Sharp energy estimates and 1D symmetry for nonlinear equations involving fractional Laplacians

Eleonora Cinti

Università degli Studi di Bologna - Universitat Politècnica de Catalunya
(joint work with Xavier Cabré)

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We consider nonlinear fractional equations of the type:

\[ (-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^n, \quad 0 < s < 1, \quad (1) \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^{1,\beta} \) function, for some \( \beta > \max\{0, 1 - 2s\} \).

The fractional Laplacian of a function \( u : \mathbb{R}^n \to \mathbb{R} \) is expressed by the formula

\[ (-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy. \]

It can also be defined using Fourier transform, in the following way:

\[ \widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi). \]
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It can also be defined using Fourier transform, in the following way:

\[\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi).\]
Case \( s=1/2 \)

We will realize the non local problem (1) in a local problem in \( \mathbb{R}^{n+1} \) with a nonlinear Neumann condition.

More precisely: \( u \) is a solution of \( (-\Delta)^{1/2} u = f(u) \) in \( \mathbb{R}^n \), if and only if its harmonic extension \( \nu(x, \lambda) \) defined on \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+ \) satisfies the problem

\[
\begin{cases}
\Delta \nu = 0 & \text{in } \mathbb{R}^{n+1}, \\
- \frac{\partial \nu}{\partial \lambda} = f(\nu) & \text{on } \mathbb{R}^n = \partial \mathbb{R}_+^{n+1}.
\end{cases}
\]
We define
\[ G(u) = \int_u^1 f. \]

If the following conditions holds we call the nonlinearity \( f \) of \textit{balanced bistable type} and the potential \( G \) of \textit{double well type}:

\begin{align*}
(H1) & \quad f \text{ is odd;} \\
(H2) & \quad G \geq 0 = G(\pm 1) \text{ in } \mathbb{R}, \text{ and } G > 0 \text{ in } (-1, 1); \\
(H3) & \quad f' \text{ is decreasing in } (0, 1).
\end{align*}

For example \( G(u) = \frac{1}{4}(1 - u^2)^2. \)
A conjecture of De Giorgi (1978)

Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution in all of $\mathbb{R}^n$ of the equation

$$-\Delta u = u - u^3,$$

such that $\partial_{x_n} u > 0$. Then, at least if $n \leq 8$, all the level sets $\{u = t\}$ of $u$ are hyperplanes, or equivalently $u$ is of the form

$$u(x) = g(a \cdot x + b) \text{ in } \mathbb{R}^n$$

for some $a \in \mathbb{R}^n$, $|a| = 1$, $b \in \mathbb{R}$. 
True for:

1. \( n = 2 \) (Ghoussoub and Gui, 1998),
2. \( n = 3 \) (Ambrosio and Cabré, 2000 - Alberti, Ambrosio, and Cabré, 2001),
3. \( 4 \leq n \leq 8 \) if, in addition, \( u \to \pm 1 \) for \( x_n \to \pm \infty \) (Savin 2009),
4. counterexample for \( n \geq 9 \) (Del Pino, Kowalczyk and Wei).
1-D symmetry for the fractional equation: known results

- In dimension $n = 2$ the 1-D symmetry property of stable solutions for problem (1) with $s = 1/2$ was proven by Cabré and Solá-Morales.
- In dimension $n = 2$ and for every $0 < s < 1$, 1-D symmetry property for stable solutions has been proven by Cabré and Sire and by Sire and Valdinoci.
Some definitions

Consider the cylinder

\[ C_R = B_R \times (0, R) \subset \mathbb{R}^{n+1}, \]

where \( B_R \) is the ball of radius \( R \) centered at 0 in \( \mathbb{R}^n \).

