Exit times of diffusions with incompressible drift

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- **Aim:** Study certain incompressible flows which 'promote' the creation of hot spots.
- Existence of such flows is surprising.
- We can prove very little.

Incompressible flows usually ‘help’ mixing.

- If $\nabla \cdot u = 0$, principal eigenvalue of $L^u = (u \cdot \nabla) - \triangle$ (with Dirichlet B.C.) is larger than that of $L^0 = -\triangle$.

  Follows immediately by minimising the Raleigh quotient: If $\phi \in H_0^1(\Omega)$ and $L^u \phi = \lambda^u \phi$, with $\int_\Omega \phi^2 = 1$ then

$$\lambda^u \int_\Omega \phi^2 = \int_\Omega \phi L^u \phi = \int_\Omega |\nabla \phi|^2 \geqslant \lambda^0$$

- Consequently, solutions to $\partial_t \theta + u \cdot \nabla \theta - \triangle \theta = 0$ approach the equilibrium state faster than solutions to $\partial_t \theta - \triangle \theta = 0$.

- In the periodic setting, the *effective diffusivity* in the presence of an incompressible drift is larger than the diffusivity without. (Fannjiang, Papanicolaou ’94).

Decrease of the explosion threshold by incompressible drift.

- Explosion problem: $$\begin{aligned} -\triangle\phi + u\cdot\nabla\phi &= \lambda e^{\phi} && \text{in } \Omega,\\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned}$$
- There exists an *explosion threshold* $\lambda_*(u)$:
  - For all $\lambda \leqslant \lambda_*(u)$ above PDE has a solution.
  - For all $\lambda > \lambda_*(u)$ above PDE has no solutions.
  - Joseph, Lundgreen '72/73; Keener, H. Keller '74; Crandall, Rabinowitz '75; Berestycki, Kiselev, Novikov, Ryzhik '09.
- Incompressible flows can *decrease* the explosion threshold!
  - Berestycki, Kagan, Joulin, Sivashinsky '97: Numerical example in a long rectangle.
  - Usually expect stirring to avoid 'hot spot' creation, and *increase* the explosion threshold!

The exit time problem.

- Let $\nabla \cdot u = 0$ and consider $\begin{cases} -\triangle \tau^u + u \cdot \nabla \tau^u = 1 & \text{in } \Omega, \\ \tau^u = 0 & \text{on } \partial\Omega \end{cases}$
- τ^u is the expected exit time of the diffusion

$$dX_t = u(X_t)\,dt + \sqrt{2}\,dW_t$$

from the domain Ω.

Main problem: Under the constraints

$$\nabla \cdot u = 0 \quad \text{and} \quad u \cdot \hat{n} = 0 \text{ on } \partial\Omega,$$

what drift maximizes τ^u in some sense.

A few remarks

- Without the divergence free constraint, can make τ^u arbitrarily large by a strong inward stirring.
- Using fast incompressible cellular flows, we can always make τ^u arbitrarily small.
- If incompressible stirring only 'helps' mixing, then $u \equiv 0$ should produce the largest τ^u.
- Surprisingly(?) this is false.

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be nice*[1] *Then $u \equiv 0$ maximises $\|\tau^u\|_{L^\infty}$ if and only if Ω is a disk.*

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[1] Nice = Bounded, simply connected and Lipschitz

Exit times in a disk.

In a disk, no incompressible stirring can ever increase the expected exit time.

