A stochastic particle system for the Burgers' equation.

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Burgers' equation

$$\partial_t u_t + u\partial_x u_t = \nu\partial_x^2 u_t,\ u_0(x) \text{ is given},\ x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

- When $\nu = 0$, classical solutions exist only for finite time.
- Classical solutions can be continued as weak solutions.
- The continued weak solutions are classical on regions separated by shocks.
- They satisfy a compatibility (Rankine-Hugoniot) condition along the shocks.
- One can choose a unique such solution satisfying Laxs entropy condition.
- For $\nu > 0$ global existence of classical solutions is known.

Method of characteristics for inviscid Burgers' equation

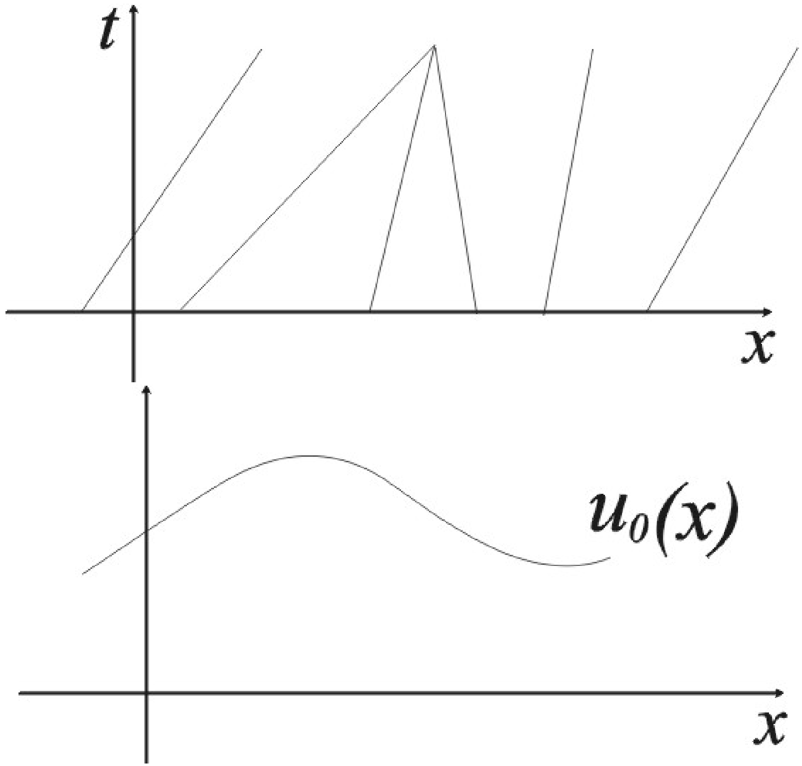
$$\partial_t u_t + u_t \partial_x u_t = 0,\ u_0(x) \text{ is given},\ x \in \mathbb{R},\ t \in \mathbb{R}^+$$

