1. (a) Show that for any triplet of distinct points \((z_1, z_2, z_3)\) there exists a unique fractional linear transformation \(f(z)\) such that \(f(z_1) = 0\), \(f(z_2) = 1\) and \(f(z_3) = \infty\).

Proof. First, let us observe that any fractional linear transformation that fixes three or more points must be the identity. Write \(f(z) = (az + b)/(cz + d)\). First, suppose at least three of the fixed points lie in \(\mathbb{C}\). For \(w \in \mathbb{C}\), \(f(w) = w \Leftrightarrow aw + b = (cw + d)w \Leftrightarrow cw^2 + (d - a)w - b = 0\). A polynomial of degree two can have at most two roots. So if \(f\) fixes three points, \(c = 0, d = a\) and \(b = 0 \Rightarrow f(z) = z\). When \(w = \infty\) is a fixed point, then \(c = 0\). The remaining fixed points must be roots of \(az + b\) which also implies \(f(z) = z\).

Existence: Let 
\[
  f(z) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3}.
\]
Then \(f\) is a fractional linear transformation that sends \(z_1 \mapsto 0, z_2 \mapsto 1, z_3 \mapsto \infty\).

Uniqueness: If \(g(z)\) is any other fractional linear transformation satisfying the same conditions, \(f^{-1}g\) is an FLT that fixes three points, namely \(z_1, z_2, z_3\). Hence \(f^{-1}g = id\), that is, \(g = f\).

(b) Find a fractional linear transformation that maps the domain bounded by the circles \(|z| = 2\) and \(|z + i| = 1\) onto a strip bounded by two lines parallel to the real axis.

Proof. Let \(w(z)\) be the unique fractional linear transformation that sends \(-2i \mapsto \infty\), \(0 \mapsto 0\) and \(2i \mapsto 2i\). Observe that the intersection of both of these circles is \(\{-2i\}\). Because fractional linear transformations are isomorphisms of \(\mathbb{C}\), the images of the circles will only intersect at infinity. Using that FLTs preserve circles \(\overline{\mathbb{C}}\)-circles, the images of the circles must be parallel lines. Next, \(w\) sends three points on the imaginary axis to the imaginary axis. Again using that FLTs preserve circles, the imaginary axis must be sent to the imaginary axis. Then noting that these circles intersect the imaginary axis at right angles, their images must be perpendicular to the imaginary axis and therefore, parallel to the real axis. Finally, to see that the domain bounded by the circles is mapped into the strip bounded by these lines, one can use that FLTs preserve conjugate points or write the explicit map and use a similar argument as above.

2. (a) Construct a fractional linear transformation \(f(z)\) that maps the unit disk \(|z| \leq 1\) onto the upper half-plane \(\text{Im}z > 0\) so that \(f(i) = \infty\) and \(f(1) = 1\).

Proof. Solving this is equivalent to finding a FLT that maps the upper half plane to the disk and sends \(\infty \mapsto i\) and \(1 \mapsto 1\) and taking its inverse. But we know all such FLTs are of the form
\[
  w(z) = e^{i\theta} \frac{z - a}{z - \bar{a}}
\]
where \( \text{Im}(a) > 0 \). We see the conditions to be satisfied if \( \theta = \pi/2 \) and \( a = i \).

(b) Show that if \( f(z) = 1/z \) then any circle passing through the points \( z = -1 \) and \( z = 1 \) is mapped onto itself under the map \( f \).

**Proof.** Let \( C \) be a circle passing through \(-1, 1\). Then since \( f \) maps \(-1 \mapsto 1, \) and \( 1 \mapsto 1, \) \( f(C) \) is a \( \mathbb{C} \) circle mapping through \(-1, 1\). But since \( C \) does not intersect 0, \( f(C) \) is a circle in the classical sense. Notice \( f \) sends the real axis to the real axis. Looking at tangent vectors, we see that \( f \) will preserve angles between the circle and the real axis. Since there is a unique circle that passes through the real axis with specified angles, \( f(C) = C \).

3. (a) State the Cauchy-Riemann equations.
   
   Set \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \). The CREQs are:
   
   \[
   \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
   \]

   (b) Suppose that \( f \) is analytic on an open set \( A \) and that \( f(z) \) is real for all \( z \in A \). Show that \( f(z) = \text{const.} \)

   **Proof.** If \( f = u + iv \), then \( v = 0 \) on \( A \). The CREQs dictate \( u_x = \frac{\partial u}{\partial x} = 0 \) and \( u_y = \frac{\partial u}{\partial y} = 0 \) on \( A \). Hence, \( u \) must be constant on \( A \) and therefore, so is \( f \).

   (c) Show that \( u(x, y) = x^3 - 3xy^2 \) and \( v = 3x^2y - y^3 \) satisfy the Cauchy-Riemann equations.

   **Proof.** \( u_x = 3x^2 - 3y^2 = v_y \) and \( u_y = -6xy = -v_x \).

   (d) Find a holomorphic function \( f(z) \) such that \( f(z) = u(z) + iv(z) \) with \( u \) and \( v \) as in part (c).

   **Proof.** \( f(z) = z^3 \) since \( (x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 \).

4. Let \( p(z) \) be a polynomial and let \( \gamma = \{|z - z_0| = R\} \) be a circle. Show by evaluating the integral that

   \[
   \frac{1}{2\pi i} \int_\gamma \frac{p(z)}{z - z_0} = p(z_0).
   \]

   **Proof.** Write \( p(z) = \sum_{i=0}^n a_i z^i \). As stated in class, we can find \( b_i \) such that \( p(z) = \sum_{i=0}^n b_i (z - z_0)^i \). Then linearity of the integral provides

   \[
   \frac{1}{2\pi i} \int_\gamma \frac{p(z)}{z - z_0} dz = \sum_{i=0}^n \frac{b_i}{2\pi i} \int_\gamma (z - z_0)^{i-1} dz
   \]

   Next, an easy direct computation shows

   \[
   \int_\gamma (z - z_0)^{i-1} = \begin{cases} 0 & i \geq 1; \\ 2\pi i & i = 0. \end{cases}
   \]

   Substituting this into the above finishes the proof.

5. State the baby Cauchy theorem. Use it to prove that if \( f(z) \) is holomorphic in a domain \( D \) then its integral over any rectangle contained in \( D \) is equal to zero.
Proof. Draw something like this

\[ \gamma_3 \]
\[ \gamma_4 \]
\[ \omega \]
\[ \gamma_2 \]
\[ \omega^- \]
\[ \gamma_1 \]

Denote the square as \( \gamma = \sum \gamma_i \), the lower triangle as \( \sigma_1 = \gamma_1 + \gamma_2 + \omega^- \) and the upper triangle as \( \sigma_2 = \omega + \gamma_3 + \gamma_4 \). Then by properties of the integral and (Baby) Cauchy’s theorem provide

\[
\int_{\gamma} f = \sum \int_{\gamma_i} f \\
= \sum \int_{\gamma_i} f + \int_{\omega} f + \int_{\omega^-} f \\
= \int_{\sigma_1} f + \int_{\sigma_2} f = 0.
\]

\qed