sciences that had been created shortly before his arrival. Nevertheless he started to work in mathematics, and with remarkable productivity on top of that: he published more than 50 papers during his first fourteen year long stay at Saint Petersburg, being also actively involved in teaching and various practical problems.

Euler moved to Berlin in 1741 where he worked until 1766 but he kept his ties to the Saint Petersburg Academy, publishing more than 100 papers and books in its publications. Then he returned to Saint Petersburg where he stayed until his death. Despite almost complete blindness Euler prepared more than 400 papers during his second seventeen year long stay in Saint Petersburg.

In his famous monographies "Introductio in analys in infinitorum" (1748), "Institutiones calculi integralis" (1755) and "Institutiones calculi integralis" (1768-70) Euler has developed mathematical analysis as a branch of mathematical science for the first time. He was the creator of calculus of variations, theory of partial differential equations and differential geometry and obtained outstanding results in number theory.

Euler was actively involved in applied problems alongside his theoretical work. For instance he took part in the creation of geographic maps of Russia and in the expert analysis of the project of a one-arc bridge over the Neva river proposed by I. Kulibin, he studied the motion of objects through the air and computed the critical stress of columns. His books include "Mechanica" (1736-37), a book on Lunar motion (1772) and a definitive book on navigation (1778). Euler died in 1783 and was buried in Saint Petersburg. His descendants stayed in Russia: two of his sons were members of the Russian Academy of Sciences and a third was a general in the Russian army.

Euler has introduced the elementary functions of a complex variable in the books mentioned above and found relations between them, such as the Euler formula $e^{i\phi} = \cos\phi + i\sin\phi$ mentioned previously and systematically used complex substitutions for computations of integrals. In his book on the basics of fluid motion (1755) Euler related the components u and v of the flow to expressions $udy - vdx$ and $udx + vdy$. Following D'Alembert who published his work three years earlier Euler formulated conditions that turn the above into exact differential forms:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2.26}$$

He found the general form of a solution of such system:

$$\begin{aligned} u - iv &= \frac{1}{2}\phi(x+iy) - \frac{i}{2}\psi(x+iy) \\ u + iv &= \frac{1}{2}\phi(x-iy) + \frac{i}{2}\psi(x-iy), \end{aligned}$$

where ϕ and ψ are arbitrary (according to Euler) functions. Relations (2.26) are simply the conditions for complex differentiability of the function $f = u - iv$ and have a simple physical interpretation (see the next section). Euler has also written down the usual conditions of differentiability (2.22) that differ from (2.26) by a sign. In 1776 the 69 year old Euler wrote a paper where he pointed out that these conditions imply that the expression $(u+iv)(dx+idy)$ is an exact differential form, and in 1777 he pointed out their application to cartography. Euler was the first mathematician to study systematically the functions of complex variables and their applications in analysis, hydrodynamics and cartography.

However, Euler did not have the total understanding of the full implications of complex differentiability. The main progress in this direction was started by the work of Cauchy 70 years later and then by Riemann 30 years after Cauchy. The two conditions of $\mathbb{C}$-differentiability,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the Cauchy-Riemann equations, though historically they should probably be called D'Alembert-Euler equations.

2.3 Geometric and Hydrodynamic Interpretations

The differentials of an $\mathbb{R}$-differentiable and, respectively, a $\mathbb{C}$-differentiable function at a point z have form

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}, \quad df = f'(z)dz. \tag{2.27}$$

The Jacobians of such maps are given by (see (2.14))

$$J_f(z) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2, \quad J_f(z) = |f'(z)|^2. \tag{2.28}$$

Let us assume that f is $\mathbb{R}$-differentiable at z and z is not a critical point of f, that is, $J_f(z) \neq 0$. The implicit function theorem implies that locally f is a homeomorphism, that is, there exists a neighborhood U of z so that f maps U continuously and one-to-one onto a neighborhood of $f(z)$. Expressions (2.28) show that in general J_f may have an arbitrary sign if f is just $\mathbb{R}$-differentiable. However, the critical points of a $\mathbb{C}$-differentiable map coincide with the points where derivative vanishes, while such maps preserve orientation at non-critical points: $J_f(z) = |f'(z)'|^2 > 0$.

Furthermore, an $\mathbb{R}$-differentiable map is said to be conformal at $z \in \mathbb{C}$ if its differential df at z is a non-degenerate transformation that is a composition of dilation and rotation. Since the latter property characterizes $\mathbb{C}$-linear maps we obtain the following geometric interpretation of $\mathbb{C}$-differentiability:

Complex differentiability of f at a point z together with the condition $f'(z) \neq 0$ is equivalent to f being a conformal map at z.

A map $f : D \to \mathbb{C}$ conformal at every point $z \in D$ is said to be conformal in D. It is realized by a holomorphic function in z with no critical points ($f'(z) \neq 0$ in D). Its differential at every point of the domain is a composition of a dilation and a rotation, in particular it conserves angles. Such mappings were first considered by Euler in 1777 in relation to his participation in the project of producing geographic maps of Russia. The name "conformal mapping" was introduced by F. Schubert in 1789.

So far we have studied differentials of maps. Let us look now at how the properties of the map itself depend on it being conformal. Assume that f is conformal in a neighborhood U of a point z and that f' is continuous in U[1]. Consider a smooth path $\gamma : I = [0, 1] \to U$ that starts at z, that is, $\gamma'(t) \neq 0$ for all $t \in I$ and $\gamma(0) = z$. Its image $\gamma_* = f \circ \gamma$ is also a smooth path since

$$\gamma_*'(t) = f'[\gamma(t)]\gamma'(t), \quad t \in I, \tag{2.29}$$

and f' is continuous and different from zero everywhere in U by assumption.

Geometrically $\gamma'(t) = \dot{x}(t) + i\dot{y}(t)$ is the vector tangent to γ at the point $\gamma(t)$, and $|\gamma'(t)|dt = \sqrt{\dot{x}^2 + \dot{y}^2}dt = ds$ is the differential of the arc length of γ at the same point. Similarly, $|\gamma_*(t)|dt = ds_*$ is the differential of the arc length of γ_* at the point $\gamma_*(t)$. We conclude from (2.29) at $t = 0$ that

$$|f'(z)| = \frac{|\gamma_*'(0|}{|\gamma'(0)|} = \frac{ds_*}{ds}. \tag{2.30}$$

Thus the modulus of $f'(z)$ is equal to the dilation coefficient at z under the mapping f.

The left side does not depend on the curve γ as long as $\gamma(0) = z$. Therefore under our assumptions all arcs are dilated by the same factor. Therefore a conformal map f has a circle property: it maps small circles centered at z into curves that differ from circles centered at $f(z)$ only by terms of the higher order.