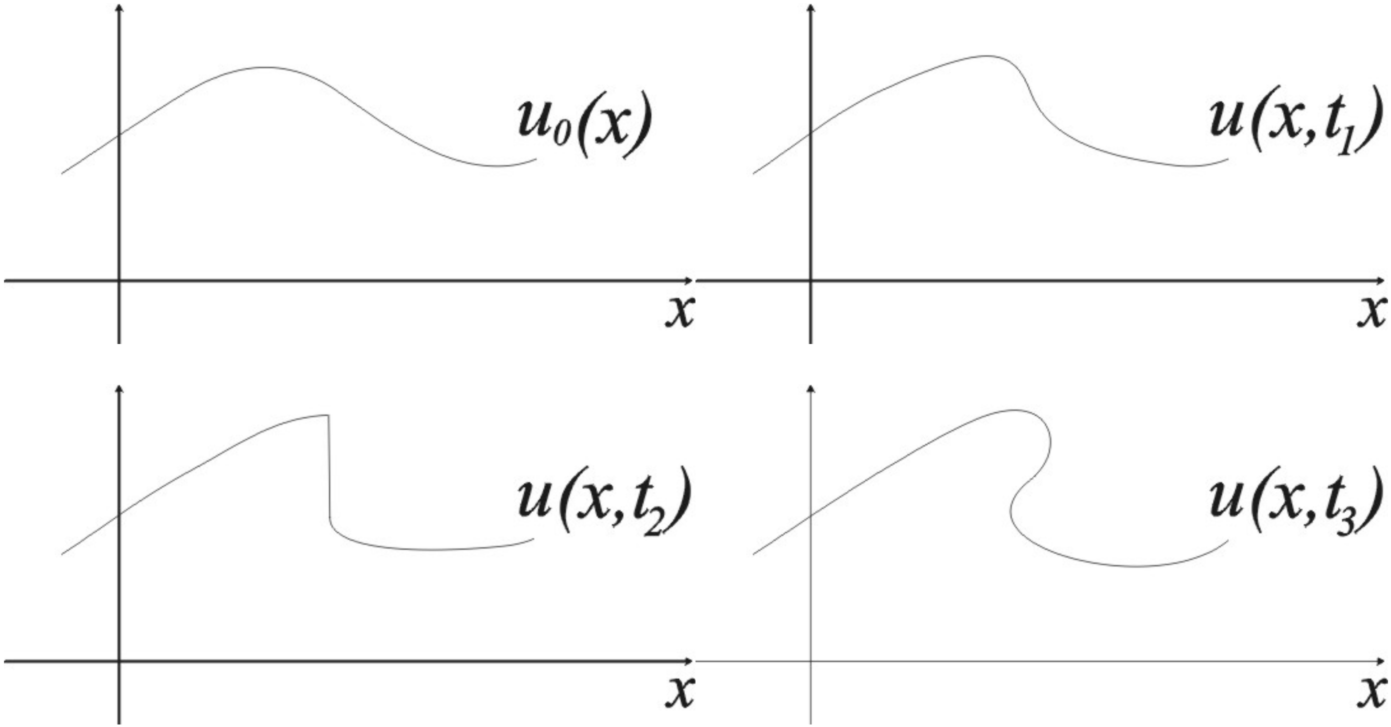
On the line

$$x = x_0 + u_0(x_0)t$$

the function $u(x, t)$ is constant.

$$x = x_0 + u_0(x_0)t$$





u0(x)
x
u(x,t1)
x
u(x,t2)
x
u(x,t3)
x

A stochastic Lagrangian approach to viscous Burgers' equation

$$\partial_t u_t + u_t \partial_x u_t = \nu \partial_x^2 u_t, \; u_{t=0}(x) = u_0(x), x \in \mathbb{R}, \; t \in \mathbb{R}^+.$$

Method of "random" characteristics:

$$dX_t = u_t(X_t)dt + \sqrt{2\nu}\, dW_t, \; u_t = \mathbb{E}\left[u_0 \circ (X_t^{-1}) \right].$$

Note that W_t is independent of x.

Idea of proof

$$dX_t = u_t(X_t)dt + \sqrt{2\nu}\, dW_t,\ \ X_0(a) = a.$$

Itô: If θ_t is constant along trajectories of X_t, then θ_t satisfies

$$d\theta_t + \left(u_t \partial_x \theta_t - \nu \partial_x^2 \theta_t\right) dt + \sqrt{2\nu}\, \partial_x \theta_t \, dW_t = 0.$$

In particular $A_t = X_t^{-1}$ and $\theta_t = \theta_0 \circ A_t$ both are constant along trajectories.

Thus $\bar{\theta}_t = \mathbb{E}\left[\theta_t\right]$ solves convection-diffusion equation:

$$d\bar{\theta}_t + u_t \partial_x \bar{\theta}_t = \nu \partial_x^2 \bar{\theta}_t.$$

We force $\bar{\theta}_t \equiv u_t$ by fixed point argument.

