Traveling waves steered by the non linearity

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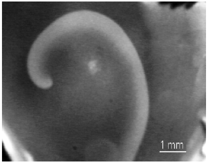
"Deterministic and stochastic front propagation", Banff.

Modeling

Brain pathophysiology: Spreading depressions

Spreading depressions(SD)

Spreading depression (SD): **Transient depolarization** of neurons that **propagates slowly** (3mm/min) through the brain.



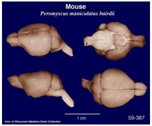
- SD are important for different pathologies:
  stroke, migraine with aura, epilepsy, head injuries.
- SD are well studied in rodent,
  never clearly observed in human!

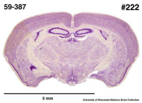
Influence of the brain morphology

Brain is composed of white and gray matter.

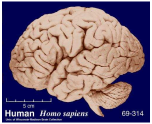
SD: reaction-diffusion mechanism in the gray matter and diffusion and absorption in the white matter.

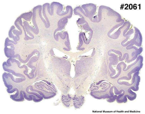
- **Rodent brain:** rather smooth and entirely composed of gray matter.





- **Humain brain:** thin layer of gray matter at the periphery, white matter inside.





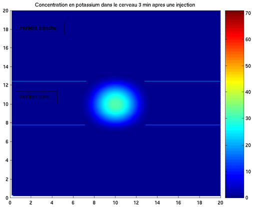
The morphology of the human brain may prevent the propagation of SD!

Effect of the white matter on SD propagation

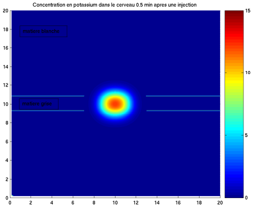
$$\partial_t u - \triangle u = \lambda u(1-u)(u-\theta)\mathbb{1}_{|y|\leq R} - \alpha u \mathbb{1}_{|y|>R}$$

where $u \in \mathcal{C}^1$ is bounded.

- If the gray matter is large



- If the gray matter is thin



Population dynamic: spatial mutation

Dynamic of a population with a quantitative trait

Let $f(t,x,v) \geq 0$ be the density of individuals who at time t and point $x \in \mathbb{R}$ possess the quantitative trait $v \in \mathbb{R}$.

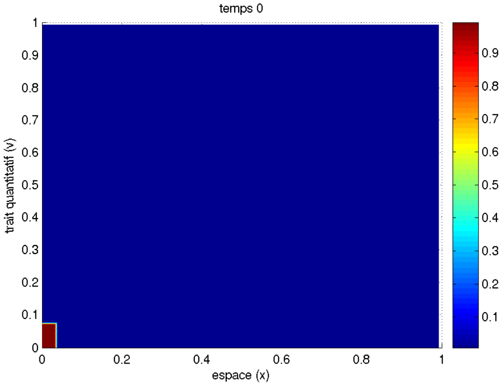
Assumptions:

- spatial diffusion,
- mutation of the trait,
- logistic growth,
- the most adapted trait depends on x.

$$\partial_t f - \nu \partial_{xx} f - \mu \int_{w \in \mathbb{R}} e^{-\frac{|v-w|^2}{\tau}} f(t,x,w)\, dw = \left((a - b\,|v - \phi(x)|^2) - \int_{w \in \mathbb{R}} f(t,x,w)\, dw \right) f(t,x,v)$$

Example of front propagation

We assume $\phi(x) = x$.



Simplifications

- Mutations are represented by diffusion of the traits
- Most adapted trait is always $y = 0$.
- Competition between individuals with the same trait y.

$$\partial_t u - \triangle u = \lambda u(1-u) - \alpha y^2 u.$$

Existence of traveling front depending on α?

Oncology: Cord tumor growth

Cord tumor

Tumoral cells need oxygen to survive and multiply.
Tumeur cords are tumour growing along a blood vessel.

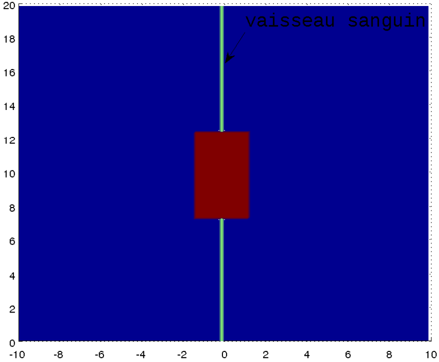
Mathematical model:

$$\begin{cases} \partial_t u - \nu \triangle u = & \lambda u(K(n) - u) \\ \partial_t n - \triangle n = & -\alpha n - \beta u \end{cases}$$

with $K(n) = n$, $n(|y| = 0) = 1$, n and u bounded.

