Existence and Non-existence of Traveling Fronts in Disordered Media

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Transition fronts for reaction-diffusion equations

We study transition fronts for the reaction-diffusion PDE

$$u_t = \Delta u + f(x, u)$$

on $\mathbb{R} \times \mathbb{R}$ with $f(x, 0) = f(x, 1) = 0$.

Transition front (generalized traveling front) is a solution $u(t, x) \in [0, 1]$ global in time and satisfying for each $t \in \mathbb{R}$,

$$\lim_{x \to -\infty} u(t, x) = 1 \quad \text{and} \quad \lim_{x \to \infty} u(t, x) = 0.$$

Defined by Berestycki-Hamel. Also Matano, Shen

This front moves to the right. Also a front moving left.

Fronts model invasions (combustion, ecology, genetics)
Reaction functions in $u_t = \Delta u + f(x, u)$

Reaction function $f : \mathbb{R} \times [0, 1] \rightarrow [0, \infty)$ is non-negative Lipschitz with $f(x, 0) = f(x, 1) = 0$ and ignition temperature

$$\theta(x) = \inf \{ u \mid f(x, u) > 0 \}$$

- **Monostable**: $\inf_x \theta(x) = 0$  
  (KPP: $f(x, u) \leq \frac{\partial f}{\partial u}(x, 0)u$)

- **Ignition**: $\inf_x \theta(x) > 0$
Homogeneous media: Traveling fronts

\[ u_t = \Delta u + f(u) \]

A **traveling front** is a solution \( u(t, x) = U(x - ct) \) such that \( U(-\infty) = 1 \) and \( U(\infty) = 0 \) (constant profile \( U \) and speed \( c \)).

- \((U, c)\) solve \( U'' + cU' + f(U) = 0 \) (gives \( c > 0 \))
- Ignition reactions: unique front speed \( c_f^* > 0 \)
- Monostable reactions: minimal front speed \( c_f^* > 0 \) and all \( c \in [c_f^*, \infty) \) are achieved (but \( c_f^* \) most physical)
- KPP: \( c_f^* = 2 \sqrt{f'(0)} \) (Kolmogorov-Petrovskii-Piskunov)

General solutions of the PDE propagate with speed \( c_f^* \).
Periodic media: Pulsating fronts

\[ u_t = \Delta u + f(x, u) \]

Assume that \( f \) is 1-periodic in \( x \). A pulsating front with speed \( c > 0 \) is a solution of the form \( u(t, x) = U(x - ct, x \mod 1) \) such that uniformly in the second argument, \( U(-\infty, x \mod 1) = 1 \) and \( U(\infty, x \mod 1) = 0 \).

- Time-periodic in a moving frame: \( u(t + \frac{1}{c}, x + 1) = u(t, x) \)
- \((U, c)\) solve a degenerate elliptic equation
- Under mild conditions on \( f \) there is again unique/minimal front speed \( c_f^* > 0 \) for ignition/monostable reactions (Xin, Berestycki-Hamel)
In general inhomogeneous media no special forms exist. Assume:

- $f(x, u)$ is Lipschitz and $f_0(u) \leq f(x, u) \leq f_1(u)$ for some reactions $f_0(u) \leq f_1(u)$ such that $f_0$ is ignition and $f_1$ is ignition or monostable.

- $f_1'(0) < (c_{f_0}^*)^2/4$ (true if $f_1$ is ignition)

  - This is equivalent to $2 \sqrt{f_1'(0)} < c_{f_0}^*$ (front is “pushed”)

- For some $\zeta < (c_{f_0}^*)^2/4$ the function $f(x, \cdot)$ is bounded away from zero (uniformly in $x$) on the interval $[\alpha_f(x), 1 - \varepsilon]$, with

$$\alpha_f(x) = \inf\{u \in (0, 1) \mid f(x, u) > \zeta u\}$$

  - I.e., $f$ cannot vanish after becoming large (except at $u = 1$)

- These conditions are “qualitatively necessary” for existence of fronts
Fronts in general inhomogeneous media
Theorem (Z.)

Assume the above hypotheses.

(i) There exists a transition front $u_+$ for

$$u_t = \Delta u + f(x, u)$$

moving to the right, with $(u_+)_t > 0$ (and $u_-$ moving to the left).

(ii) If $f_1$ is ignition, then $u_\pm$ are unique up to time shifts and general solutions with exponentially decaying initial data converge in $L^\infty_x$ to time shifts of $u_\pm$ (global attractors).

- Proved by Nolen-Ryzhik-Mellet-Roquejoffre-Sire in the case $f(x, u) = a(x)g(u)$ with $a(x) \in [a_0, a_1] \subset (0, \infty)$ and $g$ ignition reaction (constant positive ignition temperature).
- Extends to cylindrical domains $D \subset \mathbb{R}^n$ (and includes periodic case of Berestycki-Hamel, Xin).
- Bistable reaction case studied by Shen, Vakulenko-Volpert.
Non-existence of fronts for $u_t = \Delta u + f(x, u)$

If $f_1$ is KPP, then $c_{f_0}^* < c_{f_1}^* = 2\sqrt{f_1'(0)}$, so $f_1'(0) < (c_{f_0}^*)^2/4$ fails.

Let $f$ be a KPP reaction and assume

- $a(x) = \frac{\partial f}{\partial u}(x, 0) > 0$ (e.g., $f(x, u) = a(x)u(1-u)$)
- $\lambda = \sup \sigma(\Delta + a(x))$
- $\psi$ = principal eigenfunction of $\Delta + a(x)$ (if $\lambda$ is eigenvalue)

Theorem (Nolen-Roquejoffre-Ryzhik-Z.)