Monte-Carlo simulations

Exact

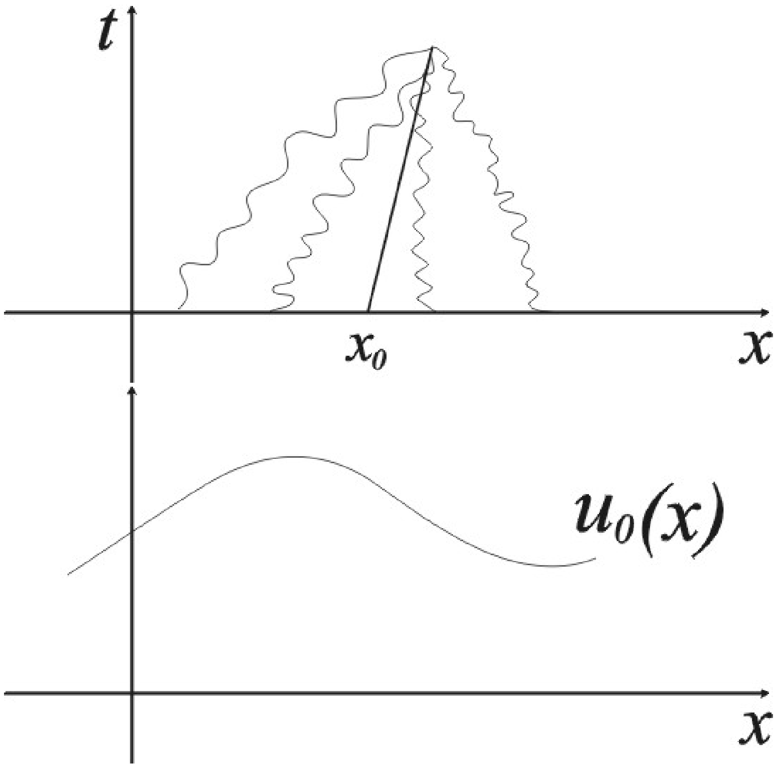
$$dX_t = u_t(X_t)dt + \sqrt{2\nu}\, dW_t,\ u_t = \mathbb{E}\left[u_0 \circ A_t\right],\ A_t = X_t^{-1}.$$

Approximate by a particle system
take N independent copies of Brownian motions

$$dX_t^i = u_t^N(X_t^i)dt + \sqrt{2\nu}\, dW_t^i,\ i = 1, ..., N,$$

$$u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ A_t^i,\ A_t^i = (X_t^i)^{-1}.$$

$$x \approx x_0 + u_0(x_0)t + \sqrt{2\nu}W_t$$



Motivation: Navier-Stokes equations

- Numerical *random vortex method* in 2 dimensions:
  time-splitting: vortex method + neat flow
  (A.Chorin, 1973, 1978 J.Goodman, 1987, D.Long, 1988).
  If $\omega = \nabla \times u$, then $u = -\Delta^{-1}\nabla \times \omega$ and we obtain
  vorticity formulation of the Navier-Stokes equations:

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega.$$

- Stochastic cascade by backward in time branching in Fourier space
  (Y. LeJan& A. S. Sznitman 1997).

- Noisy flow paths + Girsanov's change of variables
  (2D B.Busnello, 1999, 3D B.Busnello, F.Flandoli & M.Romito, 2005).

Self-contained stochastic formulation of NSE

- Noisy flow paths + Fixed point (P.Constantin&G.Iyer,2006)
- Self-contained proof of local $C^{1,\alpha}$ existence of the 2D solutions (G.Iyer,2005) and of the 3D small solutions (G.Iyer,2007).
- Further generalizations to MHD (G.Eyink, 2009).
- Self-contained proof of local $C[(0,T), W^{k+2,p}]$ existence of 3D solutions (Y.Zhang,2009).

Small probability "proof" for the Navier-Stokes equations

- We have probabilistic interpretation of the Navier-Stokes equations.
- Can we discard some "bad" set in the probability space and show that the Navier-Stokes equations are regular on its complement?
- Vlasov-McKean nonlinearity. For the stochastic-Lagrangian formulation

$$dX_t = u_t(X_t)dt + \sqrt{2\nu}\, dW_t,$$

$u_t = \mathbb{E}\left[u_0 \circ (X_t^{-1})\right]$ depends on the whole probability space.

