Traveling waves steered by the non linearity

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“Deterministic and stochastic front propagation”, Banff.
Modeling
Brain pathophysiology: Spreading depressions
Spreading depression (SD): Transient depolarization of neurons that propagates slowly (3mm/min) through the brain.

- SD are important for different pathologies: stroke, migraine with aura, epilepsy, head injuries.
- SD are well studied in rodent, never clearly observed in human!
Brain is composed of white and gray matter.

SD: reaction-diffusion mechanism in the gray matter and diffusion and absorption in the white matter.

- **Rodent brain**: rather smooth and entirely composed of gray matter.

- **Human brain**: thin layer of gray matter at the periphery, white matter inside.

The morphology of the human brain may prevent the propagation of SD!
Effect of the white matter on SD propagation

\[ \partial_t u - \Delta u = \lambda u(1 - u)(u - \theta)1_{|y| \leq R} - \alpha u 1_{|y| > R} \]

where \( u \in C^1 \) is bounded.

- If the gray matter is large
- If the gray matter is thin
Population dynamic: spatial mutation
Let $f(t, x, v) \geq 0$ be the **density of individuals** who at time $t$ and point $x \in \mathbb{R}$ possess the **quantitative trait** $v \in \mathbb{R}$.

**Assumptions:**
- spatial diffusion,
- mutation of the trait,
- logistic growth,
- the most adapted trait depends on $x$.

\[
\partial_t f - \nu \partial_{xx} f - \mu \int_{w \in \mathbb{R}} e^{-\frac{|v-w|^2}{\tau}} f(t, x, w) \, dw = \\
\left( (a - b |v - \phi(x)|^2) - \int_{w \in \mathbb{R}} f(t, x, w) \, dw \right) f(t, x, v)
\]
We assume $\phi(x) = x$. 
Simplifications

- Mutations are represented by diffusion of the traits.
- Most adapted trait is always $y = 0$.
- Competition between individuals with the same trait $y$.

\[
\partial_t u - \Delta u = \lambda u(1 - u) - \alpha y^2 u.
\]

Existence of traveling front depending on $\alpha$?
Oncology: Cord tumor growth
Tumoral cells need **oxygen** to survive and multiply. **Tumeur cords** are tumour growing along a blood vessel.

**Mathematical model:**

\[
\begin{align*}
\partial_t u - \nu \Delta u &= \lambda u(K(n) - u) \\
\partial_t n - \Delta n &= -\alpha n - \beta u
\end{align*}
\]

with \( K(n) = n \), \( n(|y| = 0) = 1 \), \( n \) and \( u \) bounded.
Here $\beta = 0$ so $n$ can be calculated explicitly.
Traveling fronts

Joint work with H. Berestycki.
\[ \partial_t u - \Delta u = \lambda u(1-u) - \alpha |y|^2 u \quad \text{with } t \in \mathbb{R} \text{ and } X = (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \]

**Traveling front:** a solution \( u(t, x, y) = u(x - ct, y) \) with a stationary state invading another state, i.e.
\[ \lim_{-\infty} u(., y) \neq \lim_{+\infty} u(., y). \]
\[ \partial_t u - \triangle u = \lambda u(1-u) - \alpha |y|^2 u \quad \text{with } t \in \mathbb{R} \text{ and } X = (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}. \]

**Traveling front:** a solution \( u(t, x, y) = u(x - ct, y) \) with a stationary state invading another state, i.e. \( \lim_{-\infty} u(., y) \neq \lim_{+\infty} u(., y) \).