We consider the energy functional

\[ E_{C_R}(v) = \int_{C_R} \frac{1}{2} |\nabla v|^2 \, dx \, d\lambda + \int_{B_R} G(v) \, dx. \] (4)
Definition

We say that a bounded solution \( v \) of (2) is \textit{stable} if the second variation of energy \( \delta^2 \mathcal{E} / \delta^2 \xi \) with respect to perturbations \( \xi \) compactly supported in \( \overline{\mathbb{R}^{n+1}} \), is nonnegative. That is, if

\[
Q_v(\xi) := \int_{\mathbb{R}^{n+1}} |\nabla \xi|^2 - \int_{\partial \mathbb{R}^{n+1}} f'(v)\xi^2 \geq 0
\]

(5)

for every \( \xi \in C_0^\infty(\overline{\mathbb{R}^{n+1}}) \).

We say that \( v \) is \textit{unstable} if and only if \( v \) is not stable.

Definition

We say that a bounded solution \( u(x) \) of (1) in \( \mathbb{R}^n \) is \textit{stable} (\textit{unstable}) if its harmonic extension \( v(x, \lambda) \) is a stable (unstable) solution for the problem (2).
Definition

We say that a bounded $C^1(\overline{\mathbb{R}^{n+1}})$ function $v$ in $\mathbb{R}^{n+1}$ is a *global minimizer* of (2) if

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(w),$$

for every bounded cylinder $C_R \subset \overline{\mathbb{R}^{n+1}}$ and every $C^\infty(\mathbb{R}^{n+1})$ function $w$ such that $w \equiv v$ in $\mathbb{R}^{n+1} \setminus \overline{C_R}$.

Definition

We say that a bounded $C^1$ function $u$ in $\mathbb{R}^n$ is a *global minimizer* of (1) if its harmonic extension $v$ is a global minimizer of (2).
Definition
We call \textit{layer solutions} for the problem (1) bounded solutions that are monotone increasing, say from $-1$ to $1$, in one of the $x$-variables.

Remark
We remind that every layer solution is a global minimizer (Cabré and Solá-Morales).
Principal ingredients in the proof of the conjecture of De Giorgi:

- Stability of solutions;
- Estimate for the Dirichlet energy:

\[ \int_{C_R} \frac{1}{2} |\nabla v|^2 \leq CR^2 \log R. \]
Principal results

Theorem (Energy estimate for minimizers in dimension $n$)

Set $c_u = \min\{G(s) : \inf v \leq s \leq \sup v\}$.

Let $f$ be any $C^{1, \beta}$ nonlinearity with $\beta \in (0, 1)$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded global minimizer of (1). Let $v$ be the harmonic extension of $u$ in $\mathbb{R}^{n+1}_+$. Then, for all $R > 2$,

$$
\int_{C_R} \frac{1}{2} |\nabla v|^2 dx d\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^{n-1} \log R,
$$

(6)

where $C_R = B_R \times (0, R)$ and $C$ is a constant depending only on $n$, $\|f\|_{C^1}$, and on $\|u\|_{L^\infty(\mathbb{R}^n)}$. 

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In particular we have that

$$\int_{C_R} \frac{1}{2} |\nabla v|^2 dxd\lambda \leq CR^{n-1} \log R.$$  \hspace{1cm} (7)

Remark

As a consequence we have that the energy estimate (15) holds for layer solutions of problem (1).
Theorem (Energy estimate for monotone solutions in dimension 3)

Let \( n = 3 \), \( f \) be any \( C^{1,\beta} \) nonlinearity with \( \beta \in (0,1) \) and \( u \) be a bounded solution of (1) such that \( \partial_{x_n} u > 0 \) in \( \mathbb{R}^3 \). Let \( v \) be its harmonic extension in \( \mathbb{R}^4_+ \).

Then, for all \( R > 2 \),

\[
\int_{C_R} \frac{1}{2} |\nabla v|^2 \, dx \, d\lambda + \int_{B_R} \{ G(u) - c_u \} \, dx \leq CR^2 \log R, \tag{8}
\]

where \( C \) is a constant depending only on \( \|u\|_{L^\infty} \) and on \( \|f\|_{C^1} \).
Theorem (1-D symmetry)

Let $n = 3$, $s = 1/2$ and $f$ be any $C^{1,\beta}$ nonlinearity with $\beta \in (0, 1)$. Let $u$ be either a bounded global minimizer of (1), or a bounded solution monotone in the direction $x_n$.