Particle Systems are free of Vlasov-McKean nonlinearity

$$dX_t^i = u_t^N(X_t^{i,N})dt + \sqrt{2\nu}\, dW_t^i,\ i = 1, \dots, N,$$

where W_t^i are N independent copies of Brownian motion, and

$$u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ A_t^i,\ A_t^i = (X_t^i)^{-1}.$$

Note that u_t^N is a random variable, defined for each elementary event $\omega \in \Omega$.

Also $u_t^N \to u_t$, the solution of the viscous Burgers' equation, as $N \to \infty$.

Same is true for the analogous Navier-Stokes system (G.Iyer&Mattingly,2009).

Shocks and other deficiencies of Particle Systems

Theorem (with G.Iyer 2009). Suppose $u_0 \in C^1(\mathbb{R})$ is decreasing, and u_t is the solution of the particle system. Let the stopping time τ be the largest time of existence of the continuous solution of the particle system. Then

$$\tau < \frac{N}{||\partial_x u_0||_{L^\infty}}.$$

Further, *entropy-type arguments do not help* in defining a weak solution past shocks, because weak formulation involves second-order spatial derivatives.

Theorem (G.Iyer&Mattingly, 2009). For the 2D Navier-Stokes analogue

$$\limsup_{t\to\infty} \mathbb{E}||\nabla u_t^N||_{L^2}^2 \geqslant \frac{C}{N}||u_0||_{L^2}^2,\ C = C(\Omega).$$

Resetting

Solve

$$dX_t^i = u_t^\delta(X_t^i)dt + \sqrt{2\nu}\,dW_t^i,\ u_t^\delta = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ (A_t^i),\ A_t^i = (X_t^i)^{-1}.$$

only on $t \in [0, \delta]$. Then reset:

$$u_\delta^\delta = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ (A_\delta^{i,N}).$$

Use u_δ^δ as initial conditions, and solve

$$dX_t^i = u_t^\delta(X_t^i)dt + \sqrt{2\nu}\,dW_t^i,\ u_t^\delta = \frac{1}{N}\sum_{i=1}^{N} u_\delta^\delta \circ A_t^i,\ A_t^i = (X_t^i)^{-1}.$$

only on $t \in [\delta, 2\delta]$, and so on.

Main difference between resetting and no-resetting

$$du_t^i + u_t^\delta \partial_x u_t^i \, dt - \nu \partial_x^2 u_t^i \, dt + \tfrac{\nu}{2} \partial_x u_t^i \, dW_t^i = 0, \quad \text{for } i \in \{1, \dots, N\}$$

$$du_t^\delta + u_t^\delta \partial_x u_t^\delta \, dt - \nu \partial_x^2 u_t^\delta \, dt + \frac{\nu}{2N} \sum_{j=1}^{N} \partial_x u_t^j \, dW_t^j = 0$$

If we reset, then $u_t^i \approx u_t^\delta$. Thus, as $\delta \to 0$, $u_t^\delta \to v_t$ and v_t satisfies

$$dv_t + v_t \partial_x v_t \, dt - \nu \partial_x^2 v_t \, dt + \nu \frac{\partial_x v_t}{2N} \sum_{j=1}^{N} dW_t^j = 0.$$

Markov property

- Original system with is Markov

$$dX_t = u_t(X_t)dt + \sqrt{2\nu}\, dW_t,\ u_t = \mathbb{E}\left[u_0 \circ (A_t)\right],\ A_t = X_t^{-1},$$

because u_t solves

$$\partial_t u_t + u\partial_x u_t = \nu \partial_x^2 u_t,\ u_0(x) \text{ is given},\ x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

- Markov property is lost for the non-reset particle system

$$dX_t^i = u_t^N(X_t^i)dt + \sqrt{2\nu}\, dW_t^i,\ u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ A_t^i,\ A_t^i = (X_t^i)^{-1}.$$

- Approximate Markov property for u_t^δ, because v_t is Markov.

Regularizing effect of resetting

Fix a small probability $\varepsilon > 0$, arbitrary time T, and sufficiently regular initial conditions $u_0 \in H^s(\mathbb{T})$, $s > 6+\frac{1}{2}$. We know a δ_t, so that the solution is smooth with probability $1-\varepsilon$.

