Some topics involving fractional powers of elliptic operators

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1 – Introduction

FRACTIONAL POWERS OF AN OPERATOR : L self-adjoint positive

$$\forall\,\beta \in (0,1), L^\beta f = \frac{1}{\Gamma(1-\beta)} \int_0^\infty t^{-\beta}\, L\, e^{-Lt} f\, dt.$$

$$\forall\,\beta \in (0,1), L^\beta f = \frac{\sin(\pi(1-\beta))}{\pi} \int_0^\infty \lambda^{\beta-1}\, L\,(L+\lambda\,\mathsf{Id})^{-1} f\, d\lambda.$$

Introduction

PURPOSES :

- DERIVE FUNCTIONAL INEQUALITIES WITH NON LOCAL R.H.S.
- SINGULAR FRACTIONAL YAMABE PROBLEM

2 – Improved functional inequalities and Gagliardo-type norms

GOAL : OBTAIN WEIGHTED POINCARE INEQUALITIES WITH NON LOCAL R.H.S. AND GENERAL MEASURES

$M > 0$, C^2 such that there exists $\lambda(M) > 0$,

$$\int_{\mathbb{R}^n} |\nabla f(y)|^2 \, M(y) \, dy \geq \lambda(M) \int_{\mathbb{R}^n} \left| f(y) - \int_{\mathbb{R}^n} f(x) \, M(x) \, dx \right|^2 M(y) \, dy. \tag{1}$$

EXAMPLES :

1. $M = (2\pi)^{-n/2} \exp\left(-|x|^2/2\right)$: Gaussian measure
2. $M(x) = e^{-|x|}$, and more generally when $M(x) = e^{-|x|^\alpha}$ with $\alpha \geq 1$.

QUESTION : IS IT POSSIBLE TO IMPROVE THE INEQUALITY BY A GAGLIARDO-TYPE R.H.S. ?

THEOREM (Mouhot,Russ,S.) Assume M is C^2, positive and L^1 and satisfies a Poincaré inequality and furthermore there exists $\varepsilon > 0$ such that

$$\frac{(1-\varepsilon)\,|\nabla V|^2}{2} - \Delta V \geq c \text{ whenever } |x| > R., \qquad M = e^{-V}. \tag{2}$$

Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that for all $f \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f(x)\, M(x)\, dx = 0$,

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, M(x)\, dx\, dy \geq$$
$$\lambda_\alpha(M) \int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |\nabla \ln M(x)|^\alpha\right) M(x)\, dx.$$

MAIN OPEN PROBLEM : L^p THEORY

STEP 1 : REWRITING THE POINCARE INEQUALITY

By the definition of L :

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, M(x)\, dx = \int_{\mathbb{R}^n} Lf(x)\, f(x)\, M(x)\, dx.$$

for all $f \in \mathcal{D}(L)$.

$$\iint_{\mathbb{R}^n} |\nabla f(x)|^2 \, M(x)\, dx \geq \lambda'(M) \int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |\nabla \ln M(x)|^2\right) \, M(x)\, dx.$$

EQUIVALENT TO

$$L \geq \lambda' \mu, \tag{3}$$

where μ is the multiplication operator by $x \mapsto 1 + |\nabla \ln M(x)|^2$, nonnegative operator on $L^2(\mathbb{R}^n, M)$

Using a functional calculus argument : for any $\alpha \in (0,2)$,

$$L^{\alpha/2} \geq (\lambda')^{\alpha/2}\,\mu^{\alpha/2},$$

$$(\lambda')^{\alpha/2} \int_{\mathbb{R}^n} |f(x)|^2\,(1+|\nabla \ln M(x)|^2)^{\alpha/2}\,M(x)\,dx \leq$$

$$\int_{\mathbb{R}^n} \left|L^{\alpha/4} f(x)\right|^2\, M(x)\,dx = \left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)}.$$

STEP 2 : Off-diagonal L^2 estimates à la Gaffney on the resolvant of the Ornstein-Ulhenbeck operator $L = -M^{-1}\nabla \cdot (M\nabla)$:

Lemma : There exists $C_1 = C_1(M) > 0$ such that : for all disjoint subsets $E, F \subset \mathbb{R}^n$ with dist$(E, F) =: d > 0$, all function $f \in L^2(\mathbb{R}^n, M)$ supported on E and all time $t > 0$,

$$\left\|(\mathsf{I} + t\,L)^{-1} f\right\|_{L^2(F,M)} + \left\|t\,L(\mathsf{I} + t\,L)^{-1} f\right\|_{L^2(F,M)} \leq 4\,e^{-C_1\,\frac{d}{\sqrt{t}}}\,\|f\|_{L^2(E,M)}\,.$$

$\longrightarrow$ Idea of the proof : this is an energy type estimate.