Going back to (2.29) we see that

$$\arg f'(z) = \arg \gamma_*'(0) - \arg \gamma'(0), \tag{2.31}$$

so that $\arg f'(z)$ is the rotation angle of the tangent lines at z under f.

The left side also does not depend on the choice of γ as long as $\gamma(0) = z$, so that all such arcs are rotated by the same angle. Thus a conformal map f preserves angles: the angle between any two curves at z is equal to the angle between their images at $f(z)$.

[1]We will later see that existence of f' implies its continuity and, moreover, existence of derivatives of all orders.

If f is holomorphic at z but z is a critical point then the circle property holds in a degenerate form: the dilation coefficient of all curves at z is equal to 0. Angle preservation does not hold at all, for instance under the mapping $z \to z^2$ the angle between the lines $\arg z = \alpha_1$ and $\arg z = \alpha_2$ doubles! Moreover, smoothness of curves may be violated at a critical point. For instance a smooth curve $\gamma(t) = t+it^2$, $t \in [-1,1]$ is mapped under the same map $z \to z^2$ into the curve $\gamma_*(t) = t^2(1-t^2) + 2it^3$ with a cusp at $\gamma_*(0) = 0$.

Exercise 2.16 Let $u(x,y)$ and $v(x,y)$ be real valued $\mathbb{R}$-differentiable functions and let $\nabla u = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$, $\nabla v = \frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}$. Find the geometric meaning of the conditions $(\nabla u, \nabla v) = 0$ and $|\nabla u| = |\nabla v|$, and their relation to the $\mathbb{C}$-differentiability of $f = u+iv$ and the conformity of f.

Let us now find the hydrodynamic meaning of complex differentiability and derivative. We consider a steady two-dimensional flow. That means that the flow vector field $v = (v_1, v_2)$ does not depend on time. The flow is described by

$$v = v_1(x,y) + iv_2(x,y). \tag{2.32}$$

Let us assume that in a neighborhood U of the point z the functions v_1 and v_2 have continuous partial derivatives. We will also assume that the flow v is irrotational in U, that is,

$$\text{curl} v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \tag{2.33}$$

and incompressible:

$$\text{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{2.34}$$

at all $z \in U$.

Condition (2.33) implies the existence of a potential function ϕ such that $v = \nabla\phi$, that is,

$$v_1 = \frac{\partial \phi}{\partial x}, \quad v_2 = \frac{\partial \phi}{\partial y}. \tag{2.35}$$

The incompressibility condition (2.34) implies that there exists a stream function ψ so that

$$v_2 = -\frac{\partial \psi}{\partial x}, \quad v_1 = \frac{\partial \psi}{\partial x}. \tag{2.36}$$

We have $d\psi = -v_2 dx + v_1 dy = 0$ along the level set of ψ and thus $\frac{dy}{dx} = \frac{v_2}{v_1}$. This shows that the level set is an integral curve of v.

Consider now a complex function

$$f = \phi + i\psi, \tag{2.37}$$

that is called the complex potential of v. Relations (2.35) and (2.36) imply that ϕ and ψ satisfy

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.38}$$

The above conditions coincide with (2.22) and show that the complex potential f is holomorphic at $z \in U$.

Conversely let the function $f = \phi + i\psi$ be holomorphic in a neighborhood U of a point z, and let the functions ϕ and ψ be twice continuously differentiable. Define the vector field $v = \nabla\phi = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$. It is irrotational in U since $\mathrm{curl} v = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$. It is also incompressible since $\mathrm{div} v = \frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$. The complex potential of the vector field v is clearly the function f.

Therefore the function f is holomorphic if and only if it is the complex potential of a steady fluid flow that is both irrotational and incompressible.

It is easy to establish the hydrodynamic meaning of the derivative:

$$f' = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = v_1 - iv_2, \tag{2.39}$$

so that the derivative of the complex potential is the vector that is the complex conjugate of the flow vector. The critical points of f are the points where the flow vanishes.

Example 2.17 Let us find the complex potential of an infinitely deep flow over a flat bottom with a line obstacle of height h perpendicular to the bottom. This is a flow in the upper half-plane that goes around an interval of length h that we may consider lying on the imaginary axis.

The boundary of the domain consists, therefore, of the real axis and the interval $[0, ih]$ on the imaginary axis. The boundary must be the stream line of the flow. We set it to be the level set $\psi = 0$ and will assume that $\psi > 0$ everywhere in D. In order to find the complex potential f it suffices to find a conformal mapping of D onto the upper half-plane $\psi > 0$. One function that provides such a mapping may be obtained as follows. The mapping $z_1 = z^2$ maps D onto the plane without the half-line $\mathrm{Re} z_1 \geq -h^2$, $\mathrm{Im} z_1 = 0$. The map $z_2 = z_1 + h^2$ maps this half-line onto the positive semi-axis $\mathrm{Re} z_2 \geq 0$, $\mathrm{Im} z_2 = 0$. Now the mapping $w_2 = \sqrt{z_2} = \sqrt{|z_2|}e^{i(\arg z_2)/2}$ with $0 < \arg z_2 < 2\pi$ maps the complex plane without the positive semi-axis onto the upper half-plane. It remains to write explicitly the resulting map

$$w = \sqrt{z_2} = \sqrt{z_1 + h^2} = \sqrt{z^2 + h^2} \tag{2.40}$$

that provides the desired mapping of D onto the upper half-plane. We may obtain the equation for the stream-lines of the flow by writing $(\phi + i\psi)^2 = (x + iy)^2 + h^2$. The streamline $\psi = \psi_0$ is obtained by solving

$$\phi^2 - \psi_0^2 = h^2 + x^2 - y^2, \quad 2\phi\psi_0 = 2xy.$$

This leads to $\phi = xy/\psi_0$ and

$$y = \psi_0 \sqrt{1 + \frac{h^2}{x^2 + \psi_0^2}}. \tag{2.41}$$

The magnitude of the flow is $|v| = \left|\frac{dw}{dz}\right| = \frac{|z|}{\sqrt{|z|^2 + h^2}}$ and is equal to one at infinity. The point $z = 0$ is the critical point of the flow. One may show that the general form of the solution is

$$f(z) = v_\infty \sqrt{z^2 + h^2}, \tag{2.42}$$

where $v_\infty > 0$ is the flow speed at infinity.

3 Properties of Fractional Linear Transformations

We will now study some simplest classes of functions of a complex variable.