Example of front propagation

Here $\beta = 0$ so n can be calculated explicitely.



Traveling fronts

Joint work with H. Berestycki.

Mathematical framework

$$\partial_t u - \triangle u = \lambda u(1-u) - \alpha |y|^2 u \quad \text{with } t \in \mathbb{R} \text{ and } X = (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1}.$$

Traveling front: a solution $u(t,x,y) = u(x-ct,y)$ with a stationary state invading another state, i.e. $\lim_{-\infty} u(.,y) \neq \lim_{+\infty} u(.,y)$.

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So we are looking for $u \in \mathcal{C}^2(\mathbb{R}^N)$ bounded and $c \in \mathbb{R}$ solution of

$$\begin{cases} \triangle u + c\partial_x u + f(y, u) = 0 & \text{on } \mathbb{R}^N, \\ u(-\infty, y) = \varphi(y) > 0, & u(+\infty, y) = 0, \end{cases}$$

with $f(y, u) = \lambda u(1 - u) - \alpha |y|^2 u$.

Invading asymptotic profile

Asymptotic profiles

If $u \xrightarrow{x\to\pm\infty} \varphi$ in $\mathcal{C}^1_{loc}$,

$$\triangle\varphi + \lambda\varphi(1-\varphi) - \alpha|y|^2\varphi = 0, \qquad y \in \mathbb{R}^{n-1}.$$

0 is a trivial solution. We are looking for solution $\varphi \geq 0$, $\varphi \not\equiv 0$.

Linearizing around 0

Let $\mathcal{H} = \{u \in H^1(\mathbb{R}^{N-1}), |y|u \in L^2\}$, $\mathcal{H} \hookrightarrow L^2$ **compact**.

$$\triangle\psi + (1 - \alpha|y|^2\psi) + \lambda\psi = 0 \qquad \text{(L)}$$

1. (L) has a principal eigenvalue
$$\lambda_\alpha = \min_{\psi \in \mathcal{H}, \int \psi^2 = 1} \int |\nabla\psi|^2 + (\alpha|y|^2 - 1)\psi^2.$$

- $\alpha \mapsto \lambda_\alpha$ is **continuous, increasing, concave**.
- $\lambda_\alpha \xrightarrow{\alpha \to 0} -1$ and for α large enough $\lambda_\alpha \geq 0$.
- There exists $\alpha_0 > 0$ such that $\lambda_\alpha < 0$ if $\alpha < \alpha_0$ and $\lambda_\alpha \geq 0$ if $\alpha \geq \alpha_0$.

2. If $\alpha \geq \alpha_0$, **0 is the unique bounded solution** of

$$\triangle\varphi + \lambda\varphi(1-\varphi) - \alpha|y|^2\varphi = 0, \qquad y \in \mathbb{R}^{N-1}.$$

3. If $\alpha < \alpha_0$, **there exists a unique bounded solution** $\varphi > 0$.

Energy of φ

For $w \in \mathcal{H}$, we define

$$H(w) = \int_{\mathbb{R}^{N-1}} \left(\frac{1}{2} |\nabla w|^2 - F(y, w) \right) dy$$

where $F(y, s) = \int_0^s f(y, \rho) d\rho$.

If $\alpha < \alpha_0$,

$$H(\varphi) = \min_{w \in \mathcal{H}} H(w) < H(0) = 0.$$

Existence of traveling front for $\alpha < \alpha_0$

For $-a < 0 \leq b$,

$$\begin{cases} \triangle u + c\partial_x u + f(y,u) = 0 & (x,y) \in]-a,b[\times\mathbb{R}^{N-1}, \\ u(-a,.) = \varphi, \quad u(b,.) = 0. \end{cases}$$

- There exists $u_{a,b}$ solution of this equation
  φ is supersolution and 0 is subsolution.
- $u_{a,b}$ is unique and decreasing in x with $c \in \mathbb{R}$ fixed:
  Slinding method.
- $c \mapsto u^c_{a,b}$ is continuous and decreasing.
  $u^c_{a,b}$ is supersolution of the equation with $c' > c$.
- $a \mapsto u_{a,b}$ is decreasing and $b \mapsto u_{a,b}$ is increasing.
  $u_{a,b}$ is subsolution of the problem with $a' < a$.

