Traveling Wave Foliations of Allen-Cahn Equation Near Eternal Solutions of Mean Curvature Flow and Jacobi-Toda System

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March 21-26 2010, Deterministic and Stochastic Fronts, BIRS, Canada

Traveling wave problem for the bistable equation

We will consider the following equation:

$$\Delta u + c\partial_{x_{N+1}} u + f(u) = 0, \quad \text{in } \mathbb{R}^{N+1}, \tag{0.1}$$

which is the traveling wave problem for:

$$u_t = \Delta u + f(u), \quad \text{in } \mathbb{R}^{N+1}, \quad t > 0,$$

where $u(x', x_{N+1}, t) = u(x', x_{N+1} - ct)$.
Traveling wave solution is an eternal solution since it exists for all time t.

In this talk we mostly assume that

$$f(u) = u(1-u^2) \Longrightarrow \int_{-1}^{1} f(u)\, du = 0,$$

i.e. f is a bistable, balanced nonlinearity.
We will mention results of (0.1) with (bistable, unbalanced nonlinearity)

$$f(u) = u(1-u^2) + a(1-u^2) \Longrightarrow \int_{-1}^{1} f(u)\, du = \frac{4}{3}a \neq 0.$$

The potential corresponding to the two cases is of the form:

$$W(u) = \frac{1}{2}(1-u^2)^2 - au(1-\frac{1}{3}u^2).$$

If $a = 0$ then $W(-1) = 0 = W(1)$. If $a > 0$ then $W(-1) > W(1)$, hence the phase $u = 1$ is more stable then $u = -1$.

We will first discuss the case $a > 0$. In one dimension we have:

$$\Phi'' + c\Phi' + f(\Phi) = 0.$$

This problem has a unique solution such that $\Phi(\pm\infty) = \pm 1$, $\Phi' > 0$. This corresponds to a traveling wave that moves to the left; the more stable phase invades the less stable one.
When $a = 0$ there exists a unique (heteroclinic) solution:

$$H'' + H(1 - H^2) = 0,$$

such that $H(\pm\infty) = \pm 1$, H is odd and $H' > 0$. Notice also that $-H$ is a solution connecting the two stable phases.

Finally we observe that in both situation one can define a planar front solution in $\mathbb{R}^{N+1}$ to (0.1):

$$u(x', x_{N+1}) = \begin{cases} \Phi(x_{N+1}), & \text{unbalanced}, (c \neq 0) \\ H(x_{N+1}), & \text{balanced}, (c = 0). \end{cases}$$

- For planar fronts see the book of Fife.
- In the unbalanced case there are other solutions with asymptotically planar, V-shaped, fronts (Ninomiya-Taniguchi (in $N+1=2$), Hamel-Monneau-Roquejoffre ($N+1>2$)), pyramidal fronts (Taniguchi, $N=3$).
- Stability of planar fronts (Levermore-Xin, Kapitula, Xin, Matano-Nara-Taniguchi).
- Related results in the monostable (KPP) case (Bonnet-Hamel, Hamel-Monneau-Roquejoffre, Hamel-Nadirashvili).

Rotationally symmetric traveling waves

From now on we will consider only the balanced case in $\mathbb{R}^{N+1}$:

$$\Delta u + c\partial_{x_{N+1}} u + u(1 - u^2) = 0. \tag{0.2}$$

A solution of this problem represents a wave traveling with speed c in the direction the x_{N+1} axis.
In this problem the velocity c is not determined by the difference in the heights of the potential wells.

Parabolic De Giorgi Conjecture

Consider eternal solutions of parabolic Allen-Cahn equation

$$u_t = \Delta u + f(u), (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Assuming their monotonicity in the x_{N+1} direction:

$$\partial_{x_{N+1}} u > 0, \quad \lim_{x_{N+1} \to \pm\infty} u(x', x_{N+1}, t) = \pm 1, \quad t \in \mathbb{R}$$

then u is one-dimensional.

This conjecture is false even in dimension $N+1=2$.