Then, $u$ depends only on one variable, i.e., there exists $a \in \mathbb{R}^3$ and $g : \mathbb{R} \to \mathbb{R}$, such that $u(x) = g(a \cdot x)$ for all $x \in \mathbb{R}^3$, or equivalently the level sets of $u$ are planes.
Some remarks

- Energy estimate (15) is sharp because it is optimal for 1-D solutions (Cabré, Solá-Morales).

- In dimension $n = 1$ energy estimate (15), for layer solutions, has been proved by Cabré and Solá-Morales; more precisely they give estimates for kinetic and potential energies separately:

$$\int_{C_R} |\nabla v|^2 dx d\lambda \leq C \log R, \quad \int_{-\infty}^{+\infty} G(v(x, 0)) dx < \infty.$$  

In Theorem 5 we have a weaker estimate because we cannot prove that the potential energy in dimension $n$ is bounded by $R^{n-1}$. 


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Sketch of the proof of Theorem 5

The proof of energy estimates for global minimizer is based on a comparison argument. It can be resumed in 3 steps:

- Construct the comparison function $w$, which takes the same value of $v$ on $\partial C_R \cap \{\lambda > 0\}$ and thus, such that

$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(w),$$
Sketch of the proof of Theorem 5

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$$\mathcal{E}_{C_R}(v) \leq \mathcal{E}_{C_R}(w),$$
use the rescaled $H^{1/2}(\partial C_1) \rightarrow H^1(C_1)$ estimate in the cylinder of radius 1 and height 1:

$$
\int_{C_1} |\nabla \overline{w}|^2 \leq C \|w\|_{L^2(\partial C_1)}^2 + C \int_{\partial C_1} \int_{\partial C_1} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}},
$$

where $w$ is the trace of $\overline{w}$ on $\partial C_1$,

and give the key estimate

$$
\int_{\partial C_R} \int_{\partial C_R} \frac{|w(x) - w(\overline{x})|^2}{|x - \overline{x}|^{n+1}} d\sigma_x d\sigma_{\overline{x}} \leq CR^{n-1} \log R.
$$
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$$\int_{C_1} |\nabla \bar{w}|^2 \leq C ||w||^2_{L^2(\partial C_1)} + C \int_{\partial C_1} \int_{\partial C_1} \frac{|w(x) - w(\bar{x})|^2}{|x - \bar{x}|^{n+1}} d\sigma_x d\sigma_{\bar{x}},$$

where $w$ is the trace of $\bar{w}$ on $\partial C_1$,

give the key estimate

$$\int_{\partial C_R} \int_{\partial C_R} \frac{|w(x) - w(\bar{x})|^2}{|x - \bar{x}|^{n+1}} d\sigma_x d\sigma_{\bar{x}} \leq CR^{n-1} \log R.$$
The comparison function $\overline{w}$ satisfies:

$$\begin{cases}
\Delta \overline{w} = 0 & \text{in } C_R \\
\overline{w}(x, 0) = 1 & \text{on } B_{R-1} \times \{\lambda = 0\} \\
\overline{w}(x, \lambda) = \nu(x, \lambda) & \text{on } \partial C_R \cap \{\lambda > 0\}.
\end{cases} \tag{9}$$
Sketch of the proof of 1-D symmetry result in dimension 3

1-D symmetry of minimizers and of monotone solutions in dimension 3 follows by our energy estimate and the following Liouville type Theorem due to Moschini:

**Proposition (Moschini)**

Let \( \varphi \in L^\infty_{\text{loc}}(\mathbb{R}^{n+1}_+) \) be a positive function. Suppose that \( \sigma \in H^1_{\text{loc}}(\mathbb{R}^{n+1}_+) \) satisfies

\[
\begin{aligned}
-\sigma \text{div}(\varphi^2 \nabla \sigma) &\leq 0 \quad \text{in } \mathbb{R}^{n+1}_+ \\
-\sigma \partial_\lambda \sigma &\leq 0 \quad \text{on } \partial \mathbb{R}^{n+1}_+ 
\end{aligned}
\]  

(10)

in the weak sense. If

\[
\int_{C_R} (\varphi \sigma)^2 \, dx \leq CR^2 \log R
\]

for some finite constant \( C \) independent of \( R \), then \( \sigma \) is constant.
Sketch of the proof of 1-D symmetry result in dimension 3

Suppose \( v_{x_3} > 0 \); set \( \varphi = v_{x_3} \) and for \( i = 1, \ldots, n - 1 \) fixed, consider the function:

\[
\sigma_i = \frac{v_{x_i}}{\varphi}.
\]

We prove that \( \sigma_i \) is constant in \( \mathbb{R}^{n+1}_+ \), using the Liouville result due to Moschini and our energy estimate.
the function $\sigma_i$ satisfies

$$\begin{cases} 
-\sigma_i \text{div}(\varphi^2 \nabla \sigma_i) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
-\sigma_i \partial_\lambda \sigma_i = 0 & \text{in } \partial \mathbb{R}^{n+1}_+.
\end{cases}$$

(11)

by our energy estimate, we get

$$\int_{C_R} (\varphi \sigma_i)^2 \leq \int_{C_R} |\nabla \nu|^2 \leq CR^2 \log R,$$

by Proposition (4) we deduce $\sigma_i = c_i$ is constant then $\nu$ depends only on $\lambda$ and the variable parallel to the vector $(c_1, c_2, c_3, 0)$ and then $u(x) = \nu(x, 0)$ is 1-D.
the function $\sigma_i$ satisfies

$$
\begin{cases}
-\sigma_i \text{div}(\varphi^2 \nabla \sigma_i) = 0 & \text{in } \mathbb{R}^{n+1} \\
-\sigma_i \partial_{\lambda} \sigma_i = 0 & \text{in } \partial \mathbb{R}^{n+1}
\end{cases}
$$

by our energy estimate, we get

$$
\int_{C_R} (\varphi \sigma_i)^2 \leq \int_{C_R} |\nabla v|^2 \leq C R^2 \log R,
$$

by Proposition (4) we deduce $\sigma_i = c_i$ is constant then $v$ depends only on $\lambda$ and the variable parallel to the vector $(c_1, c_2, c_3, 0)$ and then $u(x) = v(x, 0)$ is 1-D.
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\end{cases} \tag{11}$$

by our energy estimate, we get

$$\int_{C_R} (\varphi \sigma_i)^2 \leq \int_{C_R} |\nabla v|^2 \leq CR^2 \log R,$$

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Energy estimate for global minimizers of \((-\Delta)^s u = f(u), \text{ with } 0 < s < 1\)

Local problem:

\(u\) is a solution of

\[ (-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n, \quad (12) \]

if and only if, \(v\) defined on \(\mathbb{R}^{n+1}_+ = \{(x, \lambda) : x \in \mathbb{R}^n, \lambda > 0\}\), is a solution of the problem

\[
\begin{aligned}
\text{div}(\lambda^{1-2s}\nabla v) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\
v(x, 0) &= u(x) \quad \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \\
-\lim_{\lambda \to 0} \lambda^{1-2s} \partial_\lambda v &= f(v).
\end{aligned}
\]  

(13)
The energy functional associated to problem (13) is given by

\[ E_{s, c_R}(v) = \int_{C_R} \frac{1}{2} \lambda^{1-2s} |\nabla v|^2 \, dx \, d\lambda + \int_{B_R} G(v) \, dx. \]  

(14)