So we are looking for \( u \in C^2(\mathbb{R}^N) \) bounded and \( c \in \mathbb{R} \) solution of

\[
\begin{cases}
\triangle u + c \partial_x u + f(y, u) = 0 & \text{on } \mathbb{R}^N, \\
u(-\infty, y) = \varphi(y) > 0, & u(+\infty, y) = 0,
\end{cases}
\]

with \( f(y, u) = \lambda u(1-u) - \alpha |y|^2 u. \)
Invading asymptotic profile
Asymptotic profiles

If \( u \xrightarrow{x \to \pm \infty} \varphi \) in \( C^1_{loc} \),

\[
\Delta \varphi + \lambda \varphi (1 - \varphi) - \alpha |y|^2 \varphi = 0, \quad y \in \mathbb{R}^{n-1}.
\]

0 is a trivial solution. We are looking for solution \( \varphi \geq 0, \varphi \not\equiv 0 \).
Let $\mathcal{H} = \{u \in H^1(\mathbb{R}^{N-1}), |y|u \in L^2\}$, $\mathcal{H} \hookrightarrow L^2$ compact.

$$\triangle \psi + (1 - \alpha |y|^2 \psi) + \lambda \psi = 0 \quad (L)$$

(L) has a principal eigenvalue

$$\lambda_\alpha = \min_{\psi \in \mathcal{H}, \int \psi^2 = 1} \int |\nabla \psi|^2 + (\alpha |y|^2 - 1) \psi^2.$$ 

- $\alpha \mapsto \lambda_\alpha$ is continuous, increasing, concave.
- $\lambda_\alpha \xrightarrow{\alpha \to 0} -1$ and for $\alpha$ large enough $\lambda_\alpha \geq 0$.
- There exists $\alpha_0 > 0$ such that $\lambda_\alpha < 0$ if $\alpha < \alpha_0$ and $\lambda_\alpha \geq 0$ if $\alpha \geq \alpha_0$. 

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2. If $\alpha \geq \alpha_0$, 0 is the unique bounded solution of

$$\Delta \varphi + \lambda \varphi (1 - \varphi) - \alpha |y|^2 \varphi = 0, \quad y \in \mathbb{R}^{N-1}.$$ 

3. If $\alpha < \alpha_0$, there exists a unique bounded solution $\varphi > 0$. 
For $w \in \mathcal{H}$, we define

$$H(w) = \int_{\mathbb{R}^{N-1}} \left( \frac{1}{2} |\nabla w|^2 - F(y, w) \right) dy$$

where $F(y, s) = \int_0^s f(y, \rho) d\rho$.

If $\alpha < \alpha_0$, $H(\varphi) = \min_{w \in \mathcal{H}} H(w) < H(0) = 0$. 
Existence of traveling front for $\alpha < \alpha_0$
For $-a < 0 \leq b$,

\[
\begin{cases}
\triangle u + c \partial_x u + f(y, u) = 0 \quad (x, y) \in ] -a, b[ \times \mathbb{R}^{N-1}, \\
u(-a, .) = \varphi, \quad u(b, .) = 0.
\end{cases}
\]

- There exists $u_{a,b}$ solution of this equation
  \(\varphi\) is supersolution and 0 is subsolution.

- $u_{a,b}$ is unique and decreasing in $x$ with $c \in \mathbb{R}$ fixed:
  Sliding method.

- $c \mapsto u^{c}_{a,b}$ is continuous and decreasing.
  $u^{c}_{a,b}$ is supersolution of the equation with $c' > c$.

- $a \mapsto u_{a,b}$ is decreasing and $b \mapsto u_{a,b}$ is increasing.
  $u_{a,b}$ is subsolution of the problem with $a' < a$.\]
We can define $u^b = \lim_{a \to +\infty} u_{a,b}$ and as well $u_a = \lim_{b \to +\infty} u_{a,b}$. The function $u^b$ is solution of

$$\begin{cases}
\triangle u^b + c \partial_x u^b + f(y, u^b) = 0 & \text{on } ] - \infty, b] \times \mathbb{R}^{N-1} \\
u^b(b, .) = 0, & \partial_x u^b \leq 0.
\end{cases}$$

2 cases depending on $c$: $u^b \equiv 0$ or $\lim_{x \to -\infty} u^b(x, y) = \varphi$.