- In 2007 Chen, Guo, Hamel, Ninomiya, Roquejoffre showed the existence of solutions to (0.2) of the form $u(x', x_{N+1}) = U(r, x_{N+1})$, $r = |x'|$, $N \geq 1$. Functions U have paraboloid-like profiles of their nodal sets Γ.
- We concentrate on the case $N > 1$. In the same paper the asymptotic profiles of the fronts are given:

$$\lim_{\substack{x_{N+1}\to+\infty \\ (x', x_{N+1})\in\Gamma}} \frac{r^2}{2x_{N+1}} = \frac{N-1}{c}, \quad \text{if } N > 1.$$

- When $N=1$ the ends of the fronts become asymptotically parallel.

Traveling Wave De Giorgi Conjecture

Let u be a bounded solution of equation

$$(AC)_{TW} \qquad \Delta u + u - u^3 + c u_{x_{N+1}} = 0 \quad \text{in } \mathbb{R}^{N+1}.$$

which satisfies

$$\partial_{x_{N+1}} u > 0$$

Then, u must be radially symmetric in x'.

Gui: $N = 2$

Parabolic Allen-Cahn Equation and Mean Curvature Flow

Consider the mean curvature flow for a hypersurface $\Sigma = \Sigma(t)$:

$$\frac{\partial \Sigma}{\partial t} = H_{\Sigma}\nu, \tag{0.3}$$

where ν is the normal to the surface and H_{Σ} is its mean curvature. It is known that the evolution of zero-level set of $\epsilon-$version of the Allen-Cahn equation

$$\epsilon u_t = \epsilon \Delta u + \frac{1}{\epsilon} u(-u^2) \tag{0.4}$$

can be reduced to (0.3)
Evans-Spruck 1995, T.Illmanen 1993, Y. Tonegawa 2003, ...

Eternal Solutions of Mean Curvature Flow

Surfaces which are translated by the mean curvature (MC) flow with constant velocity (say 1) in a fixed direction satisfy:

$$H_{\Sigma} = \nu_{N+1}, \quad (x_{N+1} \text{ direction}). \tag{0.5}$$

Let $\Sigma = \Sigma(t)$ be such a surface and consider its scaling Σ_{ε}:

$$y \in \Sigma_{\varepsilon}(t) \Longleftrightarrow \varepsilon y \in \Sigma(t).$$

Then, denoting the mean curvatures of these surfaces by H_{Σ}, $H_{\Sigma_{\varepsilon}}$:

$$H_{\Sigma} = \nu_{N+1}, \quad H_{\Sigma_{\varepsilon}} = \varepsilon \nu_{N+1}. \tag{0.6}$$

- Translating solutions to the MC flow are called eternal solutions since they exists for all $t \in (-\infty, \infty)$.
- Convex eternal solutions are important in the study of singularities for the MC flow (Huisken-Sinestrari, also Wang, Wang-Sheng, B.White).
- Examples by Altschuler-Wu, Clutterbuck-Schnurer-Schulze, Nguyen.

Rotationally symmetric eternal solution to the MC flow

We describe the result of Clutterbuck-Schnurer-Schulze.
When $\Sigma(t) = \{F(x') + t\}$, where $F\colon \mathbb{R}^N \to \mathbb{R}$, is a smooth function then

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \qquad (0.7)$$

There exists a unique radially symmetric solution F of (0.7):

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \qquad (0.8)$$

The first term in this asymptotic behavior coincides with the asymptotic behavior of the nodal set of solutions to (0.2) found by Chen, Guo, Hamel, Ninomiya, Roquejoffre.

- The rotationally symmetric graphs are stable.
- They find other solutions, which are still rotationally symmetric, have the same asymptotic behavior, but are not graphs.
- Nguyen (2008) found other embedded traveling surfaces by desingularizing the Scherk surface, following ideas of Kapoulous (1997)

Bernstein Type Conjecture for MC Solitons

Let F be a solution of

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}} \text{ in } \mathbb{R}^N. \tag{0.9}$$

Then F is rotationally or cylindrically symmetric.

A natural critical dimension seems to be $N = 8$. However

X-J Wang , 2004, claimed to have the existence of non-radial eternal convex graphs when $N \geq 3$.