3.1 Fractional Linear Transformations

Fractional linear transformations are functions of the form

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \tag{3.1}$$

where a, b, c, d are fixed complex numbers, and z is the complex variable. The condition $ad - bc \neq 0$ is imposed to exclude the degenerate case when $w = \text{const}$ (if $ad - bc = 0$ then the numerator is proportional to the denominator for all z). When $c = 0$ one must have $d \neq 0$, then the function (3.1) takes the form

$$w = \frac{a}{d}z + \frac{b}{d} = Az + B \tag{3.2}$$

and becomes an *entire linear function*. Such function is either constant if $A = 0$, or a composition of a shift $z \to z' = z + B/A$ and dilation and rotation $z' \to w = Az'$, as can be seen from the factorization $w = A(z + B/A)$ if $A \neq 0$.

The function (3.1) is defined for all $z \neq -d/c, \infty$ if $c \neq 0$, and for all finite z if $c = 0$. We define it at the exceptional points setting $w = \infty$ at $z = -d/c$ and $w = a/c$ at $z = \infty$ (it suffices to set $w = \infty$ at $z = \infty$ if $c = 0$). The following theorem holds.

Theorem 3.1 *A fractional linear transformation (3.1) is a homeomorphism (that is, a continuous and one-to-one map) of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$.*

Proof. We assume that $c \neq 0$ - the simplifications in the case $c = 0$ are obvious. The function $w(z)$ is defined everywhere in $\overline{\mathbb{C}}$. We may solve (3.1) for z to obtain

$$z = \frac{dw - b}{a - cw} \tag{3.3}$$

and find that each $w \neq a/c, \infty$ has exactly one pre-image. Moreover, the extension of $w(z)$ to $\overline{\mathbb{C}}$ defined above shows that $\infty = w(-d/c)$ and $a/c = w(\infty)$. Therefore the function (3.1) is bijection of $\overline{\mathbb{C}}$ onto itself. It remains to show that (3.1) is continuous. However, its continuity is obvious at $z \neq -d/c, \infty$. The continuity of (3.1) at those points follows from the fact that

$$\lim_{z \to -d/c} \frac{az+b}{cz+d} = \infty, \quad \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}. \square$$

We would like to show now that the map (3.1) preserves angles everywhere in $\overline{\mathbb{C}}$. This follows from the existence of the derivative

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

for $z \neq -d/c, \infty$. In order to establish this property for the two exceptional points (both are related to infinity: one is infinity and the other is mapped to infinity) we have to define the notion of the angle at infinity.

Definition 3.2 *Let γ_1 and γ_2 be two paths that pass through the point $z = \infty$ and have tangents at the North Pole in the stereographic projection. The angle between γ_1 and γ_2 at $z = \infty$ is the angle between their images Γ_1 and Γ_2 under the map*

$$z \to 1/z = Z \tag{3.4}$$

at the point $Z = 0$.

Exercise 3.3 The readers who are not satisfied with this formal definition should look at the following problems:
(a) Show that the stereographic projection $\mathbb{C} \to S$ preserves angles, that is, it maps a pair of intersecting lines in $\mathbb{C}$ onto a pair of circles on S that intersect at the same angle.
(b) Show that the mapping $z \to 1/z$ of the plane $\mathbb{C}$ corresponds under the stereographic projection to a rotation of the sphere S around its diameter passing through the points $z = \pm 1$. (Hint: use expressions (1.14).)

Theorem 3.4 *Fractional linear transformations (3.1) are conformal[2] everywhere in $\overline{\mathbb{C}}$.*

Proof. The theorem has already been proved for non-exceptional points. Let γ_1 and γ_2 be two smooth (having tangents) paths intersecting at $z = -d/c$ at an angle α. The angle between their images γ_1^* and γ_2^* by definition is equal to the angle between the images Γ_1^* and Γ_2^* of γ_1^* and γ_2^* under the map $W = 1/w$ at the point $W = 0$. However, we have

$$W(z) = \frac{cz+d}{az+b},$$

[2] A map is conformal at $z = \infty$ if it preserves angles at this point.

so that Γ_1^* and Γ_2^* are the images of γ_1 and γ_2 under this map. The derivative

$$\frac{dW}{dz} = \frac{bc - ad}{(az+b)^2}$$

exists at $z = -d/c$ and is different from zero. Therefore the angle between Γ_1^* and Γ_2^* at $W = 0$ is equal to α, and the theorem is proved for $z = -d/c$. It suffices to apply the same consideration to the inverse function of (3.1) that is given by (3.3) in order to prove the theorem at $z = \infty$. □

We would like now to show that fractional linear transformations form a group. Let us denote the collection of all such functions by Λ. Let L_1 and L_2 be two fractional linear transformations:

$$L_1 : \ z \to \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0$$

$$L_2 : \ z \to \frac{a_2 z + b_2}{c_2 z + d_2}, \quad a_2 d_2 - b_2 c_2 \neq 0.$$

Their *product* is the composition of L_1 and L_2:

$$L : \ z \to L_1 \circ L_2(z).$$

The map L is clearly a fractional linear transformation (this may be checked immediately by a direct substitution)

$$L : \ w = \frac{az+b}{cz+d},$$

and, moreover, $ad - bc \neq 0$ since L maps $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ and does not degenerate into a constant.

We check that the group axioms hold.

(a) *Associativity*: for any three maps $L_1, L_2, L_3 \in \Lambda$ we have

$$L_1 \circ (L_2 \circ L_3) = (L_1 \circ L_2) \circ L_3. \tag{3.5}$$

Indeed, both sides of (3.5) represent the fractional linear transformation $L_1(L_2(L_3(z)))$.

(b) *Existence of unity*: the unity is clearly the identity transformation

$$E : \ z \to z. \tag{3.6}$$

(c) *Existence of the inverse*: for any $L \in \Lambda$ there exists an inverse map $L^{-1} \in \Lambda$ so that

$$L^{-1} \circ L = L \circ L^{-1} = E. \tag{3.7}$$

Indeed, the inverse to (3.1) is given by the map (3.3).

Therefore we have proved the following theorem.

Theorem 3.5 *Fractional linear transformations form a group with respect to composition.*

The group Λ is not commutative. For instance, if $L_1(z) = z + 1$, $L_2(z) = 1/z$, then $L_1 \circ L_2(z) = \dfrac{1}{z} + 1$ while $L_2 \circ L_1(z) = \dfrac{1}{z+1}$.

The entire linear transformations (3.2) with $A \neq 0$ form a subgroup $\Lambda_0 \subset \Lambda$ of mappings from Λ that have $z = \infty$ as a fixed point.