STEP 3 : NON LOCAL BOUND

GOAL :

$$\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \leq C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|f(x)-f(y)\right|^2}{\left|x-y\right|^{n+\alpha}} M(x)\, dx\, dy.$$

FIRST : Quadratic estimate (Spectral theory) :

There exists $C_3 = C_3(M) > 0$ such that for all $f \in \mathcal{D}(L)$,

$$\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \leq C_3 \int_0^{+\infty} t^{-1-\alpha/2} \left\|t\, L\, (\mathsf{I} + t\, L)^{-1} f\right\|^2_{L^2(\mathbb{R}^n,M)}\, dt.$$

$\longrightarrow$ Now bound $\left\|t\, L\, (\mathsf{I} + t\, L)^{-1} f\right\|^2_{L^2(\mathbb{R}^n,M)}$

SECOND : DECOMPOSITION

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} Q\left(x_j^t, \sqrt{t}\right).$$

For fixed j, one has

$$t\,L\,(\mathsf{I} + t\,L)^{-1} f = t\,L\,(\mathsf{I} + t\,L)^{-1}\, g^{j,t}$$

where, for all $x \in \mathbb{R}^n$,

$$g^{j,t}(x) := f(x) - m^{j,t}$$

and $m^{j,t}$ is defined by

$$m^{j,t} := \frac{1}{\left|Q\left(x_j^t, 2\sqrt{t}\right)\right|} \int_{Q\left(x_j^t, 2\sqrt{t}\right)} f(y)\,dy$$

$$\begin{aligned}\left\|t\,L\,(\mathsf{I}+t\,L)^{-1}f\right\|^2_{L^2(\mathbb{R}^n,M)} &= \sum_{j\in\mathbb{N}}\left\|t\,L\,(\mathsf{I}+t\,L)^{-1}f\right\|^2_{L^2\left(Q\left(x_j^t,\sqrt{t}\right),M\right)} \\ &= \sum_{j\in\mathbb{N}}\left\|t\,L\,(\mathsf{I}+t\,L)^{-1}g^{j,t}\right\|^2_{L^2\left(Q\left(x_j^t,\sqrt{t}\right),M\right)},\end{aligned}$$

and we are left with the task of estimating

$$\left\|t\,L\,(\mathsf{I}+t\,L)^{-1}g^{j,t}\right\|^2_{L^2\left(Q\left(x_j^t,\sqrt{t}\right),M\right)}.$$

$\longrightarrow$ USE OFF-DIAGONAL ESTIMATES AND

For all $k \geq 1$,

$$\left\|g_k^{j,t}\right\|^2_{L^2(C_k^{j,t},M)} \leq$$

$$\frac{\bar{C}}{(2^k\sqrt{t})^n}\int_{x\in Q(x_j^t,2^{k+1}\sqrt{t})}\int_{y\in Q(x_j^t,2^{k+1}\sqrt{t})}|f(x)-f(y)|^2\;M(x)\,dx\,dy.$$

We deduce that

$$\int_0^\infty t^{-1-\alpha/2} \left\| t\, L\, (\mathsf{I} + t\, L)^{-1} f \right\|^2_{L^2(\mathbb{R}^n, M)} \, dt \leq$$

$$C_4 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \, M(x)\, dx\, dy$$

IMPORTANT REMARK :THERE IS NO LINK BETWEEN THE SINGULARITY OF THE RIESZ KERNEL AND THE MEASURE M

THEOREM (Russ, S.)