Traveling wave foliations

- We want to find solutions to

$$\Delta u + \varepsilon \partial_{x_{N+1}} u + u(1-u^2) = 0, \quad \text{in } \mathbb{R}^{N+1}, \tag{0.10}$$

  connecting -1 to -1,i.e., $\lim_{x_N \to \pm\infty} u(x', x_{N+1}) = -1$.
- For each ε by Σ_ε we denote the eternal graph:

$$\Sigma_\varepsilon = \{x_{N+1} = \varepsilon^{-1} F(\varepsilon r)\}.$$

- We look for solutions depending on just two variables $r = |x'|$, $x' \in \mathbb{R}^N$ and x_{N+1}.
- We want the nodal set of such solution to have multiple components which resemble in some sense the eternal graph. For simplification we will consider $k = 2$ component case.
- By the normal graph of a function ψ over a surface we mean the surface

$$\Gamma \ni p \mapsto p + \nu(p)\psi(p).$$

Theorem (del Pino, Kowalczyk, Wei, 2009)

For each small $\varepsilon > 0$ and $N \geq 2$ there is a solution u_ε to (0.10) whose 0-level set consists of 2 hypersurfaces given as normal graphs over Σ_ε of smooth functions $f_{\varepsilon,j}(r)$, $j = 1, 2$. Denoting

$$\mathcal{U}_\varepsilon = f_{\varepsilon,2} - f_{\varepsilon,1} > 0, \quad \mathcal{V}_\varepsilon = f_{\varepsilon,2} + f_{\varepsilon,1},$$

we have

$$\mathcal{U}_\varepsilon(r) = \log \frac{\alpha_0}{\varepsilon^2 b(\varepsilon r)} + \mathcal{O}\big(\log \log \frac{\alpha_0}{\varepsilon^2 b(\varepsilon r)} \big),$$

where

$$b(s) \sim \frac{N-1}{s^2}, \quad s \gg 1,$$

while $\mathcal{V}_\varepsilon = \mathcal{O}(\varepsilon^\tau)$, where $\alpha_0, \tau > 0$ are constants.

- The nodal sets, up to small terms, have form:

$$\Gamma_{\varepsilon,1} = \Sigma_\varepsilon - \{\frac{1}{2}\mathcal{U}_\varepsilon(r)\nu_\varepsilon(r)\},$$
$$\Gamma_{\varepsilon,2} = \Sigma_\varepsilon + \{\frac{1}{2}\mathcal{U}_\varepsilon(r)\nu_\varepsilon(r)\}.$$

- Asymptotically we have

$$\mathcal{U}_\varepsilon(r) \sim \log \frac{2+\varepsilon^2 r^2}{\varepsilon^2}.$$

- The ends of the nodal sets diverge logarithmically in r along the ends of Σ_ε but this growth is small relative to the asymptotic behavior of Σ_ε at ∞ which is $\varepsilon^{-1}F(\varepsilon r) \sim \varepsilon r^2$.

- Our proof also gives a new proof of single traveling interface in the case of $\epsilon << 1$. Accurate information can be obtained. This can be useful for the uniqueness and stability question.
- There is no such analogue result for the mean curvature flow. This result shows that the parabolic Allen-Cahn equation is really different from the mean curvature flow. It has richer structures.

- With some extra (technical) effort a similar result can be proven in case of $k > 2$ foliating traveling waves.
- The travelling wave solution we construct connects the stable phase -1 (minimum of the potential $W(u) = \frac{1}{4}(1-u^2)^2$) with itself, as it is common two *distinct* stable phases -1 and 1. This is counterintuitive for this nonlinearity (bistable not monostable).
- We refer to this phenomenon as foliation by traveling waves. This is motivated by the apparent analogy with the foliation by constant mean curvature submanifolds (Ye, Mazzeo-Pacard, Mahmoudi-Mazzeo-Pacard), and foliations by interfaces (del Pino, Kowalczyk, Wei, Yang).
- These phenomena seem to be quite different at the end.

The mechanism of foliations

- To explain we observe that the "single" traveling wave is stable (though no proofs yet) and so is the eternal solution.
- The speed of the eternal solutions is very sensitive to the asymptotic profiles of their ends. In fact there is a continuous family of eternal solutions parametrized by their speeds. They foliate the space.
- The middle parts of the two components of the multiple front traveling wave are "attracted" by an eternal solution with the given speed $c = \varepsilon$, while their ends "approach" the ends of eternal solutions with different speeds: the bottom one is slightly slower while the upper one is slightly faster.
- Foliating traveling waves "lie" on the boundary of the basin of attraction of the wave whose speed is ε.

