RESEARCH STATEMENT

ROBIN KOYTCHEFF

INTRODUCTION

My research interests lie generally in the areas of algebraic and geometric topology and more specifically in the study of algebro-topological and homotopy-theoretic aspects of knot theory. My work has been inspired by the configuration space integrals of Bott and Taubes [3], which provide one way to construct the finite-type invariants first developed by Vassiliev [22] and recast combinatorially by Bar-Natan [2]. These configuration space integrals also give classes of arbitrary degree in the real cohomology of spaces of knots. The key ingredient of the Bott–Taubes construction is integration of differential forms along the fiber of a bundle. An important idea in algebraic topology is to replace such integration by a Pontrjagin–Thom construction. So far I have successfully used this idea to produce classes in the cohomology of the space of knots with arbitrary coefficients, rather than just real coefficients as in the original Bott–Taubes construction. I have also proved that these classes satisfy a product formula with respect to connect-sum. These results appear in a paper that I’ve written, “A homotopy-theoretic view of Bott–Taubes integrals and knot spaces”, published in 2009 in Algebraic and Geometric Topology 9(3), pp. 1467-1501 [12] (see also arXiv:0810.1785).

Presently I am investigating how my homotopy theoretic Bott–Taubes integrals is related to other tools of algebraic topology that have been used to study spaces of knots. These include the Taylor tower and Sinha’s cosimplicial model [19] for the space of knots (both coming from the Goodwillie–Weiss embedding calculus [20] [11]); Salvatore’s little 2-cubes action on the space of long knots in $\mathbb{R}^d$ ($d > 3$) [18], which followed from work of Sinha [20] and of McClure and Smith [16]; and Budney’s little 2-cubes operad action on the space of long knots in $\mathbb{R}^3$ [6] and the computation of the homology of this space by Budney and F. Cohen [7]. Lambrechts, Turchin, and Volić used the ideas of the embedding calculus to compute the rational homology of certain spaces of long knots in codimension > 2 [13], and I am currently working towards making similar explicit calculations involving the classes I have constructed. I expect to have these calculations completed in my PhD thesis which will be submitted in June.

It is an open conjecture that finite-type invariants approximate all knot invariants [22] (or equivalently that they separate knots), but Volić showed that they factor through the Taylor tower for the knot space [24]. I thus hope to ultimately study finite-type invariants using my homotopy-theoretic construction. Eventually, I would like to generalize my construction to spaces of links, braids, string-links, and higher-dimensional knots and to use homotopy-theoretic integration along the fiber to study other geometric-topological problems.

BACKGROUND AND COMPLETED RESEARCH

In more detail, the space of knots $\text{Emb}(S^1, \mathbb{R}^3)$ is the space of embeddings of a circle into $\mathbb{R}^3$. A connected component of this space corresponds to an isotopy class of knots, so elements of $H^0(\text{Emb}(S^1, \mathbb{R}^3))$ correspond to knot invariants, and cohomology classes of
higher degree can be thought of as generalizations of knot invariants. Other interesting knot spaces include $\text{Emb}(S^1, \mathbb{R}^d)$ and the space of long knots $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ ($d \geq 3$), which is the space of embeddings of $\mathbb{R}$ into $\mathbb{R}^d$ with fixed behavior outside the unit interval. The constructions of cohomology classes in my research apply to both long and closed knots in any $\mathbb{R}^d$, though the results related to the connect-sum operation apply only to long knots.

**Bott–Taubes integrals.** In [3], Bott and Taubes considered a bundle over $\text{Emb}(S^1, \mathbb{R}^3)$ such that a point in the total space $E = E_{q,t}$ is a knot together with $q + t$ points in $\mathbb{R}^3$ such that the first $q$ points lie on the knot. The fiber $F$ over a knot is thus a space of $q + t$ points in $\mathbb{R}^3$, $q$ of which lie on the knot, and is thus a subspace of the compactified configuration space $C_{q+t}[\mathbb{R}^3]$ developed in [10, 1]. Bott and Taubes integrated differential forms coming from $H^*(C_{q+t}[\mathbb{R}^3])$ along the fiber of this bundle. Integration along the fiber when the fiber has boundary does not automatically produce a closed form, so one of their main results was that a certain 0-dimensional form obtained by integration along the fiber is closed, hence represents a knot invariant. This knot invariant had been previously found through Chern–Simons theory, but the approach of Bott and Taubes was used by D. Thurston to construct finite-type invariants [21, 25] and by Cattaneo, Cotta-Ramusino, and Longoni to produce nontrivial classes of higher degrees in the real cohomology of $\text{Emb}(S^1, \mathbb{R}^d)$ for $d > 3$ [8].