Assume that $a(x) \geq 1$ (so $\lambda \geq 1$) and $\lim_{x \to \pm \infty} a(x) = 1$.

(i) If $\lambda > 2$, then there is a unique entire solution (up to a time shift) strictly between 0 and 1. It satisfies $u(t, x) = e^{\lambda t}\psi(x)$ for $t \ll -1$ (the bump). In particular, no transition front exists.

(ii) If $\lambda < 2$, then there exists a (right-moving) transition front for each speed $c \in (2, \frac{\sqrt{\lambda}}{\lambda-1})$. If $\lambda \in (1, 2)$, the bump also exists.

- First general result of non-existence of fronts (based on an unpublished ignition-KPP example by Roquejoffre-Z.)
Proof of (i): non-existence of front for \( u_t = \Delta u + f(x, u) \)

Lemma

For each \( \kappa \in (2, \frac{\lambda}{\sqrt{\lambda-1}}) \) there is \( C_{\kappa} \) such that for \((t, x) \in \mathbb{R}^- \times \mathbb{R}, \)

\[
u(t, x) \leq C_{\kappa} e^{\frac{|x|}{\kappa}|t|} u(0, 0)\]

Sufficient to show \( u(t, x) \lesssim e^{\sqrt{\lambda-1}(|x| - \kappa|t|)} u(0, 0) \) for \( |x| \leq \kappa|t| \).
Assume the contrary (by Harnack also for any \( y \) near \( x \)) and consider \( x < 0 \). Let \( \beta = \frac{|x|}{2\sqrt{\lambda-1}|t|} \leq \frac{\kappa}{2\sqrt{\lambda-1}} < 1 \). Then

\[
u(t + \beta|t|, 0) \gtrsim e^{\beta|t|} e^{-\frac{|x|^2}{4\beta|t|}} e^{\sqrt{\lambda-1}(|x| - \kappa|t|)} u(0, 0) = e^{(\lambda \beta - \sqrt{\lambda-1} \kappa)|t|} u(0, 0)\]

if \( u_t = \Delta u + u \). Still holds, with \( e^{(1-\varepsilon)\beta|t|} \), because \( 2\beta|t| < |x| \).
Same estimate for any \( y \) near 0, so if \( \psi(0) = ||\psi||_{\infty} \leq 1 \), then

\[
u(0, 0) \gtrsim e^{\lambda(1-\beta)|t|} e^{(\lambda \beta - \sqrt{\lambda-1} \kappa - \varepsilon \beta)|t|} u(0, 0) = e^{(\lambda - \sqrt{\lambda-1} \kappa - \varepsilon \beta)|t|} u(0, 0)\]

This is a contradiction if \( \varepsilon > 0 \) is small.
Proof of (i): non-existence of front for $u_t = \Delta u + f(x, u)$

So for $(t, x) \in \mathbb{R}^- \times \mathbb{R}^-$ we have

$$u(t, x) \leq C_\kappa e^{-x+\kappa t} u(0, 0)$$

Assume $a(x) - 1$ is supported on $\mathbb{R}^+$, pick any $\tau < 0$, and let

$$v^{(\tau)}(t, x) = C_\kappa e^{-x+(\kappa-2)\tau+2t} u(0, 0) + C_\kappa e^{x+2t} u(0, 0).$$

Then $v^{(\tau)}$ solves

$$v^{(\tau)}_t = \Delta v^{(\tau)} + v^{(\tau)} \geq \Delta v^{(\tau)} + f(x, v^{(\tau)})$$

on $\mathbb{R} \times \mathbb{R}^-$, with $v^{(\tau)}(\tau, x) \geq u(\tau, x)$ for $x < 0$ and $v^{(\tau)}(t, 0) \geq u(t, 0)$ for $t \in [\tau, 0]$. So for $(t, x) \in \mathbb{R}^- \times \mathbb{R}^-$,

$$u(t, x) \leq \lim_{\tau \to -\infty} v^{(\tau)}(t, x) = C_\kappa e^{-|x|+2t} u(0, 0)$$

Same for $x \geq 0$, so $u$ is a bump.
Proof of (ii): existence of fronts for $u_t = \Delta u + f(x, u)$

Assume $a$ compactly supported and $f(x, u) = a(x)u$ for $u \leq \theta$.

For $\gamma \in (\lambda, 2)$ let $\phi_\gamma$ be the generalized eigenfunction of $\Delta + a(x)$ with eigenvalue $\gamma$ and $\phi_\gamma(x) = e^{-\sqrt{\gamma-1}x}$ for $x \gg 1$.

Then $\phi_\gamma > 0$ and $\phi_\gamma(x) \approx \alpha_\gamma e^{-\sqrt{\gamma-1}x}$ for $x \ll -1$ (with $\alpha_\gamma > 0$).

$$v(t, x) = e^{\gamma t} \phi_\gamma(x)$$

solves $v_t = \Delta v + a(x)v$ so $v$ is a supersolution of the original PDE, “moving” with speed $c = \gamma / \sqrt{\gamma - 1}$ for $|x| \gg 1$.

Let $\varepsilon > 0$ be small and $\varepsilon' = (\sqrt{1 + \frac{\varepsilon}{\gamma-1}} - 1)\gamma$, so that $\varepsilon' > \varepsilon$ by $\frac{\gamma}{2(\gamma - 1)} > 1$. Then

$$w(t, x) = e^{\gamma t} \phi_\gamma(x) - Ae^{(\gamma + \varepsilon')t} \phi_{\gamma + \varepsilon'}(x)$$

“moves” with speed $c$, has a “constant” in $t$ maximum, and is a subsolution where $w \geq 0$ if $A \gg 1$ (so that $\sup w \leq \theta$).