Derivation of the Jacobi-Toda system

- We introduce the Fermi coordinates around Σ_ε,

$$x = p + z\nu_\varepsilon(p), \quad p \in \Sigma_\varepsilon,$$

- Since we seek solutions that depend on (r, x_{N+1}) only, we can assume that the Fermi coordinates depend on (r, z) only.
- We build an approximate solution of the form:

$$\begin{aligned} u(r,z) &= H(z - f_{\varepsilon,1}(\varepsilon r)) - H(z - f_{\varepsilon,2}(\varepsilon r)) - 1 \\ &\equiv H_{\varepsilon,1} - H_{\varepsilon,2} - 1, \end{aligned}$$

  where functions $f_{\varepsilon,j}$ are to be determined and H is the heteroclinic:

$$H'' + H(1 - H^2) = 0, \quad H(\pm\infty) = \pm 1.$$

- In these coordinates,

$$\Delta = \partial_{zz} + \Delta_{\Sigma_\varepsilon} - H_{\Sigma_\varepsilon}\partial_z$$

$$\partial_{N+1} f = \nabla f \cdot \nabla x_{N+1}$$

- The error of the approximate solution

$$\begin{aligned} S(u) \sim \sum_{j=1}^{2} \{\partial_{zz} H_{\varepsilon,j} + f(H_{\varepsilon,j})\} \\ + \sum_{j=1}^{2} \{(\varepsilon \nu_{N+1} - H_{\Sigma_\varepsilon})\partial_z H_{\varepsilon,j}\} \\ + \sum_{j=1}^{2} \{\big(\Delta_{\Sigma_\varepsilon} - z|A_{\Sigma_\varepsilon}|^2 \partial_z\big) H_{\varepsilon,j} \\ + \varepsilon \nabla_{\Sigma_\varepsilon} H_{\varepsilon,j} \cdot \nabla_{\Sigma_\varepsilon}(x_{N+1})\} \\ + f(\sum_{j=1}^{2} H_{j,\varepsilon} - 1) - \sum_{j=1}^{2} f(H_{j,\varepsilon}). \end{aligned}$$

- the interaction term

$$f(\sum_{j=1}^{2} H_{j,\varepsilon} - 1) - \sum_{j=1}^{2} f(H_{j,\varepsilon}) \sim e^{-\sqrt{2}|f_{\epsilon,1} - f_{\epsilon,2}|}$$

- Projection of the error onto $\partial_z H_{\varepsilon,j}$ gives formally the Jacobi-Toda system

$$\varepsilon^2 \alpha_0 \left[(\Delta_\Sigma + |A_\Sigma|^2) f_{\varepsilon,j} + \nabla_\Sigma f_{\varepsilon,j} \cdot \nabla_\Sigma (x_{N+1}) \right]$$

$$- \quad e^{\sqrt{2}(f_{\varepsilon,j-1} - f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j} - f_{\varepsilon,j+1})} = 0, \qquad (0.11)$$

where we always agree that $f_{\varepsilon,0} = -\infty$, $f_{\varepsilon,m+1} = \infty$. Here $\alpha_0 > 0$ is a universal constant.

- Here Jacobi operator

$$\mathcal{J}_\Sigma(\psi) = \Delta_\Sigma \psi + |A_\Sigma|^2 \psi + \nabla_\Sigma \psi_{\varepsilon,j} \cdot \nabla_\Sigma (x_{N+1}).$$

- Infinite dimensional reduction is used to justify this rigorously.

Other Jacobi-Toda Systems

Allen-Cahn equation on a compact N-dimensional Riemannian manifold $(M, \tilde{g})$

$$(AC)_M \quad \varepsilon^2 \Delta_{\tilde{g}} u + (1 - u^2)u = 0 \quad \text{in } M, \tag{0.12}$$

where $\Delta_{\tilde{g}}$ is the Laplace-Beltrami operator on M.

- Pacard and Ritoré : single interface on non-degenerate minimal $(N-1)$-dimensional submanifold of M.
- del Pino-Kowalczyk-Wei-Yang 2009: Assume that

$$|A|^2 + Ric > 0 \tag{0.13}$$

For any fixed integer $K \geq 2$, there exists a positive sequence $(\varepsilon_i)_i$ approaching 0 such that problem $(AC)_M$ has a solution u_ε with K phase transition layers with mutual distance $O(\varepsilon|\ln \varepsilon|)$.