3.2 Geometric properties

Let us present two elementary properties of fractional linear transformations. In order to formulate the first one we introduce the convention that a circle in $\overline{\mathbb{C}}$ is either a circle or a straight line on the complex plane $\mathbb{C}$ (both are mapped onto circles under the stereographic projection).

Theorem 3.6 *Fractional linear transformations map a circle in $\overline{\mathbb{C}}$ onto a circle in $\overline{\mathbb{C}}$.*

Proof. The statement is trivial if $c = 0$ since entire linear transformations are a composition of a shift, rotation and dilation that all have the property stated in the theorem. If $c \neq 0$ then the mapping may be written as

$$L(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = A + \frac{B}{z+C}. \tag{3.8}$$

Therefore L is a composition $L = L_1 \circ L_2 \circ L_3$ of three maps:

$$L_1(z) = A + Bz, \quad L_2(z) = \frac{1}{z}, \quad L_3(z) = z + C.$$

It is clear that L_1 (dilation with rotation followed by a shift) and L_3 (a shift) map circles in $\overline{\mathbb{C}}$ onto circles in $\overline{\mathbb{C}}$. It remains to prove this property for the map

$$L_2(z) = \frac{1}{z}. \tag{3.9}$$

Observe that any circle in $\overline{\mathbb{C}}$ may be represented as

$$E(x^2+y^2) + F_1 x + F_2 y + G = 0, \tag{3.10}$$

where E may vanish (then this is a straight line). Conversely, any such equation represents a circle in $\overline{\mathbb{C}}$ that might degenerate into a point or an empty set (we rule out the case $E = F_1 = F_2 = G = 0$). Using the complex variables $z = x + iy$ and $\bar{z} = x - iy$, that is, $x = (z+\bar{z})/2$, $y = \frac{1}{2i}(z-\bar{z})$ we may rewrite (3.10) as

$$Ez\bar{z} + Fz + \bar{F}\bar{z} + G = 0, \tag{3.11}$$

with $F = (F_1 - iF_2)/2$, $\bar{F} = (F_1 + iF_2)/2$.

In order to obtain the equation for the image of the circle (3.11) under the map (3.9) it suffices to set $z = 1/w$ in (3.11) to get

$$E + F\bar{w} + \bar{W}w + Gw\bar{w} = 0. \tag{3.12}$$

This is an equation of the same form as (3.11). The cases when such an equation degenerates to a point or defines an empty set are ruled out by the fact that (3.9) is a bijection. Therefore the image of the circle defined by (3.10) is indeed a circle in $\overline{\mathbb{C}}$.

We have seen above that a holomorphic function f at a non-critical point z_0 maps infinitesimally small circles centered at z_0 onto curves that are close to circles centered at $f(z_0)$ up to higher order corrections. Theorem 3.6 shows that fractional linear transformations map *all* circles in $\overline{\mathbb{C}}$ onto circles exactly. It is easy to see, however, that the center of a circle is not mapped onto the center of the image.

In order to formulate the second geometric property of the fractional linear transformations we introduce the following definition.

Definition 3.7 *Two points z and z^* are said to be conjugate with respect to a circle $\Gamma = \{|z - z_0| = R\}$ in $\mathbb{C}$ if*
(a) they lie on the same half-line originating at z_0 ($\arg(z - z_0) = \arg(z^* - z_0)$) and $|z - z_0||z^* - z_0| = R^2$, or, equivalently,
(b) any circle γ in $\overline{\mathbb{C}}$ that passes through z and z^* is orthogonal to Γ.

The equivalence of the two definitions is shown as follows. Let z and z^* satisfy part (a) and γ be any circle that passes through z and z^*. Elementary geometry implies that if ζ is the point where the tangent line to γ that passes though z_0 touches γ, then $|\zeta - z_0|^2 = |z - z_0||z^* - z_0| = R^2$ and hence $\zeta \in \Gamma$ so that the circles γ and Γ intersect orthogonally. Conversely, if any circle that passes through z and z^* is orthogonal to Γ then in particular so is the straight line that passes through z and z^*. Hence z_0, z and z^* lie on one straight line. It easy to see that z and z^* must lie on the same side of z_0. Then the same elementary geometry calculation implies that $|z - z_0||z^* - z_0| = R^2$.

The advantage of the geometric definition (b) is that it may be extended to circles in $\overline{\mathbb{C}}$: if Γ is a straight line it leads to the usual symmetry. Definition (a) leads to a simple formula that relates the conjugate points: the conditions

$$\arg(z - z_0) = \arg(z^* - z_0), \quad |z - z_0||z^* - z_0| = R^2,$$

may be written as

$$z^* - z = \frac{R^2}{\overline{z - z_0}}. \tag{3.13}$$

The mapping $z \to z^*$ that maps each point $z \in \overline{\mathbb{C}}$ into the point z^* conjugate to z with respect to a fixed circle Γ is called inversion with respect to Γ.

Expression (3.13) shows that inversion is a function that is complex conjugate of a fractional linear transformation. Therefore inversion is an anticonformal transformation in $\overline{\mathbb{C}}$: it preserves "absolute value of angles" but changes orientation.

We may now formulate the desired geometric property of fractional linear transformations and prove it in a simple way.

Theorem 3.8 *A fractional linear transformation L maps points z and z^* that are conjugate with respect to a circle Γ onto points w and w^* that are conjugate with respect to the image $L(\Gamma)$.*

Proof. Consider the family $\{\gamma\}$ of all circles in $\overline{\mathbb{C}}$ that pass through z and z^*. All such circles are orthogonal to Γ. Let γ' be a circle that passes through w and w^*.

According to Theorem 3.6 the pre-image $\gamma = L^{-1}(\gamma')$ is a circle that passes through z and z^*. Therefore the circle γ is orthogonal to Γ. Moreover, since L is a conformal map, $\gamma' = L(\gamma)$ is orthogonal to $L(\Gamma)$, and hence the points w and w^* are conjugate with respect to $L(\Gamma)$. □

3.3 Fractional linear isomorphisms and automorphisms

The definition of a fractional linear transformation

$$L(z) = \frac{az+b}{cz+d} \tag{3.14}$$

involves four complex parameters a, b, c and d. However, the mapping really depends only on *three* parameters since one may divide the numerator and denominator by one of the coefficients that is not zero. Therefore it is natural to expect that three given points may be mapped onto three other given points by a unique fractional linear transformation.