Proposition. *Let* $\Omega \subset \mathbb{R}^n$ *be nice, and* v *be any divergence free vector field which is tangential on* $\partial\Omega$. *Then*

$$\|\tau^v\|_{L^p(\Omega)} \leqslant \left\|\tau^{0,D}\right\|_{L^p(D)}$$

where $D \subset \mathbb{R}^n$ *is a disk with* $|D| = |\Omega|$, *and* $\tau^{0,D}$ *is the expected exit time from* D *with* 0 *drift.*

Proof

- Given any $\tau = \tau^v$, consider the symmetric rearrangement τ^*:
  - D is a disk with $|D| = |\Omega|$, and $\tau^* : D \to \mathbb{R}^+$ is radial.
  - For all h, $|\{\tau > h\}| = |\{\tau^* > h\}|$.
- Let $\Omega_h = \{\tau > h\}$, $\Omega_h^* = \{\tau^* > h\}$. Then

$$\int_{\partial\Omega_h^*} |\nabla\tau^*| \, d\sigma \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|} \, d\sigma = |\partial\Omega_h^*|^2 \leqslant |\partial\Omega_h|^2 \leqslant \int_{\partial\Omega_h} |\nabla\tau| \, d\sigma \int_{\partial\Omega_h} \frac{1}{|\nabla\tau|} \, d\sigma.$$

- Co-area implies $\int_{\partial\Omega_h} \frac{1}{|\nabla\tau|} \, d\sigma = -\frac{d}{dh}|\Omega_h| = -\frac{d}{dh}|\Omega_h^*| = \int_{\partial\Omega_h^*} \frac{1}{|\nabla\tau^*|} \, d\sigma$
- $\implies \displaystyle\int_{\partial\Omega_h^*} |\nabla\tau^*| \, d\sigma \leqslant \int_{\partial\Omega_h} |\nabla\tau| \, d\sigma = |\Omega_h^*|$.
- Since τ^* is radial $\implies$ QED.

Increasing the exit times for non-circular domains.

- Consider 'infinite amplitude' flows:
  - For $A \in \mathbb{R}$, $\nabla \cdot u = 0$, let τ^{Au} solve

$$\begin{aligned} -\triangle \tau^{Au} + Au \cdot \nabla \tau^{Au} &= 1 \quad && \text{in } \Omega, \\ \tau^{Au} &= 0 && \text{on } \partial\Omega \end{aligned}$$

  - Let $\bar{\tau}^u \stackrel{\text{def}}{=} \lim_{A \to \infty} \tau^{Au}$ (convergence is uniform in Ω).
  - The limit $\bar{\tau}^u$ satisfies the *Freidlin problem.*
  - If $u = \nabla^\perp \psi \stackrel{\text{def}}{=} \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}$, and ψ has 'one hill', then $\bar{\tau}^u$ is given explicitly by

$$\bar{\tau}^u(y) \stackrel{\text{def}}{=} \lim_{A \to \infty} \tau^{Au}(y) = -\int_0^{\psi(y)} \frac{|\Omega_{\psi,h}|}{\int_{\Omega_{\psi,h}} \triangle \psi \, dx} \, dh$$

where $\Omega_{\psi,h} = \{x \mid \psi(x) > h\}$.

- If D is not a disk, we will show that there is some ψ such that for $u = \nabla^\perp \psi$, $\|\bar\tau^u\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.
  - Will of course imply that for large A, $\|\tau^{Au}\|_{L^\infty} > \|\tau^0\|_{L^\infty}$.
- **Main idea:** Let $I(\psi) = \|\bar\tau^{\nabla^\perp \psi}\|_{L^\infty}$. Set up a variational principle for 'one hill' stream functions using the explicit solution of the Freidlin problem. Show τ^0 is not a critical point.
  - If τ^0 doesn't have 'one hill', then reduce the domain to a level set of τ^0 near it's maximum. Increasing expected exit time from this domain will increase it from the larger domain.
  - If τ^0 is not a critical point of the said variational principle, then for some u, large A,

$$\left\|\tau^{Au}\right\|_{L^\infty} > \|\bar\tau^u\|_{L^\infty} - \varepsilon > \left\|\bar\tau^{\nabla^\perp \tau^0}\right\|_{L^\infty} = \left\|\tau^0\right\|_{L^\infty}.$$