- Near Γ, u_ε can be approximated by

$$u_\varepsilon \approx \sum_{k=1}^{K} \mathtt{w}\left(\frac{t - \varepsilon f_j(y)}{\varepsilon} \right) + \frac{1}{2}\left((-1)^{K+1} - 1\right),.$$

The functions f_j satisfy

$$\|f_j\|_\infty \le C|\ln\varepsilon|, \quad f_{j+1} - f_j = O(|\ln\varepsilon|), \quad 1 \le j \le K-1,$$

- and solve the Toda system,

$$\varepsilon^2\Big(\triangle_\Gamma f_j + \left(|A_\Gamma|^2 + \mathrm{Ric}_g(\nu_\Gamma,\nu_\Gamma)\right) f_j\Big) \tag{0.14}$$

$$= a_0\left[e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}\right]$$

in Γ, for $j = 1, \ldots, K$, for a universal constant $a_0 > 0$.

Remarks

- Resonance Condition: unlike Pacard-Ritore's result, which is true for all $\epsilon << 1$, here the result is true for a selected sequence. This is related to a resonance phenomena due the combined interaction of the interfaces and the curvature. In fact, we believe that the Morse index of our solution approaches $+\infty$ as $\varepsilon \to 0$.
  Similar phenomena has appeared in other problems: Malchiodi-Montenegro (2004), del Pino-Kowalczyk-Wei (2006), ...
  However, here the resonance phenomena is different from all the above. In all the above, resonance phenomena exists even for a single interface. Here the resonance phenomena only exists for interfaces, due to an intricate play between the curvature and the interaction of interfaces.

Why Resonance?

Before we go to the general case, let us consider the simplest of situation: $K = 2$. The system becomes

$$\varepsilon^2\Big(\triangle_\Gamma f_1 + (|A|^2 + Ric)f_1\Big) + e^{-\sqrt{2}(f_2 - f_1)} = \varepsilon^{2+\sigma} h_1 \quad \text{on} \quad \Gamma,$$

$$\varepsilon^2\Big(\triangle_\Gamma f_2 + (|A|^2 + Ric)f_2\Big) - e^{-\sqrt{2}(f_2 - f_1)} = \varepsilon^{2+\sigma} h_2 \quad \text{on} \quad \Gamma$$

Adding the above equations, we have

$$\triangle_\Gamma (f_1 + f_2) + (|A|^2 + Ric)(f_1 + f_2) = \varepsilon^\sigma (h_1 + h_2) \quad \text{on } \Gamma$$

By our nondegeneracy condition,

$$f_1 + f_2 = O(\varepsilon^\sigma)$$

Now let $u = \sqrt{2}(f_1 - f_2) + \log 2$, we arrive at the following simple PDE

$$(JT1) \quad \varepsilon^2(\Delta_g u + (|A|^2 + Ric)u) + e^u = \varepsilon^{2+\sigma} h \text{ on } \Gamma$$

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What are the difficulties in solving (JT1)?

- 1.variational methods, if works, can only find a solution and there is no information on asymptotic behavior of the solutions, since we have to ask

$$u << -1$$

- 2. when $N \geq 3$, e^u is supercritical, there is no way of using variational method.
- 3. A more difficult problem is the resonance phenomena.

Resonance Phenomena

Let us for simplicity we assume that

$$|A|^2 + Ric \equiv Constant = 1$$

Then equation (JT1) becomes

$$(JT2) \qquad \varepsilon^2(\Delta_g u + u) + e^u = \varepsilon^{2+\sigma} h \text{ on } \Gamma$$

When $h = 0$, it has a constant solution

$$\varepsilon^2 u_0 + e^{u_0} = 0$$

$$u_0 = \log \varepsilon^2 + \log\log \frac{1}{\varepsilon} + O(\log\log\log \frac{1}{\varepsilon})$$

Thus we take $u = u_0 + u_1$, then we are reduced to solving

$$(\Delta_g + (2\log \frac{1}{\varepsilon} + 1))u_1 = \varepsilon^\sigma h$$

The left hand operator has eigenvalues

$$\lambda_j - 2\log \frac{1}{\varepsilon}$$

As we know, by Weyl's formula,

$$\lambda_j \sim j^{\frac{2}{N-1}}$$

As $j \to +\infty$, $\lambda_j - 2\log\frac{1}{\varepsilon}$ may cross zero at large N.
This kind of resonance phenomena also appeared in