Theorem 3.9 *Given any two triplets of different points $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ and $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ there exists a unique fractional linear transformation L so that $L(z_k) = w_k$, $k = 1, 2, 3$.*

Proof. First we assume that none of z_k and w_k is infinity. The existence of L is easy to establish. We first define fractional linear transformations L_1 and L_2 that map z_1, z_2, z_3 and w_1, w_2, w_3, respectively, into the points 0, 1 and ∞:

$$L_1(z) = \frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}, \quad L_2(w) = \frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}. \tag{3.15}$$

Then the mapping

$$w = L(z) = L_2^{-1}\circ L_1(z), \tag{3.16}$$

that is determined by solving $L_2(w) = L_1(z)$ for $w(z)$:

$$\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1} = \frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}, \tag{3.17}$$

satisfies $L(z_k) = w_k$, $k = 1, 2, 3$ by construction.

We show next uniqueness of such L. Let $\lambda(z)$ be a fractional linear transformation that satisfies $\lambda(z_k) = w_k$, $k = 1, 2, 3$. Let us define $\mu(z) = L_2 \circ \lambda \circ L_1^{-1}(z)$ with L_1 and L_2 defined by (3.15). Then we have $\mu(0) = 0$, $\mu(1) = 1$, $\mu(\infty) = \infty$. The last condition implies that μ is an entire linear transformation: $\mu(z) = \alpha z + \beta$. Then $\mu(0) = 0$ implies $\beta = 0$ and finally $\mu(1) = 1$ implies that $\alpha = 1$ so that $\mu(z) = z$. Therefore we have $L_2 \circ \lambda \circ L_1^{-1} = E$ is the identity transformation and hence $\lambda = L_2^{-1} \circ L_1 = L$.

Let us consider now the case when one of z_k or w_k may be infinity. Then expression (3.17) still makes sense provided that the numerator and denominator of the fraction where such z_k or w_k appears are replaced by one. This is possible since each z_k and w_k

appears exactly once in the numerator and once in the denominator. For instance, if $z_1 = w_3 = \infty$ expression (3.17) takes the form

$$\frac{1}{z - z_2} \cdot \frac{z_3 - z_2}{1} = \frac{w - w_1}{w - w_2} \cdot \frac{1}{1}.$$

Therefore Theorem 3.9 holds for $\overline{\mathbb{C}}$. □

Theorems 3.9 and 3.6 imply that any circle Γ in $\overline{\mathbb{C}}$ may be mapped onto any other circle Γ^* in $\overline{\mathbb{C}}$: it suffices to map three points on Γ onto three points on Γ^* using Theorem 3.9 and use Theorem 3.6. It is clear from the topological considerations that the disk B bounded by Γ is mapped onto one of the two disks bounded by Γ^* (it suffices to find out to which one some point $z_0 \in B$ is mapped). It is easy to conclude from this that any disk $B \subset \overline{\mathbb{C}}$ may be mapped onto any other disk $B^* \subset \overline{\mathbb{C}}$.

A fractional linear transformation of a domain D on D^* is called a fractional linear isomorphism. The domains D and D^* for which such an isomorphism exists are called FL-isomorphic. We have just proved that

Theorem 3.10 *Any two disks in $\overline{\mathbb{C}}$ are FL-isomorphic.*

Let us find for instance all such isomorphisms of the upper half plane $H = \{\text{Im} z > 0\}$ onto the unit disk $D = \{|z| < 1\}$. Theorem 3.9 would produce an ugly expression so we take a different approach. We fix a point $a \in H$ that is mapped into the center of the disk $w = 0$. According to Theorem 3.9 the point $\bar{a}$ that is conjugate to a with respect to the real axis should be mapped onto the point $w = \infty$ that is conjugate to $w = 0$ with respect to the unit circle $\{|w| = 1\}$. However, a fractional linear transformation is determined by the points that are mapped to zero and infinity, up to a constant factor. Therefore the map should be of the form $w = k\dfrac{z - a}{z - \bar{a}}$.

We have $|z - a| = |z - \bar{a}|$ when $z = x$ is real. Therefore in order for the real axis to be mapped onto the unit circle by such $w(z)$ we should have $|k| = 1$, that is, $k = e^{i\theta}$. Thus, all FL-isomorphisms of the upper half plane $H = \{\text{Im} z > 0\}$ onto the unit disk $D = \{|z| < 1\}$ have the form

$$w = e^{i\theta}\frac{z - a}{z - \bar{a}}, \tag{3.18}$$

where a is an arbitrary point in the upper half plane ($\text{Im} a > 0$) and $\theta \in \mathbb{R}$ is an arbitrary real number. The map (3.18) depends on three real parameters: θ and two coordinates of the point a that is mapped onto the center of the disk. The geometric meaning of θ is clear from the observation that $z = \infty$ is mapped onto $w = e^{i\theta}$ - the change of θ leads to rotation of the disk.

An FL-isomorphism of a domain on itself is called an FL-automorphism. Clearly the collection of all FL-isomorphisms of a domain is a group that is a subgroup of the group Λ of all fractional linear transformations.

The set of all FL-automorphisms $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ coincides, obviously, with the group Λ. It is also clear that the collection of all FL-automorphisms $\mathbb{C} \to \mathbb{C}$ coincides with the set Λ_0 of all entire linear transformations of the form $z \to az + b$, $a \neq 0$. We compute the group of FL-automorphisms of the unit disk before we conclude.

We fix a point a, $|a| < 1$ that is mapped onto the center $w = 0$. The point $a^* = 1/\bar{a}$ that is conjugate to a with respect to the unit circle $\{|z| = 1\}$ should be mapped to $z = \infty$. Therefore any such map should have the form

$$w = k\frac{z-a}{z-1/\bar{a}} = k_1\frac{z-a}{1-\bar{a}z},$$

where k and k_1 are some constants. The point $z = 1$ is mapped onto a point on the unit circle and thus $|k_1|\left|\frac{1-a}{1-\bar{a}}\right| = |k_1| = 1$. hence we have $k_1 = e^{i\theta}$ with $\theta \in \mathbb{R}$. Therefore such maps have the form

$$w = e^{i\theta}\frac{z-a}{1-\bar{a}z}. \tag{3.19}$$

Conversely, any function of the form (3.19) maps the unit disk onto the unit disk. Indeed, it maps the points a and $1/\bar{a}$ that are conjugate with respect to the unit circle to $w = 0$ and $w = \infty$, respectively. Therefore $w = 0$ must be the center of the image $w(\Gamma)$ of the unit circle Γ (since it is conjugate to infinity with respect to the image circle). However, $|w(1)| = \left|\frac{1-a}{1-\bar{a}}\right| = 1$ and hence $w(\Gamma)$ is the unit circle. Moreover, $w(0) = -e^{i\theta}a$ lies inside the unit disk so the unit disk is mapped onto the unit disk.