Let us define

$$A_0 = \left\{ c \in \mathbb{R} \mid u^b(x, .) \xrightarrow{x \to -\infty} 0 \right\} \text{ and } A_\varphi = \left\{ c \in \mathbb{R} \mid u^b(x, .) \xrightarrow{x \to -\infty} \varphi \right\}$$

And as well

$$B_0 = \left\{ c \in \mathbb{R} \mid u_a(x, .) \xrightarrow{x \to +\infty} 0 \right\} \text{ and } B_\varphi = \left\{ c \in \mathbb{R} \mid u_a(x, .) \xrightarrow{x \to +\infty} \varphi \right\}.$$
Some properties of $\mathcal{A}$

There exists $c^* > 0$ such that $\mathcal{A}_\varphi = ]-\infty, c^*[ \text{ and } \mathcal{A}_0 = [c^*, +\infty[. $

- $\mathcal{A}_0$ and $\mathcal{A}_\varphi$ are intervals. 
  \( c \mapsto u_{a,b} \) is decreasing. If $c_0 \in \mathcal{A}_0$, $\forall c > c_0 \ c \in \mathcal{A}_0$ and if $c_0 \in \mathcal{A}_\varphi$, $\forall c < c_0 \ c \in \mathcal{A}_\varphi$.

- $0 \in \mathcal{A}_\varphi$.
  
  Energy estimate:
  
  \[
  c \int_{x_1}^{x_2} \int_{\mathbb{R}^{N-1}} \partial_x u(x, y)^2 \, dy \, dx = H(u(x_2, \cdot)) - H(u(x_1, \cdot)) \\
  + \int_{\mathbb{R}^{N-1}} \partial_x u(x_1, y)^2 \, dy - \int_{\mathbb{R}^{N-1}} \partial_x u(x_2, y)^2 \, dy
  \]

  Thus for $c = 0$, $x_1 = -a$ and $x_2 = b$,
  
  \[
  \int_{\mathbb{R}^{N-1}} \partial_x u(b, y)^2 \, dy \geq H(0) - H(\varphi) > 0 \text{ and } u \not\in \mathcal{A}_0.
  \]

- $c \in \mathcal{A}_0$ if $c$ is large enough.
  
  Construction of a supersolution on $[-a, b]$, $\frac{a}{\to} \infty \to 0.$

- $\mathcal{A}_0$ is closed.
Some properties of $\mathcal{B}$

- $\mathcal{B}_0$ and $\mathcal{B}_\varphi$ are intervals.
- $\mathcal{B}_\varphi$ is closed.
- $\mathcal{A}_\varphi \cap \mathcal{B}_0 = \emptyset$

*By using the sliding method.*

$$\Rightarrow c^* \in \mathcal{B}_\varphi$$
For $c = c^*$, there exists a traveling front.

$a = n, b = 0$: There exists $h_n \in ]0, n[\) such that $u_{n,0}(-h_n, 0) = \frac{1}{2}\varphi(0)$.

c $\in A_0 \Rightarrow h_n \xrightarrow{n \to \infty} +\infty$, c $\in B_\varphi \Rightarrow h_n - n \xrightarrow{n \to \infty} +\infty$. Hence $u_{n,0}(. - h_n, .) \xrightarrow{n \to \infty} U$ and $U$ is a traveling front.

For $c > c^*$, there exists a traveling front.

Use the traveling front with $c^*$ to build sub- and super-solution.

For $c < c^*$, there is no traveling front.

Assume there is a traveling front $U$, then $u^b < U$ by a maximum principle and there is a contradiction using the sliding method.
Uniqueness: study of the exponential decay in $x$ to initiate a sliding method.

Spreading

Tumor cords.

Influence of circumvolutions.

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Uniqueness: study of the exponential decay in $x$ to initiate a sliding method.

- Spreading
- Tumor cords.
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Thank you for your attention!