Exit times of diffusions with incompressible drift

Gautam Iyer, Carnegie Mellon University
  gautam@math.cmu.edu

Collaborators:
  Alexei Novikov, Penn. State
  Lenya Ryzhik, Stanford University
  Andrej Zlatoš, University of Chicago

Partially supported by the National Science Foundation and the Center for Nonlinear Analysis.
• **Aim:** Study certain incompressible flows which ‘promote’ the creation of hot spots.

• Existence of such flows is surprising.

• We can prove very little.
Incompressible flows usually ‘help’ mixing.

- If $\nabla \cdot u = 0$, principal eigenvalue of $L^u = (u \cdot \nabla) - \Delta$ (with Dirichlet B.C.) is larger than that of $L^0 = -\Delta$.

Follows immediately by minimising the Raleigh quotient: If $\phi \in H^1_0(\Omega)$ and $L^u \phi = \lambda^u \phi$, with $\int_\Omega \phi^2 = 1$ then

$$\lambda^u \int_\Omega \phi^2 = \int_\Omega \phi L^u \phi = \int_\Omega |\nabla \phi|^2 \geq \lambda^0$$

- Consequently, solutions to $\partial_t \theta + u \cdot \nabla \theta - \Delta \theta = 0$ approach the equilibrium state faster than solutions to $\partial_t \theta - \Delta \theta = 0$.

- In the periodic setting, the effective diffusivity in the presence of an incompressible drift is larger than the diffusivity without. (Fan-njiang, Papanicolaou ’94).
Decrease of the explosion threshold by incompressible drift.

- Explosion problem: 
  \[-\Delta \phi + u \cdot \nabla \phi = \lambda e^\phi \quad \text{in } \Omega,\]
  \[\phi = 0 \quad \text{on } \partial \Omega.\]

- There exists an explosion threshold $\lambda_*(u)$:
  - For all $\lambda \leq \lambda_*(u)$ above PDE has a solution.
  - For all $\lambda > \lambda_*(u)$ above PDE has no solutions.
  - Joseph, Lundgreen ’72/73; Keener, H. Keller ’74; Crandall, Rabinowitz ’75; Berestycki, Kiselev, Novikov, Ryzhik ’09.

- Incompressible flows can decrease the explosion threshold!
  - Usually expect stirring to avoid ‘hot spot’ creation, and increase the explosion threshold!
The exit time problem.

- Let $\nabla \cdot u = 0$ and consider
  \[
  \begin{cases}
  -\triangle \tau^u + u \cdot \nabla \tau^u = 1 & \text{in } \Omega, \\
  \tau^u = 0 & \text{on } \partial \Omega
  \end{cases}
  \]

- $\tau^u$ is the expected exit time of the diffusion
  \[dX_t = u(X_t) \, dt + \sqrt{2} \, dW_t\]
  from the domain $\Omega$.

**Main problem:** Under the constraints

\[\nabla \cdot u = 0 \quad \text{and} \quad u \cdot \hat{n} = 0 \text{ on } \partial \Omega,\]

what drift maximizes $\tau^u$ in some sense.
A few remarks

• Without the divergence free constraint, can make $\tau^u$ arbitrarily large by a strong inward stirring.

• Using fast incompressible cellular flows, we can always make $\tau^u$ arbitrarily small.

• If incompressible stirring only ‘helps’ mixing, then $u \equiv 0$ should produce the largest $\tau^u$.

• Surprisingly(?) this is false.

Theorem. Let $\Omega \subset \mathbb{R}^2$ be nice.\(^1\) Then $u \equiv 0$ maximises $\|\tau^u\|_{L^\infty}$ if and only if $\Omega$ is a disk.

\(^1\)Nice = Bounded, simply connected and Lipschitz
Exit times in a disk.

In a disk, no incompressible stirring can ever increase the expected exit time.

**Proposition.** Let $\Omega \subset \mathbb{R}^n$ be nice, and $v$ be any divergence free vector field which is tangential on $\partial \Omega$. Then

$$\|\tau^v\|_{L^p(\Omega)} \leq \|\tau^{0,D}\|_{L^p(D)}$$

where $D \subset \mathbb{R}^n$ is a disk with $|D| = |\Omega|$, and $\tau^{0,D}$ is the expected exit time from $D$ with 0 drift.
Proof

- Given any $\tau = \tau^v$, consider the symmetric rearrangement $\tau^*$:
  
  - $D$ is a disk with $|D| = |\Omega|$, and $\tau^* : D \to \mathbb{R}^+$ is radial.
  - For all $h$, $|\{\tau > h\}| = |\{\tau^* > h\}|$.