3.4 Some elementary functions

The function

$$w = z^n, \tag{3.20}$$

where n is a positive integer, is holomorphic in the whole plane $\mathbb{C}$. Its derivative $\frac{dw}{dz} = nz^{n-1}$ when $n > 1$ is different from zero for $z \neq 0$, hence (3.20) is conformal at all $z \in \mathbb{C}\backslash\{0\}$. Writing the function (3.20) in the polar coordinates as $z = re^{i\phi}$, $w = \rho e^{i\psi}$ we obtain

$$\rho = r^n, \quad \psi = n\phi. \tag{3.21}$$

We see that this mapping increases angles by the factor of n at $z = 0$ and hence the mapping is not conformal at this point.

Expressions (3.21) also show that two points z_1 and z_2 that have the same absolute value and arguments that differ by a multiple of $2\pi/n$:

$$|z_1| = |z_2|, \quad \arg z_1 = \arg z_2 + k\frac{2\pi}{n} \tag{3.22}$$

are mapped onto the same point w. Therefore, when $n > 1$ this is not a one-to-one map in $\mathbb{C}$. In order for it to be an injection $D \to \mathbb{C}$ the domain D should not contain any points z_1 and z_2 related as in (3.22).

An example of a domain D so that (3.20) is an injection from D into $\mathbb{C}$ is the sector $D = \{0 < \arg z < 2\pi/n\}$. This sector is mapped one-to-one onto the domain $D^* = \{0 < \arg z < 2\pi\}$, that is, the complex plane without the positive semi-axis.

The rational function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{3.23}$$

is called the Joukovsky function. It is holomorphic in $\mathbb{C}\backslash\{0\}$. Its derivative

$$\frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

is different from zero everywhere except $z = \pm 1$. Thus (3.23) is conformal at all finite points $z \neq 0, \pm 1$. The point $z = 0$ is mapped onto $w = \infty$. The fact that $w(z)$ is conformal at $z = 0$ follows from the existence and non-vanishing of the derivative

$$\frac{d}{dz}\left(\frac{1}{w}\right) = 2\frac{1 - z^2}{(1 + z^2)^2}$$

at $z = 0$. According to our definition the conformality of $w = f(z)$ at $z = \infty$ is equivalent to the conformality of $\tilde{w} = f(1/z)$ at $z = 0$. However, we have $\tilde{w}(z) = w(z)$ for the Joukovsky function and we have just proved that $w(z)$ is conformal at $z = 0$. Therefore it is also conformal at $z = \infty$.

The function (3.23) maps two different points z_1 and z_2 onto the same point w if

$$z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2} = (z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right) = 0,$$

that is, if

$$z_1 z_2 = 1. \tag{3.24}$$

An example of a domain where $w(z)$ is one-to-one is the outside of the unit disk: $D = \left\{z \in \overline{\mathbb{C}} : \ |z| > 1\right\}$. In order to visualize the mapping (3.23) we let $z = re^{i\phi}$, $w = u + iv$ and rewrite (3.23) as

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\phi, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\phi. \tag{3.25}$$

We see that the Joukovsky function transforms the circles $\{|z| = r_0\}$, $r_0 > 1$ into ellipses with semi-axes $a_{r_0} = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)$ and $b_{r_0} = \frac{1}{2}\left(r - \frac{1}{r}\right)$ and focal points at $w = \pm 1$ (since $a_{r_0}^2 - b_{r_0}^2 = 1$ for all r_0). Note that as $r \to 1$ the ellipses tend to the interval $[-1, 1] \subset \mathbb{R}$, while for large r the ellipses are close to the circle $\{|z| = r\}$. The rays $\{\phi = \phi_0, \ 1 < r < \infty\}$ are mapped onto parts of hyperbolas

$$\frac{u^2}{\cos^2\phi_0} - \frac{v^2}{\sin^2\phi_0} = 1$$

with the same focal points $w = \pm 1$. Conformality of (3.23) implies that these hyperbolas are orthogonal to the family of ellipses described above.

The above implies that the Joukovsky function maps one-to-one and conformally the outside of the unit disk onto the complex plane without the interval $[-1, 1]$.

The mapping is not conformal at $z = \pm 1$. It is best seen from the representation

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2. \tag{3.26}$$

This shows that (3.23) is the composition of three mappings

$$\zeta = \frac{z+1}{z-1}, \quad \omega = \zeta^2, \quad w = \frac{1+\omega}{1-\omega} \tag{3.27}$$

(the last mapping is the inverse of $\omega = \dfrac{w-1}{w+1}$). The first and the last maps in (3.27) are fractional linear transformations and so are conformal everywhere in $\overline{\mathbb{C}}$. The mapping $\omega = \zeta^2$ doubles the angles at $\zeta = 0$ and $\zeta = \infty$ that correspond to $z = \pm 1$. Therefore the Joukovsky function doubles the angles at these points.

Exercise 3.11 Use the decomposition (3.27) to show that the Joukovsky function maps the outside of a circle γ that passes through $z = \pm 1$ and forms an angle α with the real axis onto the complex plane without an arc that connects $z = \pm 1$ and forms angle 2α with the real axis. One may also show that circles that are tangent to γ at $z = 1$ or $z = -1$ are mapped onto curves that look like an airplane wing. This observation allowed Joukovsky (1847-1921) to create the first method of computing the aerodynamics of the airplane wings.

3.5 The exponential function

We define the function e^z in the same way as in real analysis:

$$e^z = \lim_{n \to +\infty} \left(1 + \frac{z}{n}\right)^n. \tag{3.28}$$

Let us show the existence of this limit for any $z \in \mathbb{C}$. We set $z = x + iy$ and observe that

$$\left|\left(1 + \frac{z}{n}\right)^n\right| = \left(1 + \frac{2x}{n} + \frac{x^2 + y^2}{n^2}\right)^{n/2}$$

and

$$\arg\left(1 + \frac{z}{n}\right)^n = n \arctan \frac{y/n}{1 + x/n}.$$

This shows that the limits

$$\lim_{n \to \infty} \left|\left(1 + \frac{z}{n}\right)^n\right| = e^x, \quad \lim_{n \to \infty} \arg\left(1 + \frac{z}{n}\right)^n = y$$

exist. Therefore the limit (3.28) also exists and may be written as

$$e^{x+iy} = e^x(\cos y + i \sin y). \tag{3.29}$$

Therefore

$$|e^z| = e^{Rez}, \quad \arg e^z = \text{Im} z. \tag{3.30}$$

We let $x = 0$ in (3.29) and obtain the Euler formula

$$e^{iy} = \cos y + i \sin y, \tag{3.31}$$

that we have used many times. However, so far we have used the symbol e^{iy} as a short-hand notation of the right side, while now we may understand it as a complex power of the number e.