Remark

- The weight \( \lambda^{1-2s} \) belongs to the Muckenhoupt class \( A_2 \), since 
  \(-1 < 1 - 2s < 1 \) [theory of Fabes-Kenig-Serapioni];
- problem (13) is invariant under translations in the \( x_i \)-directions.
Theorem (Energy estimate for minimizers in dimension n)

Let $f$ be any $C^{1,\beta}$ nonlinearity, with $\beta > \max\{0, 1 - 2s\}$, and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a global minimizer of (1). Let $v$ be the $s$-extension of $u$ in $\mathbb{R}^{n+1}_+$. Then, for all $R > 2$,

$$
\int_{C_R} \frac{1}{2} \lambda^{1-2s} |\nabla v|^2 dxd\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^{n-2s} \quad \text{if } 0 < s < 1/2
$$

$$
\left( \int_{C_R} \frac{1}{2} |\nabla v|^2 dxd\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^{n-1}\log R \quad \text{if } s = 1/2 \right) \quad (15)
$$

$$
\int_{C_R} \frac{1}{2} \lambda^{1-2s} |\nabla v|^2 dxd\lambda + \int_{B_R} \{G(u) - c_u\} dx \leq CR^{n-1} \quad \text{if } 1/2 < s < 1,
$$

where $C$ denotes different positive constants depending only on $n$, $\|f\|_{C^1}$, $\|u\|_{L^\infty(\mathbb{R}^n)}$ and $s$. 

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The proof is based on a comparison argument as before. Here a crucial ingredient is the following extension theorem.

**Theorem**

Let $\Omega$ be a bounded subset of $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial \Omega$ and $M$ a Lipschitz subset of $\partial \Omega$. For $z \in \mathbb{R}^{n+1}$, let $d_M(z)$ denote the Euclidean distance from the point $z$ to the set $M$. Let $w$ belong to $C(\partial \Omega)$.

Then, there exists an extension $\tilde{w}$ of $w$ in $\Omega$ belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, such that

\[
\int_{\Omega} d_M(z)^{1-2s} |\nabla \tilde{w}|^2 \, dz \leq C \|w\|^2_{L^2(\partial \Omega)} + C \int \int_{B_s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+2s}} \, d\sigma_z \, d\overline{\sigma}_z + C \int \int_{B_w} d_M(z)^{1-2s} \frac{|w(z) - w(\overline{z})|^2}{|z - \overline{z}|^{n+1}} \, d\sigma_z \, d\overline{\sigma}_z.
\]
The sets $B_s$ and $B_w$ are defined as follows:

$$B_s = \begin{cases} \partial \Omega \times \partial \Omega & \text{if } 0 < s < 1/2 \\ M \times M & \text{if } 1/2 < s < 1, \end{cases}$$

(17)

and

$$B_w = \begin{cases} (\partial \Omega \setminus M) \times (\partial \Omega \setminus M) & \text{if } 0 < s < 1/2 \\ (\partial \Omega \setminus M) \times \partial \Omega & \text{if } 1/2 < s < 1. \end{cases}$$

(18)

After rescaling, we apply this result for $\Omega = C_1$ and $M = B_1 \times \{0\}$.
Theorem (1-D symmetry)

Let \( n = 3, \frac{1}{2} \leq s < 1 \) and \( f \) be any \( C^{1,\beta} \) nonlinearity with \( \beta > \max\{0, 1 - 2s\} \).

Let \( u \) be either a bounded global minimizer of \((1)\), or a bounded solution monotone in the direction \( x_n \).

Then, \( u \) depends only on one variable, i.e., there exists \( a \in \mathbb{R}^3 \) and \( g : \mathbb{R} \to \mathbb{R} \), such that \( u(x) = g(a \cdot x) \) for all \( x \in \mathbb{R}^3 \), or equivalently the level sets of \( u \) are planes.
Some open problems:

- 1-D symmetry for $n = 3$ and $0 < s < 1/2$;
- 1-D symmetry for $n > 3$ and $0 < s < 1$;
- critical dimension;
- counterexample in large dimensions.