- Let $\Omega_h = \{\tau > h\}$, $\Omega^*_h = \{\tau^* > h\}$. Then

\[
\int_{\partial \Omega_h^*} |\nabla \tau^*| \, d\sigma \int_{\partial \Omega_h^*} \frac{1}{|\nabla \tau^*|} \, d\sigma = |\partial \Omega_h^*|^2 \leq |\partial \Omega_h|^2 \leq \int_{\partial \Omega_h} |\nabla \tau| \, d\sigma \int_{\partial \Omega_h} \frac{1}{|\nabla \tau|} \, d\sigma.
\]

- Co-area implies $\int_{\partial \Omega_h} \frac{1}{|\nabla \tau|} \, d\sigma = -\frac{d}{dh} |\Omega_h| = -\frac{d}{dh} |\Omega_h^*| = \int_{\partial \Omega_h^*} \frac{1}{|\nabla \tau^*|} \, d\sigma$

- $\Rightarrow \int_{\partial \Omega_h^*} |\nabla \tau^*| \, d\sigma \leq \int_{\partial \Omega_h} |\nabla \tau| \, d\sigma = |\Omega_h^*|$

- Since $\tau^*$ is radial $\Rightarrow \text{QED.}$
Increasing the exit times for non-circular domains.

- Consider ‘infinite amplitude’ flows:
  - For $A \in \mathbb{R}$, $\nabla \cdot u = 0$, let $\tau^{Au}$ solve
    
    
    
    
    
    $\tau^{Au} = 0$ on $\partial \Omega$

  - Let $\bar{\tau}^u \overset{\text{def}}{=} \lim_{A \to \infty} \tau^{Au}$ (convergence is uniform in $\Omega$).
  - The limit $\bar{\tau}^u$ satisfies the Freidlin problem.
  - If $u = \nabla \perp \psi \overset{\text{def}}{=}(\begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix})$, and $\psi$ has ‘one hill’, then $\bar{\tau}^u$ is given explicitly by
    
    
    
    

    where $\Omega_{\psi,h} = \{x \mid \psi(x) > h\}$.
• If $D$ is not a disk, we will show that there is some $\psi$ such that for $u = \nabla^\perp \psi$, $\|\bar{\tau} u\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.

  - Will of course imply that for large $A$, $\|\tau^A u\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.

• **Main idea:** Let $I(\psi) = \|\bar{\tau}^{\nabla^\perp} \psi\|_{L^\infty}$. Set up a variational principle for ‘one hill’ stream functions using the explicit solution of the Freidlin problem. Show $\tau^0$ is not a critical point.

  - If $\tau^0$ doesn’t have ‘one hill’, then reduce the domain to a level set of $\tau^0$ near it’s maximum. Increasing expected exit time from this domain will increase it from the larger domain.

  - If $\tau^0$ is not a critical point of the said variational principle, then for some $u$, large $A$,

    $$\|\tau^A u\|_{L^\infty} > \|\bar{\tau} u\|_{L^\infty} - \varepsilon > \|\bar{\tau}^{\nabla^\perp} \tau^0\|_{L^\infty} = \|\tau^0\|_{L^\infty}. $$
The variational principle

- Let $v : \Omega \to \mathbb{R}^2$ be smooth (not necessarily divergence free), with $v \cdot \hat{n} = 0$ on $\partial \Omega$. ('Direction' of the variation.)

- Let $\frac{dX_\varepsilon}{d\varepsilon} = v(X_\varepsilon)$ with $X_0(x) = x$.

- Compute $V(\psi, v) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} I(\psi \circ X_\varepsilon)$.

- Some suffering shows $V(\psi, v) = 0$ for all $v$ if and only if

$$-2\Delta \phi(x) = 1 + |\nabla \phi(x)|^2 \left( \int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla \phi|} \right) \left( \int_{\{\phi=\phi(x)\}} |\nabla \phi| \, d\sigma \right)^{-1}$$

where $\phi = \bar{\tau} \nabla^\perp \psi$. 
• From above $V(\psi, v) = 0$ for all $v$ if and only if

$$-2\triangle \phi(x) = 1 + |\nabla \phi(x)|^2 \left( \int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla \phi|} \right) \left( \int_{\{\phi=\phi(x)\}} |\nabla \phi| d\sigma \right)^{-1}$$

where $\phi = \bar{\tau} \nabla \perp \psi$.

• If $V(\tau^0, v) = 0$ for all $v$, then

$$2 = 1 + |\nabla \tau^0|^2 M(\tau^0)$$

and so $\tau^0$ solves the eikonal equation.

• If $\Omega$ is not a disk, the eikonal equation necessarily has interior singularities. However $\tau^0$ is analytic.
Simulations

(a) Maximiser $\psi$

(b) Expected exit time $\tau_0$
Simulations

(c) Maximiser $\psi$

(d) Expected exit time $\tau_0$
Open questions

- Existence/uniqueness of solutions to the previous PDE.
- Are such solutions indeed maximisers?
- An understanding of why such stirring increases the exit time.
- Maximising other norms ($L^p$). Other constraints (e.g. finite power).
- Characterize flows that increase the explosion threshold.