- Malchiodi-Montenegro: 2004 boundary layer for singularly perturbed elliptic problem
- del Pino-Kowalczyk-Wei: 2007 geodesics for nonlinear Schrodinger equation
- Malchiodi-Wei: 2007 boundary layer for Allen-Cahn equation near the boundary

Gap Condition

The problem can still be solved under some gap condition:it is possible to obtain

$$|\lambda_j - 2\log\frac{1}{\varepsilon}| \geq \delta^p$$

for p large and hence

$$\|v\| \leq C\delta^{-p}\|(\Delta_g + (2\log\frac{1}{\varepsilon} - 1))v\|$$

But the right hand error is $O(\varepsilon^\sigma)$ which controls any power of δ^{-p}.

More complicated proofs when $|A|^2 + Ric \not\equiv Constant$

A New Jacobi-Toda System for Traveling Waves

We need to solve the following new Jacobi-Toda system:

$$\varepsilon^2 \mathcal{J}_\Sigma f_{\varepsilon,j} - \ e^{\sqrt{2}(f_{\varepsilon,j-1}-f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j}-f_{\varepsilon,j+1})} = 0. \tag{0.15}$$

$$\mathcal{J}_\Sigma(\psi) = \Delta_\Sigma \psi + |A_\Sigma|^2 \psi + \nabla_\Sigma \psi_{\varepsilon,j} \cdot \nabla_\Sigma (x_{N+1}).$$

Main Result: For all ε small, problem (0.15) can be solved.

NO Resonance Needed !!! Why?
The convection term $\nabla_\Sigma \psi_{\varepsilon,j} \cdot \nabla_\Sigma (x_{N+1})$

Jacobi Operator

Our theory of solvability of the Jacobi-Toda system will be valid for functions of the radial variable r only and so we need to express the operator $\mathcal{J}$ in terms of the radial variable r. The Laplace-Betrami operator for a surface $x_{N+1} = F(r)$ acting on $v = v(r)$ is

$$\begin{aligned} \Delta_\Gamma v &= \frac{1}{r^{N-1}\sqrt{1+F_r^2}}\frac{\partial}{\partial r}\left(\frac{r^{N-1}}{\sqrt{1+F_r^2}}\frac{\partial}{\partial r}\right)v \\ &= \frac{v_{rr}}{1+F_r^2} + \left(\frac{N-1}{r} - \frac{F_r}{1+F_r^2}\right)v_r. \end{aligned} \tag{0.16}$$

The principal curvatures are given by

$$\mathbb{k}_1 = \ldots = \mathbb{k}_{N-1} = \frac{F_r}{r\sqrt{1+F_r^2}}, \quad \mathbb{k}_N = \frac{F_{rr}}{(1+F_r^2)^{3/2}}, \tag{0.17}$$

hence

$$|A_\Gamma|^2 = \sum_{j=1}^{N} \mathbb{k}_j^2 = \frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}. \tag{0.18}$$

Finally we have:

$$\nabla_\Gamma v \cdot \nabla_\Gamma F = \frac{v_r F_r}{1+F_r^2},$$

hence we find the following expression for the radial operator L_0:

$$\mathcal{J}[v] = \frac{v_{rr}}{1+F_r^2} + \frac{(N-1)v_r}{r} + \Big(\frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}\Big)v. \tag{0.19}$$

Let us change the independent variable

$$s = \int_0^r \sqrt{1 + F_r^2}\, dr. \tag{0.20}$$

The new variable s is nothing else but the arc length of the curve $\gamma = \{(r, F(r)), r > 0\}$ in $\mathbb{R}^2$. Using the asymptotic formula (0.8) for F we get that

$$s \sim r, \quad r \ll 1, \qquad s = \frac{r^2}{2(N-1)} + \mathcal{O}(\log r), \quad r \gg 1. \tag{0.21}$$

By a straightforward computation we obtain the following expression for the Jacobi operator $\mathcal{J}$:

$$\mathcal{J}[v] = v_{ss} + a(s)v_s + b(s)v \tag{0.22}$$

where

$$a(s) = \frac{F_r + \frac{N-1}{r}}{\sqrt{1+F_r^2}}, \quad b(s) = |A_\Gamma(r)|^2, \quad r = r(s). \tag{0.23}$$