Let us list some basic properties of the exponential function.

1. The function e^z is holomorphic in the whole plane $\mathbb{C}$. Indeed, letting $e^z = u+iv$ we find that $u = e^x \cos y$, $v = e^y \sin y$. The functions u and v are everywhere differentiable in the real sense and the Cauchy-Riemann equations hold everywhere:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = e^x \sin y.$$

Therefore the function (3.29) defines an extension of the real exponential function to the whole complex plane and the extended function is holomorphic. We will later see that such extension is unique.

2. The usual formula for the derivative of e^z holds. Indeed, we may compute the derivative along the direction x since we know that it exists. Therefore

$$(e^z)' = \frac{\partial}{\partial x} \left(e^x \cos y + i e^x \sin y\right) = e^z. \tag{3.32}$$

The exponential function never vanishes saince $|e^z| = e^x > 0$ and hence $(e^z)' \neq 0$ so that the mapping $w = e^z$ is conformal everywhere in $\mathbb{C}$.

3. The usual product formula holds

$$e^{z_1+z_2} = e^{z_1} e^{z_2}. \tag{3.33}$$

Indeed, setting $z_k = x_k + iy_k$, $k = 1, 2$ and using the expressions for sine and cosine of a sum we may write

$$e^{x_1}(\cos y_1 + i \sin y_1) e^{x_2}(\cos y_2 + i \sin y_2) = e^{x-1+x-2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)).$$

Thus addition of complex numbers z_1 and z_2 corresponds to multiplication of the images e^{z_1} and e^{z_2}. In other words the function e^z transforms the additive group of the field of complex numbers into its multiplicative group: under the map $z \to e^z$:

$$z_1 + z_2 \to e^{z_1} \cdot e^{z_2}. \tag{3.34}$$

4. The function e^z is periodic with an imaginary period $2\pi i$. Indeed, using the Euler formula we obtain $e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$ and hence we have for all $z \in \mathbb{C}$:

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z.$$

On the other hand, assume that $e^{z+T} = e^z$. Multiplying both sides by e^{-z} we get $e^T = 1$, which implies $e^{T_1}(\cos T_2 + i\sin T_2) = 1$, with $T = T_1 + iT_2$. Evaluating the absolute value of both sides we see that $e^{T_1} = 1$ so that $T_1 = 0$. Then the real part of the above implies that $\cos T_2 = 1$, and the imaginary part shows that $\sin T_2 = 0$. We conclude that $T = 2\pi ni$ and $2\pi i$ is indeed the basic period of e^z.

The above mentioned considerations also show that for the map $e^z : D \to \mathbb{C}$ to be one-to-one the domain D should contain no points that are related by

$$z_1 - z_2 = 2\pi in, \quad n = \pm 1, \pm 2, \dots \tag{3.35}$$

An example of such a domain is the strip $\{0 < \text{Im}z < 2\pi$. Setting $z = x + iy$ and $w = \rho e^{i\psi}$ we may write $w = e^z$ as

$$\rho = e^x, \quad \psi = y. \tag{3.36}$$

This shows that this map transforms the lines $y = y_0$ into the rays $\psi = y_0$ and the intervals $\{x = x_0,\ 0 < y < 2\pi\}$ into circles without a point $\{\rho = e^{x_0},\ 0 < \psi < 2\pi\}$. The strip $\{0 < y < 2\pi\}$ is therefore transformed into the whole plane without the positive semi-axis. The twice narrower strip $\{0 < y < \pi\}$ is mapped onto the upper half-plane $\text{Im}w > 0$.

3.6 The trigonometric functions

The Euler formula shows that we have $e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$ for all real $x \in \mathbb{R}$ so that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These expressions may be used to continue cosine and sine as holomorphic functions in the whole complex plane setting

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3.37}$$

for all $z \in \mathbb{C}$. It is clear that the right side in (3.38) is holomorphic.

All properties of these functions follow from the corresponding properties of the exponential function. They are both periodic with the period 2π: the exponential function has the period $2\pi i$ but expressions in (3.37) have the factor of i in front of z. Cosine is an even function, sine is odd. The usual formulas for derivatives hold:

$$(\cos z)' = i\frac{e^{iz} - e^{-iz}}{2} = -\sin z$$

and similarly $(\sin z)' = \cos z$. The usual trigonometric formulas hold, such as

$$\sin^2 z + \cos^2 z = 1, \quad \cos z = \sin\left(z + \frac{\pi}{2}\right),$$

etc. The reader will have no difficulty deriving these expressions from (3.37).

The trigonometric functions of a complex variable are closely related to the hyperbolic ones defined by the usual expressions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}. \tag{3.38}$$

They are related to sine and cosine by

$$\begin{aligned} \cosh z &= \cos iz, \quad \sinh z = -\sin iz \\ \cos z &= \cosh iz, \quad \sin z = -i \sinh iz \end{aligned} \tag{3.39}$$

as may be seen by comparing (3.37) and (3.38).

Using the formulas for cosine of a sum and relations (3.39) we obtain

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y,$$

so that

$$\begin{aligned} |\cos z| &= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\ &= \sqrt{\cos^2 x (1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y} = \sqrt{\cos^2 x + \sinh^2 y}. \end{aligned} \tag{3.40}$$

We see that $|\cos z|$ goes to infinity as $y \to \infty$.

Let us consider for example the map of half-strip $D = \{-\pi/2 < x < \pi/2, \quad y > 0\}$ by the function $w = \sin z$. We represent this map as a composition of the familiar maps

$$z_1 = iz, \quad z_2 = e^{z_1}, \quad z_3 = \frac{z_2}{i}, \quad w = \frac{1}{2}\left(z_3 + \frac{1}{z_3}\right).$$

This shows that $w = \sin z$ maps conformally and one-to-one the half-strip D onto the upper half-plane. Indeed, z_1 maps D onto the half-strip $D_1 = \{x_1 < 0, \ -\pi/2 < y_1 < \pi/2\}$; z_2 maps D_1 onto the semi-circle $D_2 = \{|z| < 1, \ -\pi/2 < \arg z < \pi/2\}$; z_3 maps D_2 onto the semi-circle $D_3 = \{|z| < 1, \ \pi < \arg z < 2\pi\}$. Finally, the Joukovksy function w maps D_3 onto the upper half-plane. The latter is best seen from (3.25): the interval $[0,1]$ is mapped onto the half-line $[1,+\infty)$, the interval $[-1,0)$ is mapped onto the half-line $(-\infty,1]$, and the arc $\{|z| = 1, \ \pi < \arg z < 2\pi\}$ is mapped onto the interval $(-1,1)$ of the x-axis. This shows that the boundary of D_3 is mapped onto the real axis. Furthermore, (3.25) shows that for $z_3 = \rho e^{i\phi}$ we have $\mathrm{Im}\ w = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\phi > 0$ so that the interior of D_3 is mapped onto the upper half plane (and not onto the lower one).