**Homotopy-theoretic Bott–Taubes integration.** I carried out a construction inspired by the Bott–Taubes integrals, in which I replaced integration of differential forms by a Pontrjagin–Thom map, as follows. Let $\mathcal{K} = \text{Emb}(\mathbb{R}, \mathbb{R}^d)$ where $d \geq 3$, and let $E$ be the total space of the bundle over $\mathcal{K}$ in the Bott–Taubes construction. Suppose that there were an embedding $E \hookrightarrow \mathcal{K} \times \mathbb{R}^N$. In that case, quotienting by the complement of a tubular neighborhood of $E$ would give a pre-transfer map

$$\Sigma^N \mathcal{K}_+ \to E^\nu_N$$

from the $N$-fold suspension of the knot space (with a disjoint basepoint) to the Thom space of the normal bundle $\nu_N \to E$ of the embedding $e_N$. Together with the Thom isomorphism and suspension isomorphism, this would induce a map in cohomology with coefficients in any ring $R$

$$H^*(E; R) \to H^{*-\dim F}(\mathcal{K}; R)$$

corresponding to integration along the fiber.

However, $E$ has corners coming from those in the fiber $F$, so it must be embedded into $\mathcal{K} \times \mathbb{R}^{N-L} \times [0, \infty)^L$ in a way that respects the corner structure; this allows for notions of a tubular neighborhood and a normal bundle of the embedding. I used the categorical approach to manifolds with corners given in [14] to retain the corner structure in the embedding. In my construction so far, I quotient by all boundary subspaces in order to get a map from an iterated suspension of (rather than a cone on) $\mathcal{K}$, and to get the number of suspensions such that the degree shift in cohomology is the same as in integration along the fiber. Thus the pre-transfer map is actually

$$\tau : \Sigma^N \mathcal{K}_+ \to E^\nu_N / \partial E^\nu_N.$$  

(1)

Letting $N$ approach $\infty$ gives a map of spectra, and the following is the first main result of my paper:
Theorem 1. There is a map from the suspension spectrum of $K$ to the Thom spectrum of the normal bundle to $E$ modulo its boundary

\[ \Sigma^\infty K_+ \rightarrow E'/\partial E' \]

which induces in cohomology a map

\[ H^*(E, \partial E) \rightarrow H^{*-\dim F}(K) \]

analogous to the Bott–Taubes integration along the fiber.

In the case of bona fide integration along the fiber, there is only a map on the level of forms, not cohomology, because of the presence of boundary. On the other hand, the map (3) exists for cohomology with arbitrary coefficient rings, as well as for any generalized cohomology theory with respect to which the bundle $\nu_N \rightarrow E$ is orientable. It is also worth noting that even though the maps (1) and (2) are of quotients by the entire boundary subspaces, the embeddings used for the Pontrjagin–Thom map preserve all the corner structure.

The next result of my paper was motivated by Budney’s little 2-cubes action on $\text{Emb}(\mathbb{R}, \mathbb{R}^3)$ [6]. This action can be viewed as a lifting of the connect-sum operation on isotopy classes of knots to an operation on the space of knots; the fact that the little 2-cubes operad parametrizes it reflects its homotopy-commutativity. Budney and F. Cohen then showed that $H_*(\text{Emb}(\mathbb{R}, \mathbb{R}^3))$ is generated by certain homology operations applied to the homology of the space of prime knots [7]. This motivated the problem of computing the evaluation of any Bott–Taubes cohomology class in terms of its evaluations on classes in the homology of the space of prime knots.