The variational principle

- Let $v : \Omega \to \mathbb{R}^2$ be smooth (not necessarily divergence free), with $v \cdot \hat{n} = 0$ on $\partial\Omega$. *('Direction' of the variation.)*
- Let $\dfrac{dX_\varepsilon}{d\varepsilon} = v(X_\varepsilon)$ with $X_0(x) = x$.
- Compute $V(\psi, v) = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(\psi \circ X_\varepsilon)$.
- Some suffering shows $V(\psi, v) = 0$ for all v if and only if

$$-2\triangle\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla\phi|}\right) \left(\int_{\{\phi=\phi(x)\}} |\nabla\phi|\, d\sigma\right)^{-1}$$

where $\phi = \bar{\tau}^{\nabla^\perp\psi}$.

- From above $V(\psi, v) = 0$ for all v if and only if

$$-2\triangle\phi(x) = 1 + |\nabla\phi(x)|^2 \left(\int_{\{\phi=\phi(x)\}} \frac{d\sigma}{|\nabla\phi|} \right) \left(\int_{\{\phi=\phi(x)\}} |\nabla\phi| \, d\sigma \right)^{-1}$$

where $\phi = \bar{\tau}^{\nabla^\perp\psi}$.

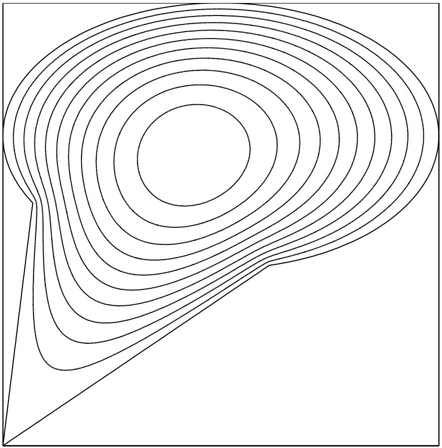
- If $V(\tau^0, v) = 0$ for all v, then

$$2 = 1 + \left|\nabla\tau^0\right|^2 M(\tau^0)$$

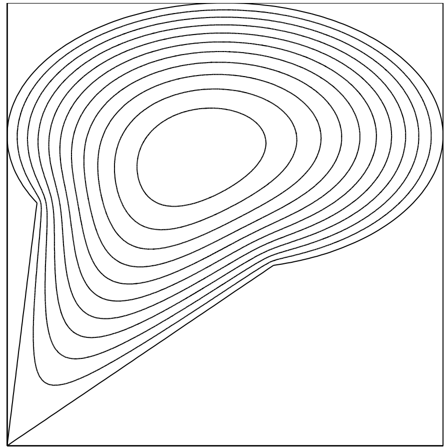
and so τ^0 solves the eikonal equation.

- If Ω is not a disk, the eikonal equation necessarily has interior singularities. However τ^0 is analytic.

Simulations

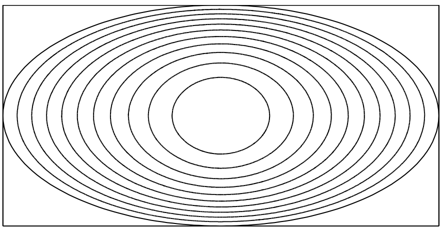


(a) Maximiser ψ

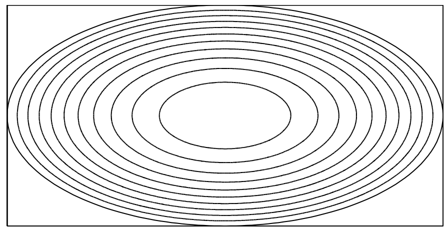


(b) Expected exit time τ_0

Simulations



(c) Maximiser ψ



(d) Expected exit time τ_0

Open questions

- Existence/uniqueness of solutions to the previous PDE.
- Are such solutions indeed maximisers?
- An understanding of why such stirring increases the exit time.
- Maximising other norms (L^p). Other constraints (e.g. finite power).
- Characterize flows that increase the explosion threshold.