Traveling waves in an inhomogeneous medium

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The reaction-diffusion equation

\[ u_t = u_{xx} + f(x, u), \quad x \in \mathbb{R}, \quad t > 0. \]

Solutions will behave like a traveling wave with moving interface. . . .

(i) How does the solution evolve at large times?

(ii) If \( f(x, u) \) is random, what are the statistical properties of \( u \)?
Pushed fronts in a homogeneous environment

Suppose \( u(t, x) \) satisfies

\[
    u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \quad t > 0
\]

\[
    u(0, x) = u_0(x) \in [0, 1]
\]

\( f(u) \) is nonlinear and \( \int_0^1 f(u) \, du > 0 \):

\begin{align*}
    f(u) & \quad \quad \quad f(u) \\
    \theta & \quad \quad \quad \theta \\
    u = 1 & \quad \quad u = 1
\end{align*}
Diffusion + Reaction = front propagation

Traveling wave solutions:

\[ \tilde{u}(t, x) = \tilde{u}(0, x - \tilde{c}t), \quad x \in \mathbb{R}, \ t \in \mathbb{R} \]
Traveling wave solutions are attractors.

If \( u(t, x) \) solves the initial value problem with appropriate “wave-like” initial data at \( t = 0 \), then for some \( \tau \in \mathbb{R} \),

\[
\sup_x |u(t, x) - \tilde{u}(t + \tau, x)| \leq Ce^{-rt}, \quad \forall \ t \geq 0
\]

The inhomogeneous environment

\[ u_t = u_{xx} + f(x, u), \quad x \in \mathbb{R}, \quad t > 0; \quad u(0, x) = u_0(x). \]

- \( f_{\min}(u) \leq f(x, u) \leq f_{\max}(u) \)
- \( \int_0^1 f_{\min}(u) \, du > 0 \)
- For example: \( f(x, u) = g(x) f_0(u), \quad g(x) > 0. \)
If $f(x,u)$ is periodic in $x$ there are **pulsed traveling waves**

$$\tilde{u}(t + \frac{L}{\tilde{c}}, x) = \tilde{u}(t, x - L)$$

For example, see Berestycki, Hamel (2002), Xin (1992, 1993).

What if we do not impose a periodic structure on $f$?
What does the solution look like?

The initial data is a step function (in black).

The plot shows $u(t, x)$ at regularly-spaced points in time. $g(x)$ was randomly generated.
The interface width does **not** spread out as $t \to \infty$.

For some universal constant $C$,

$$|X^+(t) - X^-(t)| \leq C$$

holds for all $t$ sufficiently large.
Two solutions with different initial data (in black).
A Generalized Traveling Wave:

There exists a right-moving transition-front solution $\tilde{u}(t, x)$ of

$$
\tilde{u}_t = \tilde{u}_{xx} + g(x)f_0(\tilde{u}), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.
$$

It is unique up to a time shift. Also, $\tilde{u}_t > 0$, for all $x \in \mathbb{R}, t \in \mathbb{R}$. This solution is an attractor: if $u_0(x)$ is wave-like, then there is a time shift $\tau$ and constants $C, r > 0$ such that

$$
\sup_{x \in \mathbb{R}} |u(x, t) - \tilde{u}(t + \tau, x)| \leq Ce^{-rt}
$$

holds for all $t \geq 0$.

Mellet, Roquejoffre, Sire (2009),
N., Ryzhik (2009),
Mellet, N., Ryzhik, Roquejoffre (2009)
What if $f$ is random?

Suppose that

$$f = g(x, \omega)f_0(u)$$

where $g(x, \omega) : \mathbb{R} \times \Omega \rightarrow (0, \infty)$ is a stationary random field, with suitable bounds and regularity.

Let $\{\pi_x\}_{x \in \mathbb{R}}$ be a group of measure-preserving transformations which act ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$ so that $g(x + h, \omega) = g(x, \pi_h \omega)$.

In this case, the preceding results hold with probability one.
A Law of Large Numbers for the interface

Let \( X(t, \omega) \) be the random interface position:

\[
X(t, \omega) = \sup\{x \in \mathbb{R} \mid u(t, x, \omega) = \frac{1}{2}\}.
\]

Then \( X(t, \omega) \) satisfies

\[
\lim_{t \to \infty} \frac{X(t, \omega)}{t} = \tilde{c}, \quad \text{almost surely, and in } L^1(\Omega).
\]

The constant \( \tilde{c} > 0 \) is independent of the initial data.