Note that

$$a(s) = \frac{N-1}{s}\big(1 + \mathcal{O}(s^2)\big), \quad s \ll 1, \qquad a(s) = 1 + \mathcal{O}(s^{-1}), \quad s \gg 1,$$
$$b(s) = \frac{(N-1)}{r^2} + \mathcal{O}(r^{-4}) = \frac{1}{2s} + \mathcal{O}(s^{-2}\log s), \quad s \gg 1, \tag{0.24}$$

and that in general $a(s)$, $b(s) > 0$ since Γ is convex and $F_r(0) = 0$. We also have

$$b(0) = 1, b^{'}(0) = 0, b''(0) = -\frac{N^2+4N+2}{N^4(N+2)} < 0, \quad N = 2, \ldots. \tag{0.25}$$

A non-homogeneous Jacobi-Toda system

In reality we have to deal in general with the non-homogeneous Jacobi-Toda system. Thus we will consider the following problem:

$$\varepsilon^2 \alpha_0 \mathcal{J}[f_{\varepsilon,j}] - e^{\sqrt{2}(f_{\varepsilon,j-1} - f_{\varepsilon,j})} + e^{\sqrt{2}(f_{\varepsilon,j} - f_{\varepsilon,j+1})} = \varepsilon^2 h_{\varepsilon,j}, \qquad (0.26)$$

where $f_{\varepsilon,j} = f_{\varepsilon,j}(r)$, $h_{\varepsilon,j} = h_{\varepsilon,j}(r)$. The above problem can also be seen in terms of the arc length variable s.

To describe the strategy we use we will assume for simplification that $m = 2$, and denote $u_\varepsilon = \sqrt{2}(f_{\varepsilon,2} - f_{\varepsilon,1})$ and $v_\varepsilon = \sqrt{2}(f_{\varepsilon,1} + f_{\varepsilon,2})$, and respectively $h_\varepsilon = \frac{\sqrt{2}}{\alpha_0}(h_{\varepsilon,2} - h_{\varepsilon,1})$ and $g_\varepsilon = \frac{\sqrt{2}}{\alpha_0}(h_{\varepsilon,2} + h_{\varepsilon,1})$. Then we get the following decoupled system:

$$\mathcal{J}[u_\varepsilon] - \frac{2\sqrt{2}}{\varepsilon^2 \alpha_0} e^{-u_\varepsilon} = h_\varepsilon$$

$$\mathcal{J}[v_\varepsilon] = g_\varepsilon$$

Solvability of Jacobi Operator

$$\mathcal{J}[v_\varepsilon] = g_\varepsilon$$

We will see that the right hand side of this equation satisfies:

$$g_\varepsilon \sim \varepsilon^\tau (1+s)^{-\frac{3}{2}-\beta}, \quad \tau > 0, \beta \geq 0. \tag{0.27}$$

The equation can be solved by using the nondegeneracy of the traveling graph. The key observation is that the operator L_0 has a decaying, positive element in its kernel

$$\phi_0 = \frac{1}{\sqrt{1+F_r^2}} \sim \frac{1}{r} \quad (\sim \frac{1}{\sqrt{s}}). \tag{0.28}$$

$$|v_\varepsilon(s)| \leq C\varepsilon^\tau (2+s)^{-\frac{1}{2}} \log(s+2). \tag{0.29}$$

Simple Jacobi-Toda Equation

$$\mathcal{J}[u_\varepsilon] - \frac{2\sqrt{2}}{\varepsilon^2\alpha_0}e^{-u_\varepsilon} = h_\varepsilon \tag{0.30}$$

The solvability theory for the nonlinear equation (0.30) is another story. Even when the right hand side $h_\varepsilon = 0$, we still have the nonlinear term to deal with. In general the decay rate of this term will be actually slower and in addition it is a term of order ε^{-2}. In other words the real difficulty is in solving the non-homogeneous nonlinear problem (0.30).