Let G be a unimodular Lie group with polynomial growth. Let $d\mu_M = M dx$ be a measure absolutely continuous to the Haar measure, with Haar volume $V(r)$, on G where $M = e^{-v} \in L^1(G)$. Assume that there exists constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that, for all $x \in G$,

$$\frac{1-\varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c \text{ whenever } |x| > R. \tag{4}$$

Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that, for any function $f \in \mathcal{D}(G)$ such that $\int_G f(x)\, d\mu_M(x) = 0$,

$$\iint_{G\times G} \frac{|f(x)-f(y)|^2}{V\left(|y^{-1}x|\right)|y^{-1}x|^\alpha}\, dx\, d\mu_M(y) \geq$$

$$\lambda_\alpha(M) \int_G |f(x)|^2 \left(1 + \sum_{i=1}^{k} |X_i v(x)|^2\right) d\mu_M(x).$$

→ HARDY INEQUALITIES ON CARNOT GROUPS (simply connected groups with a stratification of the Lie algebra)

THEOREM (Russ, S.) Let G be a Carnot group with homogeneneous dimension $n \geq 3$; let N be a norm on the group (i.e. $N(x) = u(x)^{1/2-n}$ for $x \neq 0$ and $N(0) = 0$, $-\Delta u = \delta$). Then for all $\alpha > 2-n$ and all $s \in (0,1)$,

$$\int_G u^2(x) \left(\frac{|\nabla_G N|}{|N|} \right)^s N^\alpha dx \lesssim \iint_{G\times G} \frac{|u(x)-u(y)|^2}{|y^{-1}x|^{n+s}} N^\alpha(x)\, dx\, dy. \qquad (5)$$

$\longrightarrow$ RIEMANNIAN MANIFOLDS (with doubling of the volume + Faber-Krahn inequality)

Assume that $\rho : M \to [0, +\infty)$ satisfies

$$|d\rho| \leq 1 \text{ on } M, \;\; \Delta\rho \leq -\frac{C}{\rho}, \tag{6}$$

where $C > 0$.

THEOREM (Russ, S.) Let M be a complete non compact Riemannian manifold. Then, if $\alpha < C - 1$ and $\rho^{-1}(0)$ has codimension greater than $2 - \alpha$, one has, for all $s \in (0, 1)$,

$$\int_M u^2(x)\rho^{s(\alpha-2)}(x)dx \lesssim \iint_{M\times M} \frac{|u(y) - u(x)|^2}{d(x, y)^{p+s}} \rho^{\alpha}(x)dxdy. \tag{7}$$

3 – Conformally covariant fractional operators

QUESTION : IS THERE A CONFORMALLY COVARIANT GEOMETRIC FRACTIONAL ORDER OPERATOR ON M ?

IF SO : DOES IT GIVE GEOMETRIC INFORMATION ON THE MANIFOLD ?

$(M^n, \bar{g})$: compact Riemannian manifold, $n \geq 3$. If $\Lambda \subset M$ is any closed set, then the 'standard' singular Yamabe problem concerns the existence and geometric properties of complete metrics $g = u^{\frac{4}{n-2}}\bar{g}$ with constant scalar curvature, i.e.

$$\Delta_{\bar{g}} u + \frac{n-2}{4(n-1)} R^{\bar{g}} u = R^g\, u^{\frac{n+2}{n-2}}, \qquad u > 0,$$

1. $R^g < 0$ exist quite generally if Λ is sufficiently large in a capacitary sense (Labutin)

2. $R^g > 0$ existence is only known when Λ is a smooth submanifold (possibly with boundary) of dimension $k \leq (n-2)/2$ (Mazzeo-Pacard)

THEOREM (Schoen-Yau)

If (M, h) is a compact manifold with locally conformally flat metric h with positive scalar curvature, then the developing map D from the universal cover $\widetilde{M}$ to S^n is injective, and moreover, $\Lambda := S^n \setminus D(\widetilde{M})$ has Hausdorff dimension less than or equal to $(n-2)/2$. More generally, they also showed that if $\Omega = S^n \setminus \Lambda$ carries a complete metric $g = u^{\frac{4}{n-2}}\bar{g}$ (where $\bar{g}$ is the standard round metric) with positive scalar curvature and bounded Ricci curvature, then $\dim \Lambda \leq (n-2)/2$.