Tangent and cotangent of a complex variable are defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z} \tag{3.41}$$

and are rational functions of the complex exponential:

$$\tan z = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \quad \cot z = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}. \tag{3.42}$$

These functions are holomorphic everywhere in $\mathbb{C}$ except for the points where the denominators in (3.42) vanish (the numerators do not vanish at these points). Let us find such points for $\cot z$. We have $\sin z = 0$ there, or $e^{iz} = e^{-iz}$ so that $z = n\pi$, $n = \pm 1, \pm 2, \ldots$ - we see that the singularities are all on the real line.

Tangent and cotangent remain periodic in the complex plane with the real period π, and all the usual trigonometric formulas involving these functions still hold. Expression (3.40) and the corresponding formula for sine shows that

$$|\tan z| = \sqrt{\frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}}. \tag{3.43}$$

The mappings realized by the functions $w = \tan z$ and $w = \cot z$ are a composition of known maps. For instance, $w = \tan z$ can be reduced to the following:

$$z_1 = 2iz, \quad z_2 = e^{z_1}, \quad w = -i\frac{z_2 - 1}{z_2 + 1}.$$

This function maps conformally and one-to-one the strip $D = \{-\pi/4 < x < \pi/4\}$ onto the interior of the unit disk: z_1 maps D onto the strip $D_1 = \{-\pi/2 < y_1 < \pi/2\}$; z_2 maps D_1 onto the half plane $D_2 = \{x_2 > 0\}$; z_3 maps the imaginary axis onto the unit circle: $\left|-i\frac{iy-1}{iy+1}\right| = \frac{|1-iy|}{|1+iy|} = 1$, and the interior point $z_2 = 1$ of D_2 is mapped onto $w = 0$, an interior point of the unit disk.

4 Exercises for Chapter 1

1. Let us define multiplication for two vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $\mathbb{R}^2$ by

$$z_1 \star z_2 = (x_1 x_2 + y_1 y_2, x_1 y_2 + x_2 y_1).$$

This corresponds to the "$i^2 = 1$" rule.
(a) Show that this set is not a field and find divisors of zero.
(b) Let $\bar{z} = (x_1, -y_1)$ and define the absolute value as $\|z\| = \sqrt{|z \star \bar{z}|}$. Find the set of points such that $\|z\| = 0$. Show that absolute value of a product is the product of absolute values. Show that $\|z\| = 0$ is a necessary and sufficient condition for z to be a divisor of zero.
(c) Given z_2 so that $\|z_\| \neq 0$ define the ratio as

$$z_1 \star\star z_2 = \frac{z_1 \star \bar{z}_2}{z_2 \star \bar{z}_2}$$

with the denominator on the right side being a real number. Show that $(z_1 \star z_2) \star\star z_2 = z_1$.
(d) Let us define a derivative of a function $w = f(z) = u + iv$ as

$$f'(z) = \lim_{\Delta z \to 0, \|\Delta z\| \neq 0} \Delta w \star\star \Delta z$$

if the limit exists. Show that in order for such a derivative to exist if f is continuously differentiable in the real sense it is necessary and sufficient that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

(e) Find the geometric properties of the maps $w = z \star z$ and $w = 1 \star \star z$.
(f) Define $e_*^z = e^x(\cosh y, \sinh y)$ and $\sin_* z = (\sin x \cos y, \cos x \sin y)$. Find the similarities and differences of these functions from the usual exponential and trigonometric functions and describe their geometric properties.

2. Prove that
(a) if the points $z_1, \ldots, z_n$ lie on the same side of a line passing through $z = 0$ then $\sum_{k=1}^{n} z_k \neq 0$.
(b) if $\sum_{k=1}^{n} z_k^{-1} = 0$ then the points $\{z_k\}$ may not lie on the same side of a line passing through $z = 0$.

3. Show that for any polynomial $P(z) = \prod_{k=1}^{n}(z - a_k)$ the zeros of the derivative $P'(z) = \sum_{k=1}^{n} \prod_{j \neq k}(z - a_j)$ belong to the convex hull of the set of zeros $\{a_k\}$ of the polynomial $P(z)$ itself.

4. Show that the set of limit points of the sequence $a_n = \prod_{k=1}^{n}\left(1 + \frac{i}{k}\right)$, $n = 1, 2, \ldots$ is a circle. (Hint: show that first that $|a_n|$ is an increasing and bounded sequence and then analyze the behavior of $\arg a_n$).

5. let $f = u + iv$ have continuous partial derivatives in a neighborhood of $z_0 \in \mathbb{C}$. Show that the Cauchy-Riemann conditions for its $\mathbb{C}$-differentiability may be written in a more general form: there exist two directions s and n such that n is the rotation of s counterclockwise by 90 degrees, and the directional derivatives of u and v are related by

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}, \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s}.$$

In particular the conditions of $\mathbb{C}$-differentiability in the polar coordinates have the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

6. Let the point z move on the complex plane according to $z = re^{it}$, where r is constant and t is time. Find the velocity of the point $w = f(z)$, where f is a holomorphic function on the circle $\{|z| = r\}$. (Answer: $izf'(z)$.)

7. Let f be holomorphic in the disk $\{|z| \leq r\}$ and $f'(z) \neq 0$ on $\gamma = \{|z| = r\}$. Show that the image $f(\gamma)$ is a convex curve if and only if $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) + 1 \geq 0$. (Hint: first

show that convexity is equivalent to $\frac{\partial}{\partial\phi}\left(\frac{\pi}{2}+\phi+\arg f'(r^{e^{i\phi}})\right)\geq 0$.)

8. Find the general form of a fractional linear transformation that corresponds to the rotation of the Riemann sphere in the stereographic projection around two points lying on the same diameter. (Answer: $\frac{w-a}{1+\bar{a}w}=e^{i\theta}\frac{z-a}{1+\bar{a}z}$.)

9. Show that a map $w=\frac{az+b}{cz+d}$, $ad-bc=1$ preserves the distances on the Riemann sphere if and only if $c=-\bar{b}$ and $d=\bar{a}$.