N., Ryzhik (2009)

See Freidlin-Gärtner (1979) for a related result with K.P.P.-type nonlinearity.
A Central Limit Theorem

If the environment is sufficiently mixing, then

(i) There is $\kappa^2 \geq 0$ such that

$$\frac{X(t, \omega) - t\tilde{c}}{\sqrt{t}} \to N(0, \kappa^2), \quad \text{as } t \to \infty.$$ 

(ii) If $\kappa^2 > 0$, the family of continuous process $\{Y_n(t)\}_{n=1}^{\infty}$ defined by

$$Y_n(t, \omega) = \frac{X(nt, \omega) - nt\tilde{c}}{\kappa \sqrt{n}}, \quad t \in [0, 1],$$

converges weakly (as $n \to \infty$) to a standard Brownian motion on $[0, 1]$, in the sense of weak convergence of measures on $C([0, 1])$ with the topology of uniform convergence.

N. (2009)
Numerical observation of Gaussian fluctuations in interface position:

Left: Histogram for the random variable $X(t, \omega)$, 13,000 samples.

Right: Quantile-quantile plot vs. normal distribution.
Bounds on the variance $\kappa^2$

One can construct random media for which $\kappa^2 > 0$. Under the scaling

$$f(x, u) \rightarrow f\left(\frac{x}{L}, u\right), \quad L > 0$$

the variance is bounded by

$$C_1 L \leq \kappa^2(L) \leq C_2 L$$

for $L$ sufficiently large, while $0 < C_3 < \tilde{c}(L) \leq C_4$. 
**Statistical invariance** of the generalized traveling wave:

We may normalize $\tilde{X}(0, \omega) = 0$, so that

$$
\tilde{u}(T_k(\omega), x + k, \omega) = \tilde{u}(0, x, \pi_k \omega), \quad \forall \, k \in \mathbb{R}
$$

$T_k = T_k(\omega)$ is the hitting time to $x = k$: $\tilde{X}(T_k, \omega) = k$.

Increments $\Delta T_k = T_{k+1} - T_k$ are stationary with respect to $k$.

In this sense, the profile is **statistically invariant** with respect to reference point $x = k$. 
How do we obtain a CLT for $X(t, \omega)$?

Consider the hitting times

$$T_k(\omega) = \inf \{ t \geq 0 | \tilde{X}(t, \omega) = k \}.$$

Then

$$\frac{T_n - \tilde{\tau}n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (\Delta T_k - \mathbb{E}[\Delta T_k]),$$

where $\Delta T_k = T_{k+1} - T_k$.

For the traveling wave, the increments $\Delta T_k$ are identically distributed, but not independent.
Stability of the wave under perturbations of the environment enables us to show that

\[ \Delta T_k = T_{k+1} - T_k \]

does not depend strongly on the distant past:

or distant future:

\[ \Delta T_k \]

depends primarily on the local environment near \( x = k \).
Many interesting problems to consider:

- Propagation in multiple dimensions
- Systems of equations, propagating pulses

Thank you for your attention!

References:

The mixing condition

Define the family of $\sigma$-algebras

$$\mathcal{F}^-_k = \sigma(g(x, \omega) | x \leq k)$$
$$\mathcal{F}^+_k = \sigma(g(x, \omega) | x \geq k)$$

$$\mathcal{F}^+_k \subset \mathcal{F}^{-}_{k+1} \subset \mathcal{F}, \quad \text{and} \quad \mathcal{F} \supset \mathcal{F}^+_k \supset \mathcal{F}^+_k$$

We say the environment is $\phi$-mixing if for all $j \geq k$ and any $\xi \in L^2(\Omega, \mathcal{F}^-_k, \mathbb{P})$ and $\eta \in L^2(\Omega, \mathcal{F}^+_j, \mathbb{P})$,

$$|\mathbb{E}[\xi \eta] - \mathbb{E}[\xi] \mathbb{E}[\eta]| \leq \sqrt{\phi(j - k)} \left(\mathbb{E}[\xi^2] \mathbb{E}[\eta^2]\right)^{1/2}$$

for $\phi(n) : \mathbb{Z}^+ \to [0, \infty)$ is nonincreasing. If $\sum_{n \geq 1} \sqrt{\phi(n)} < \infty$, then the invariance principle holds.