I was able to lift the space-level connect-sum $\mu : K \times K \rightarrow K$ to a multiplication on the $E_{q,t}$. The compatibility of these multiplications with the Thom collapse maps showed that the map (2) is a map of ring spectra. It also allowed me to prove the second main result of my paper. The formula referenced below ultimately allows calculations in terms of the generators of $H^*(C_{q+t}[\mathbb{R}^d])$, but for simplicity I refrain from writing it here. It can be considered as an analogue of Theorem 9(2) in [2] which says that a certain class of Vassiliev invariants are additive with respect to connect-sum. The interested reader may consult [12] for the actual formula.

Theorem 2. Let $a, b \in H_\ast K$. There is a product formula which expresses the the evaluation of a Bott–Taubes class on $\mu_\ast(a \otimes b)$ in terms of evaluations of “smaller” Bott–Taubes classes on $a$ and $b$.

Ongoing and future research

Homotopy-theoretic Bott–Taubes integration and the embedding calculus. Now that I have a homotopy-theoretic construction of Bott–Taubes integration, I would like to see how it fits into other algebro-topological approaches to knot theory, especially those coming from the Goodwillie–Weiss embedding calculus. The embedding calculus gives a “Taylor tower”

\[ T_1 K \leftarrow ... \leftarrow T_{n-1} K \leftarrow T_n K \leftarrow ... \]

with maps $K \rightarrow T_n K$, where each $T_n K$ is essentially the space of $n$-times punctured knots. For $d > 3$, the map from $K$ to the $n$th stage of Taylor tower is $(n - 1)(d - 3)$-connected [11]. So in this case the tower converges to the original embedding functor, i.e., the homotopy limit is weakly equivalent to $K$. Although convergence of the tower to the knot space not known for $d = 3$, all finite-type invariants factor through the Taylor tower in this case [24].
Closely related to the Taylor tower is a cosimplicial model $X^\bullet$ for $\mathcal{K}$ developed by Sinha in [19] (see also [20]). This model is analogous to the cosimplicial model for the loop space, and its levels $X^k$ are roughly configuration spaces of points in $\mathbb{R}^d$. Sinha showed that the homotopy-invariant partial totalization $\widetilde{\text{Tot}}^n X^\bullet$ is homotopy equivalent to $T_n \mathcal{K}$. This implies that for $d > 3$ the full totalization $\widetilde{\text{Tot}}X^\bullet$ is weakly equivalent to $\mathcal{K}$.

I have recently developed and am currently studying a “cosimplicial” version $E^n$ of the total space $E$ in the Bott–Taubes construction. The $E^n$ themselves are not cosimplicial spaces, but they fiber over the $\widetilde{\text{Tot}}^n X^\bullet$. There is a tower of fibrations

\[
\begin{array}{c}
\vdots \\
E^{n+1} \\
\downarrow \\
E^n \\
\downarrow \\
E^{n-1} \\
\downarrow \\
\vdots
\end{array}
\]

and the map from $E$ to the limit of the $E^n$ is an equivalence in the case of knots in $\mathbb{R}^d$ for $d > 3$. Despite the lack of smoothness, a Pontrjagin–Thom-type construction can be carried out for the fibration $E^n \to \widetilde{\text{Tot}}^n X^\bullet$ by quotienting by the complement of a neighborhood of an embedding. The Pontrjagin–Thom constructions for the $E^n$ are compatible with the one for $E$, and this compatibility together with the fact that the limit of the $E^n$ is $E$ provide an analogue of the Thom isomorphism. Hence the collapse map induces a map

\[H^*(E^n, \partial E^n) \to H^{*-\dim F}(\widetilde{\text{Tot}}^n X^\bullet)\]

in cohomology, extending my homotopy-theoretic Bott–Taubes construction to the cosimplicial setting. I am currently working on proving a product formula similar to Theorem 2 in this cosimplicial setting. Together with the Bousfield–Kan spectral sequence [4, 5] this should yield explicit calculations involving the Bott–Taubes classes. More generally, applying the embedding calculus to my construction should help make computations because the layers of the Taylor tower (i.e., the homotopy fibers of $T_n \mathcal{K} \to T_{n-1} \mathcal{K}$) are well understood.