Result: There exists a solution u_ϵ such that

$$u_\varepsilon(s) = \log\frac{2\sqrt{2}}{\varepsilon^2\alpha_0 b(s)} + \mathcal{O}(\log\log\frac{1}{\varepsilon^2 b(s)}), \quad s\to\infty \tag{0.31}$$

Defining $v(s) = u_\varepsilon(s)$ and $\frac{2\sqrt{2}}{\varepsilon^2 \alpha_0} = \frac{1}{\delta^2}$, we solve first

$$v_{ss} + a(s) v_s + b(s) v - \frac{1}{\delta^2} e^{-v} = 0, \quad \delta \sim \varepsilon \ll 1. \tag{0.32}$$

As $\epsilon \to 0$ the solution of this equation approaches the solution of the nonlinear ODE equation:

$$a(s) v_{0,s} + b(s) v_0 = \frac{1}{\delta^2} e^{-v_0},$$

from which the asymptotic formula

$$v_0(s) \sim \log \frac{1}{\delta^2 b(s)},$$

follows. This is very different from the way the classical Toda system is solved. Analysis of (0.32) is rather delicate (need to solve an inhomogeneous version of it).

Definition of the linearized operator $\mathfrak{L}_\delta$

From the above considerations we see that linearization around the approximate solution v_0 is the following operator

$$\mathfrak{L}_\delta[h] = h_{ss} + a(s)h_s + p_\delta(s)h = g(s). \tag{0.33}$$

$$0 < p_\delta(s) \leq C \log \frac{1}{\delta^2}, \tag{0.34}$$

while when $s \geq \bar{s}$ then $p_\delta(s)$ satisfies

$$\frac{1}{C(2+s)} \log \big(\frac{2+s}{\delta^2}\big) < p_\delta(s) \leq \frac{C}{2+s} \log \big(\frac{2+s}{\delta^2}\big), \quad s \geq 0. \tag{0.35}$$

A Model ODE Problem

The key is to solve the following problem

$$h_{ss} + a(s)h_s + \frac{\lambda}{1+s}h = g(s). \tag{0.36}$$

where $\lambda >> 1$,

$$a(s) \sim -1 + O(s^{-1})$$

To begin we make the following transformation:

$$\hat{h} = \exp\left(\frac{1}{2}\int_1^s a(\tau)\,d\tau\right)h. \tag{0.37}$$

Then near $s \sim 0$, $\hat{h} = s^{(N-1)/2}h$ and near $s \to +\infty$, $\hat{h} \sim e^{s/2}h$, by (0.24). Equation (0.36) is transformed to

$$\hat{h}'' + (p_\delta(s) - \hat{a}(s))\hat{h}(s) = \hat{g}, \tag{0.38}$$

where

$$\hat{a} = \frac{1}{2}a' + \frac{1}{4}a^2, \quad \hat{g} = \exp\left(\frac{1}{2}\int_1^s a(\tau)\,d\tau\right)g.$$

We mainly work with the transformed equation (0.38). A simplified version is

$$\mathfrak{L}_\lambda[h] = \hat{h}'' + (\frac{\lambda}{1+s} - \frac{1}{4})\hat{h}(s) = O(e^{\frac{s}{2}}(1+s)^{-3})), \qquad (0.39)$$

The problem: any kernel of

$$h^{''} + (\frac{\lambda}{1+s} - \frac{1}{4})h = 0$$

has large number of zeroes. We have to estimate the growth of the amplitudes.

The idea of is the following: we will consider the inner and the outer problem separately, construct suitable inverses of $\mathfrak{L}_\lambda$ for these problems and then "glue" the solutions. The situation now is more complicated since we have to consider the full second order problem. It is at this level that we take full advantage of some special properties of the eternal solution to the mean curvature flow.

We introduce the following weighted Hölder norms for functions $u\colon \mathbb{R}_+ \to \mathbb{R}$:

$$\|u\|_{\mathcal{C}^{\ell}_{\beta,\mu}(\mathbb{R}_+)} := \sum_{j=0}^{\ell} \sup_{s>1} \Big\{ (2+s)^{\beta+j} \Big[\log \Big(\frac{2+s}{\delta^2} \Big) \Big]^{\mu} \|u\|_{\mathcal{C}^{j}((s-1,s+1))} \Big\},$$

where β, and $\mu \geq 0$.

The point is to construct a right inverse of (0.33) which is bounded in the weighted norm defined above. More precisely we will show:

Lemma

Suppose that $\beta > 1$. Then there exists a constant $C > 0$ and a solution to (0.33) such that

$$\|h\|_{\mathcal{C}^{2,\mu}_{\beta}(\mathbb{R}_+)} \leq C\lambda \|g\|_{\mathcal{C}^{0,\mu}_{\beta+1}(\mathbb{R}_+)}. \tag{0.40}$$