Complex Analysis - Quiz #1 Solutions

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1. *Prove that if* $|z_1| = |z_2| = |z_3| = 1$*, then*

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \frac{1}{2} \arg \frac{z_2}{z_1}$$

To compute the argument of a difference of complex numbers of unit modules, we use the following identities.

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \qquad \sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

These can be deduced from the angle addition formulas. It follows that

$$\begin{aligned} e^{ix} - e^{iy} &= -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) + i2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \\ &= 2 \sin\left(\frac{x-y}{2}\right) \left[-\sin\left(\frac{x+y}{2}\right) + i \cos\left(\frac{x+y}{2}\right) \right] \\ &= 2i \sin\left(\frac{x-y}{2}\right) e^{i\left(\frac{x+y}{2}\right)} = 2 \sin\left(\frac{x-y}{2}\right) e^{i\left(\frac{\pi + x + y}{2}\right)}. \end{aligned}$$

Write $z_j = e^{i\theta_j}$ where $\theta_j = \arg z_j$ for $(j = 1, 2, 3)$. Then,

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \arg \left[\frac{\sin\left(\frac{\theta_3 - \theta_2}{2}\right)}{\sin\left(\frac{\theta_3 - \theta_2}{2}\right)} e^{\frac{i}{2}(\theta_2 - \theta_1)} \right].$$

If the quotient of sines is a positive number, then we have the equality as stated and when the quotient of sines is negative, there is an extra factor of π. We remark that the quotient can indeed be negative. One example is when $\theta_1 = \pi$, $\theta_2 = \pi/4$, and $\theta_3 = \pi/2$.

2. *Let* $l(z) : \mathbb{C} \to \mathbb{C}$ *be a* $\mathbb{R}$*-linear map. Show that there exist two complex numbers* $a, b \in \mathbb{C}$ *so that* $l(z) = az + b\bar{z}$ *for any* $z \in \mathbb{C}$*.*

The linearity of l, and evaluation of l at the basis vectors $1, i$ of $\mathbb{C}$ as a $\mathbb{R}$-vector space provide the choices

$$a = \frac{1}{2}\left[l(1) - il(i)\right] \qquad b = \frac{1}{2}\left[l(1) + il(i)\right].$$

Take any $z \in \mathbb{C}$. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Then,

$$\begin{aligned}
az + b\bar{z} &= \frac{1}{2}\left[l(1) - il(i)\right](x + iy) + \frac{1}{2}\left[l(1) + il(i)\right](x - iy) \\
&= \frac{1}{2}\left[l(x) - il(ix) + il(y) + l(iy)\right] + \frac{1}{2}\left[l(x) + il(ix) - il(y) + l(iy)\right] \\
&= \frac{1}{2}\left[2l(x) + 2l(iy)\right] \\
&= l(x + iy) = l(z).
\end{aligned}$$