$(X, \bar{g})$: Riemannian manifold, $(\partial X = M, g)$ conformal infinity of X

$$\bar{g} = \frac{dx^2 + g_x}{x^2}$$

with

$$g_0 = g$$

See Graham-Jenne-Mason-Sparling, Fefferman-Graham, Graham-Zworski, Fefferman-Graham : For $s \notin n/2 + \mathbb{N}$

$$\begin{cases} -\Delta_{\bar{g}} u - s(n-s)u = 0 & \text{in } X \\ u = f & \text{on } \partial X = M. \end{cases} \tag{8}$$

$$u = \mathcal{P}(s) f = x^{n-s} F + x^s G, \quad F|_M = f$$

Conformally covariant fractional operators

Scattering operator :

$$f \to \mathcal{S}(s)f = G|_M$$

This is a meromorphic family of pseudo-diff operators for $Re(s) > n/2$ with poles at $s = \frac{n}{2} + \mathbb{N}$ of finite rank AND

$$symbol_p(\mathcal{S}(s)) = C_{n,s} symbol_p((-\Delta)^s)$$

Define : $\gamma \in (0,1)$ and $\gamma = s - n/2$

$$P_\gamma f = d_{n,s}\mathcal{S}(n/2+\gamma).$$

THIS IS A GEOMETRIC OPERATOR on M : if $g_v = v^{\frac{4}{n-2\gamma}} g_0$ then

$$P_\gamma^{g_v} f = v^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g_0}(vf)$$

REMARK :

– $P_1 = L_g = \Delta_g + \frac{n-2}{4(n-1)} R_g(x)$. Conformal laplacian

– $P_2 = \Delta^2 + \delta(a_n R_g g + b_n Ric)d - c_n Q$. Paneitz operator

GEOMETRIC CONSEQUENCES ?

Definition : The Q-curvature of order γ associated to a metric g is defined by

$$Q_\gamma^g = P_\gamma^g(1). \tag{9}$$

Fractional Yamabe problem : given a background metric $\bar{g}$, find $u > 0$ so that if $g = u^{4/(n-2\gamma)}\bar{g}$, then Q_γ^g is constant. This amounts to solving

$$P_\gamma^{\bar{g}} u = Q_\gamma^g u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0, \tag{10}$$

for $Q_\gamma^g = \text{const}$.

Model of a complete metric : $\mathbb{S}^n \backslash \mathbb{S}^k$ with the product metric of

$$\mathbb{S}^{n-k-1} \times \mathbb{H}^{k+1}.$$

THEOREM (Gonzalez, Mazzeo, S.) Suppose that $(X^n, \bar{g})$ is compact and $g = u^{\frac{4}{n-2\gamma}}\bar{g}$ is a complete metric on $\Omega = X \setminus \Lambda$, where Λ is a smooth k-dimensional submanifold and u is polyhomogeneous along Λ. If $0 < \gamma < \frac{n}{2}$ and $Q^g_\gamma > 0$ everywhere, then n, k and γ are restricted by the inequality

$$\Gamma(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2})/\Gamma(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}) > 0. \tag{11}$$

Remark : $k < (n-2\gamma)/2 \to \Gamma(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2})/\Gamma(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}) > 0$ and equivalence for $\gamma = 1$.

Conformally covariant fractional operators

OPEN PROBLEMS :

1. Definition of these operators on general complete manifolds
2. Case when the singular set is *a priori* rectifiable.
3. Geometrization problem : Let (M, g) be a compact manifold. Is there a conformal metric such that Q_γ is constant ? See Qing-Raske...

Existence theorems :

THEOREM (Gonzalez, Mazzeo, S.) Let Γ be a discrete subgroup of SO$(n+1,1)$ which acts discretely and properly discontinuously on $\mathbb{H}^{n+1}$. Suppose that the Poincaré exponent $\delta(\Gamma)$ lies in the interval $(0, n/2)$, and let $\Lambda = \Lambda(\Gamma)$ be the limit set of Γ in S^n. If $\dim \Lambda < \frac{1}{2}(n-2\gamma)$, then $\Omega = S^n \setminus \Lambda$ admits a complete conformally related metric g with $Q_\gamma^g > 0$.

THEOREM (Gonzalez, Mazzeo, S.) Let X be any n-manifold with nonnegative Yamabe constant and Λ a k-dimensional submanifold with $k < \frac{1}{2}(n-2)$. Then for all γ in some range $(1-\epsilon, 1+\epsilon)$, there exists a solution to

$$P_\gamma^{\bar{g}} u = Q_\gamma^g u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0.$$