We can define $u^b = \lim_{a\to+\infty} u_{a,b}$ and as well $u_a = \lim_{b\to+\infty} u_{a,b}$. The function u^b is solution of

$$\begin{cases} \triangle u^b + c\partial_x u^b + f(y, u^b) = 0 & \text{on }]-\infty, b] \times \mathbb{R}^{N-1} \\ u^b(b, .) = 0, \quad \partial_x u^b \leq 0. \end{cases}$$

2 cases depending on c: $u^b \equiv 0$ or $\lim_{x\to-\infty} u^b(x, y) = \varphi$.

Let us define

$$\mathcal{A}_0 = \left\{ c \in \mathbb{R} | u^b(x, .) \xrightarrow{x\to-\infty} 0 \right\} \text{ and } \mathcal{A}_\varphi = \left\{ c \in \mathbb{R} | u^b(x, .) \xrightarrow{x\to-\infty} \varphi \right\}$$

And as well

$$\mathcal{B}_0 = \left\{ c \in \mathbb{R} | u_a(x, .) \xrightarrow{x\to+\infty} 0 \right\} \text{ and } \mathcal{B}_\varphi = \left\{ c \in \mathbb{R} | u_a(x, .) \xrightarrow{x\to+\infty} \varphi \right\}.$$

Some properties of $\mathcal{A}$

There exists $c^* > 0$ such that $\mathcal{A}_\varphi =]-\infty, c^*[$ and $\mathcal{A}_0 = [c^*, +\infty[$.

- $\mathcal{A}_0$ and $\mathcal{A}_\varphi$ are intervals.
  $c \mapsto u_{a,b}$ is decreasing. If $c_0 \in \mathcal{A}_0$, $\forall c > c_0 \;\; c \in \mathcal{A}_0$ and if $c_0 \in \mathcal{A}_\varphi$, $\forall c < c_0 \;\; c \in \mathcal{A}_\varphi$.
- $0 \in \mathcal{A}_\varphi$.
  Energy estimate:

$$c \int_{x_1}^{x_2} \int_{\mathbb{R}^{N-1}} \partial_x u(x,y)^2 dydx = H(u(x_2,.)) - H(u(x_1,.)) + \int_{\mathbb{R}^{N-1}} \partial_x u(x_1,y)^2 dy - \int_{\mathbb{R}^{N-1}} \partial_x u(x_2,y)^2 dy$$

  Thus for $c = 0$, $x_1 = -a$ and $x_2 = b$,
  $\int_{\mathbb{R}^{N-1}} \partial_x u(b,y)^2 dy \geq H(0) - H(\varphi) > 0$ and $u \notin \mathcal{A}_0$.
- $c \in \mathcal{A}_0$ if c is large enough.
  Construction of a supersolution on $[-a,b]$, $\xrightarrow{a\to\infty} 0$.
- $\mathcal{A}_0$ is closed.

Some properties of $\mathcal{B}$

- $\mathcal{B}_0$ and $\mathcal{B}_\varphi$ are intervals.
- $\mathcal{B}_\varphi$ is closed.
- $\mathcal{A}_\varphi \cap \mathcal{B}_0 = \emptyset$

  By using the sliding method.

$$\Rightarrow c^* \in \mathcal{B}_\varphi$$

Construction of traveling fronts

1. **For $c = c^*$, there exists a traveling front.**
   $a = n$, $b = 0$: There exists $h_n \in]0, n[$ such that $u_{n,0}(-h_n, 0) = \frac{1}{2}\varphi(0)$. $c \in \mathcal{A}_0 \Rightarrow h_n \xrightarrow{n\to\infty} +\infty$, $c \in \mathcal{B}_\varphi \Rightarrow h_n - n \xrightarrow{n\to\infty} +\infty$. Hence $u_{n,0}(. - h_n, .) \xrightarrow{n\to\infty} U$ and U is a traveling front.
2. **For $c > c^*$, there exists a traveling front.**
   Use the traveling front with c^* to build sub- and super-solution.
3. **For $c < c^*$, there is no traveling front.**
   Assume there is a traveling front U, then $u^b < U$ by a maximum principle and there is a contradiction using the sliding method.

Work in progress

- Uniqueness: study of the exponential decay in x to initiate a sliding method.
- Spreading
- Tumor cords.
- Influence of circumvolutions.
- ...

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Thank you for your attention!