Because of Volić’s result that finite-type invariants factor through the Taylor tower [24], my planned avenue of research should shed light on these invariants, which are not completely understood. In particular, it is an open conjecture that they approximate all knot invariants or equivalently that they separate knots. At the same time, my classes are more general than the ones that other researchers have studied so far. For example, the abovementioned result of Volić is only known for real coefficients. Since my construction works for arbitrary coefficient rings, it could very likely lead to generalizations of it.
Other refinements to the homotopy-theoretic Bott–Taubes construction. I also plan to more closely examine certain aspects of my homotopy-theoretic Bott–Taubes construction. The first such direction relates to the corner structure. The maps (1) and (2) come from embeddings which preserve all the corner structure, leaving much to be investigated. For example, considering various strata in \( E \hookrightarrow \mathcal{K} \times \mathbb{R}^{N-L} \times [0, \infty)^L \) and quotienting by their boundaries would give maps similar to (1), but from lower suspensions of \( \mathcal{K} \) to Thom spaces of strata of \( E \). Because I have so far quotiented by all boundary subspaces, my construction is not exactly parallel to that of Bott and Taubes: they consider classes which come from the absolute cohomology of configuration spaces (rather than relative to boundary), and they produce a closed form by showing that certain boundary contributions cancel (instead of contributing zero). I plan to pursue a modification of my construction where I would glue the boundaries of certain fibers \( F_{q,t} \) to each other instead of quotienting by all the boundaries. This would produce classes which could be much more easily compared to the original Bott–Taubes classes.

The other aspect I plan to refine relates to lifting the 2-cubes action on \( \mathcal{K} \) to the total space \( E \). In the work \([12]\), I attempted to prove similar formulae to the one referenced in Theorem 2 with the product \( \mu_* \) replaced by the Browder operation (a.k.a., the bracket) on \( H_*(\mathcal{K}; \mathbb{Q}) \) or the Dyer–Lashof operations in \( H_*(\mathcal{K}; \mathbb{Z}/p) \) coming from the 2-cubes action. A very useful step towards such formulae would have been to lift the little 2-cubes action to the total space of the bundle. However, I observed that the multiplication on the \( E_{q,t} \) does not extend to a little 2-cubes action because multiplication of ordered configurations of points is noncommutative in homology. Thus it would be interesting to investigate whether replacing ordered configurations by unordered configurations allows for a lifting of the 2-cubes action to the total space.

Other embedding spaces and further generalizations. Other possible future directions of inquiry include using my construction to study spaces of links, braids, and homotopy string-links. Volić has studied cohomology of these spaces using (bona fide) configuration space integrals \([23]\), and he and Munson have developed cosimplicial models for these spaces \([17]\). Homotopy-theoretic integration may produce interesting integral classes which do not appear in the real cohomology, and the cosimplicial models should be conducive to explicit calculations, as expected in the case of knot spaces. Another likely extension of my research would be to study spaces of embeddings of \( S^m \) into \( S^n \).

More generally, integration along the fiber via the Pontrjagin–Thom construction has been used to solve various problems in geometry and topology. For example, R. Cohen and Jones used it to give a homotopy-theoretic formulation of the Chas–Sullivan string topology product, which allowed for generalizations and calculations \([9]\). Another example is in Madsen and Weiss’ proof of the Mumford conjecture on the cohomology of the mapping class group \([15]\), in which they showed that all its cohomology classes come from a Pontrjagin–Thom construction. The conjecture that finite-type invariants approximate all knot invariants can be considered as an analogous statement for just the degree-0 cohomology of the space of knots, since Bott–Taubes integrals give finite-type invariants via integration along the fiber. Thus I am hopeful that my homotopy-theoretic construction will help make progress towards answering such questions. In future research, I may pursue other geometric and topological problems to which homotopy-theoretic integration along the fiber can be applied.
References