Theorem (with G.Iyer 2009). There exists δ_0, that depends only on the above, so that for $\delta \leqslant \delta_0$ there exists a spatially independent stopping time τ with

$$P(\tau > T) > 1 - \varepsilon, \text{ and } u^\delta_{t\wedge\tau} \in C^6([0,\tau];\mathbb{T}),$$

where u^δ_t is the reset process:

$$dX^i_t = u^\delta_t(X^i_t)dt + \sqrt{2\nu}\, dW^i_t, \; u^\delta_t = \frac{1}{N}\sum_{i=1}^N u^\delta_{k\delta} \circ A^i_t, \; A^i_t = (X^i_t)^{-1}$$

on $t \in [k\delta, (k+1)\delta]$.

Idea of proof

- Show that as $\delta \to 0$, $u_t^\delta \to v_t$ and v_t satisfies

$$dv_t + v_t \partial_x v_t \, dt - \nu \partial_x^2 v_t \, dt + \nu \frac{\partial_x v_t}{2N} \sum_{j=1}^{N} dW_t^j = 0.$$

- SPDE above is dissipative for $N > 1$! Prove a strong norm of v is uniformly bounded in time. (Fourier series estimate).
- Show $\sup_{t \leqslant T} \mathbb{E} ||u_t^\delta - v_t||_{H^s}^2 \leqslant C\sqrt{\delta}$ almost surely.
- Gives a uniform in time bound on $||u_t^{\delta_t}||_{C^1}$ with large probability.
- Local existence depends only on $||u_t^{\delta_t}||_{C^1}$. Thus global existence.

Deficiencies of proof

- No $\mathbb{R}$, only $\mathbb{T}$. Due to Fourier series argument.

- We do not know how to handle boundaries: Stopping loses invertibility.

- Numerically even **large** stopping δ prevents formation of shocks.

- No geometry! Fourier series = Sobolev spaces. Sobolev embedding gives C^1.

THANK YOU!

Shocks in Particle Systems

Theorem (with G.Iyer 2009). Suppose $u_0 \in C^1(\mathbb{R})$ is decreasing, and u_t is the solution of the particle system. Let the stopping time τ be the largest time of existence of the continuous solution of the particle system. Then

$$\tau < \frac{N}{||\partial_x u_0||_{L^\infty}}.$$

Further, *entropy-type arguments do not help* in defining a weak solution past shocks, because weak formulation involves second-order spatial derivatives.

Shocks for the Particle System. Idea of proof

- The method characteristics defines a family of smooth maps

$$X_t^{i,N} : \mathbb{R} \to \mathbb{R}, X_t^{i,N} : a \to X_t^{i,N}(a).$$

- This map is bijection, thus $\partial_x X_t^{i,N} > 0$, $\partial_x A_t^{i,N} > 0$ (1D argument).
- This map stops to be bijection exactly when characteristics meet, or, alternatively, when the inverse function theorem to X_t could not be applied. Thus we must estimate the first time T when $\partial_x X_T^{i,N} = 0$.
- Monotonicity of u_0 imply that $\partial_x u_0 < 0$. Say $\partial_x u_0|_0 = -1$.

$$d_t(\partial_x X_t^{1,N})\Big|_0 = \partial_x u_t^N\Big|_{X_t^{1,N}} \partial_x X_t^{1,N}\Big|_0 \text{ (noise does not depend on } x\text{)}$$

$$= \frac{1}{N}\left[\partial_x u_0\Big|_0 + \sum_{i=2}^{N} \partial_x u_0\Big|_{A_t^{i,N} \circ X_t^{1,N}} \cdot \partial_x A_t^{i,N}\Big|_{X_t^{1,N}} \partial_x X_t^{1,N}\Big|_0\right] \leqslant -1/N.$$

Further work

- Quasigeostrophic equation

- Small probability existence of Navier-Stokes

- Resetting for Navier-Stokes removes phenomenon

$$\limsup_{t\to\infty} \mathbb{E}||\nabla u_t^N||_{L^2}^2 \geqslant \frac{C}{N}||u_0||_{L^2}^2, \ C = C(\Omega).$$

- ...