ASYMPTOTICS FOR WILF'S PARTITIONS WITH DISTINCT MULTIPLICITIES

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Abstract. We give an asymptotic for the number of partitions of n for which the nonzero multiplicities of its parts are all distinct. The results rely on an analysis of the distribution of the quantity $\sum_{k=1}^{m} k\pi(k)$ as π varies over the symmetric group on $\{1, 2, \cdots, m\}$.

1. Statement of Results

Wilf, in his list of unsolved problems [6], asks for any interesting theorems concerning the following set of constrained partitions of n: Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different. Write $f(n) = |T(n)|$. Zeilberger [7] discussed an algorithm to compute $f(n)$ and asked for an asymptotic for $f(n)$. In this note we prove the following asymptotic.

Theorem 1.1. *As $n \to \infty$ we have*

$$\log(f(n)) = \frac{1}{3}(6n)^{\frac{1}{3}} \log(n) - \frac{1}{2}(6n)^{\frac{1}{3}} \log(\log(n)) + O(n^{\frac{1}{3}}).$$

Remark. The leading order term was independently obtained by Fill, Janson, and Ward [2]. Their method is similar, but perhaps more direct.

Additionally, we prove the following theorem reinterpreting the partitions in $T(n)$ as partitions with constrained difference between consecutive parts. Let $DG(n)$ denote the set of all partitions of n with parts $\lambda_1 \geq \lambda_2 \geq \cdots$ such that the nonzero distances $\mu_i = \lambda_i - \lambda_{i+1}$ are distinct. We call $DG(n)$ the set of partitions of n with "distinct gaps".

Theorem 1.2. *There is a bijection between $T(n)$ and $DG(n)$.*

For example, under this bijection the partition $\lambda = (1, 3, 3, 4, 4, 4, 4)$ is an element of $T(23)$ and it corresponds to the partition $(1 + 3 + 4, 3 + 4, 4, 4) = (8, 7, 4, 4) \in DG(23)$, since it has nonzero gap sizes 1 and 3.

The asymptotic of Theorem 1.1 relies on the fact that every partition $\lambda \in T(n)$ is constructed uniquely from the following datum:

(1) a subset $P = \{p_1, \cdots, p_k\}$ of $\{1, 2, \cdots, n\}$ with $p_1 > \cdots > p_k$ (the part sizes)
(2) a subset $M = \{m_1, \cdots, m_k\}$ of $\{1, 2, \cdots, n\}$ with $m_1 > \cdots > m_k$ (the multiplicities)

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(3) a permutation $\pi : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$ such that

$$\sum_{j=1}^{k} m_{\pi(j)} p_j = n.$$

Consequentially, we are led to consider the distribution of the quantity $\sum_{k=1}^{m} k\pi(k)$ as π ranges over S_m, the symmetric group on m elements. In particular, the quantity $(6n)^{\frac{1}{3}}$ in Theorem 1.1 comes from the fact that the minimum of $\sum_{k=1}^{m} k\pi(k)$ as π ranges over S_m is $\frac{m(m+1)(m+2)}{6} \sim \frac{m^3}{6}$. The Hoeffding Combinatorial Lemma [3] implies that the distribution of $\sum_k k\pi(k)$ is approximately normal for large n (see also [4]). However, we exploit the fact that for any fixed n the tails are slightly heavier than one might expect. It is an interesting question to give precise results for the distribution of $\sum_{k=1}^{m} k\pi(k)$ as π ranges over S_m.

The data of Maciej Ireneusz Wilczynsk for $f(n)$ with $n \leq 508$ available on Solane's Online Encyclopedia of Integer Sequences **A098859**, suggests that $\log(f(n)) \sim C\sqrt{n}$ for some constant C. This may be due to the fact that $\frac{n^{\frac{1}{3}}\left(\log(n) - \frac{3}{2}\log(\log(n))\right)}{n^{\frac{1}{2}}}$ obtains an absolute maximum near $n = 2955$.

In Section 2 we prove Theorem 1.2. In Section 3 we prove Theorem 1.1. Finally, we mention that it would be interesting to analyze the constant for the term of size $n^{\frac{1}{3}}$ in Theorem 1.1. It seems that the constant is affected by the expected part sizes of such a partition.

2. Proof of Theorem 1.2

In this section we describe the bijection of Theorem 1.2.

Proof of Theorem 1.2. Let $\lambda \in T(n)$. Let $a_1 > a_2 > \cdots > a_k$ be the distinct parts of λ and let $m_1 > m_2 > \cdots > m_k$ be the distinct multiplicities of the parts so that $n = a_1 m_{\pi(1)} + a_2 m_{\pi(2)} + \cdots + a_k m_{\pi(k)}$ with π in S_k, the symmetric group on k elements. Note that

$$n = m_1 a_{\pi^{-1}(1)} + \cdots + m_k a_{\pi^{-1}(k)} = \sum_{i=1}^{n} \sum_{s:\ m_s \geq i} a_{\pi^{-1}(s)}.$$

Define λ^* to be a partition of n with parts λ_i given by $\sum_{s:\ m_s \geq i} a_{\pi^{-1}(s)}$. It is clear that $\lambda_i \leq \lambda_{i-1}$ for each i and that if we do not have equality then the difference is equal to a_r for some r and that this is the only such i with difference a_r. Therefore $\lambda^* \in DG(n)$. This map is clearly a bijection. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by counting elements of a larger set, namely

$$T^*(n) := \cup_{m=1}^{n} T(m).$$

Every partition $\lambda \in T^*(n)$ is constructed uniquely from the following datum:

(1) a subset $P = \{p_1, \cdots, p_k\}$ of $\{1, 2, \cdots, n\}$ with $p_1 > \cdots > p_k$ (the part sizes)
(2) a subset $M = \{m_1, \cdots, m_k\}$ of $\{1, 2, \cdots, n\}$ with $m_1 > \cdots > m_k$ (the multiplicities)

(3) a permutation $\pi : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$ such that

$$\sum_{j=1}^{k} m_{\pi(j)} p_j \le n.$$

We define the following

$$f^*(n) := \sum_{m=1}^{n} f(n). \tag{3.1}$$

We will prove that $\log(f(n)) \sim \log(f^*(n))$. We begin with the following lemma restricting the number of parts in a partition of $T(n)$.

Lemma 3.1. *If $\lambda \in T(n)$ then λ contains at most $3n^{\frac{1}{3}} + 1$ distinct parts.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s) \in T(n)$. Let $a_1 \ge a_2 \ge \cdots \ge a_k$ be the distinct parts of λ and m_i the multiplicity of a_i. Then there are at least $\lfloor k/2 \rfloor$ values of m_i with $m_i \ge \frac{k}{2}$. Suppose that $m_{i_1}, \cdots, m_{i_{\lfloor \frac{k}{2} \rfloor}} \ge \frac{k}{2}$. Then

$$n = m_1 a_1 + \cdots + m_k a_k \ge \frac{k}{2} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} a_{i_j} \ge \frac{k}{2} \left(\lfloor \frac{k}{2} \rfloor (\lfloor \frac{k}{2} \rfloor + 1) \right) \frac{1}{2} > \frac{k^2(k-1)}{16} > \frac{(k-1)^3}{27}.$$

So we have $k < 3n^{\frac{1}{3}} + 1$. □

Remark. Erdös and Lehner [1] proved that the number of parts in a random unrestricted partition of n has, with high probability, about $\frac{1}{2\pi}(6n)^{\frac{1}{2}} \log(n)$ parts. Lemma 3.1 implies that partitions in $T(n)$ are very rare amongst the set of all partitions of n.

From Lemma 3.1 we immediately deduce a crude upper bound for $f(n)$.

Proposition 3.2. *We have $f(n) < 2\exp\left(2n^{\frac{1}{3}} \log(n) + 2\log(n)\right)$.*

Proof. Let $A = \lfloor 3n^{\frac{1}{3}} + 1 \rfloor$. Then for $n > 1$ the number of partitions in $T(n)$ is bounded above by

$$\sum_{k=1}^{A} \sum_{m_1, \cdots, m_k \in \{1, 2, \cdots, n\}} \sum_{a_1, \cdots, a_k \in \{1, 2, \cdots, n\}} 1 = n^2 \frac{n^{2A} - 1}{n^2 - 1} < 2\exp\left(2n^{\frac{1}{3}} \log(n) + 2\log(n)\right).$$

□

The next lemma shows that to establish Theorem 1.1 it is enough to compute the asymptotic of $f^*(n)$.

Lemma 3.3. *For $n \gg 1$*

$$f^*(n) > f(n) > \frac{f^*(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)}{n^2}$$

Proof. The upper bound is clear. To prove the lower bound, first consider $T^{**}(n)$, the set of all partitions of n with distinct multiplicities and no parts of size 1 or 2. Let $f^{**}(n) = |T^{**}(n)|$. We prove $f(n) > f^{**}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)$. To see this let $\lambda \in T^{**}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)$. We may add m_1 1s and m_2 2's to the partition λ with $m_1 + 2m_2 = \lfloor 101 n^{\frac{2}{3}} \rfloor$, to make a partition of n. There are at least $\lfloor \frac{101}{2} n^{\frac{2}{3}} \rfloor$ choices for the pair (m_1, m_2). Since there are at most $3n^{\frac{1}{3}} + 1$ multiplicity sizes in λ (see Lemma 3.1, appropriately modified) we see that there is a choice of (m_1, m_2) that produces a partition in $T(n)$. Moreover, each of these partitions is distinct since the partition without 1s and 2s are distinct.

Finally we prove that $f^{**}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor) \geq f^{*}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)/n^2$. To see this note that from each partition in $T^{*}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)$ we may delete the parts of size 1 and 2 to produce an element of $T^{**}(n - \lfloor 101 n^{\frac{2}{3}} \rfloor)$. This map is at most n^2 to 1 since there are at most n^2 choices for the multiplicities of m_1 and m_2. □

Lemma 3.4. *The number of choices for subsets $P = \{p_i\}_{i=1}^{k}$ and $M = \{m_i\}_{i=1}^{k}$ such that there is at least one permutation with $\sum_{j=1}^{k} m_{\pi(j)} a_j \leq n$ is $\exp\left(O\left(n^{\frac{1}{3}}\right)\right)$.*

Proof. Assume that P and M is such a pair of sets. Then P contains at most one element of size $> n/3$, because if it contained two then by the distinctness of the multiplicities the partition would have to contain at least three parts of size $> n/3$ which is a contradiction. Likewise, there are $\leq r$ parts of size $> \frac{n}{(r+1)(r+2)/2}$.

Therefore, with $\alpha := \lfloor \frac{\log_2(n)}{3} \rfloor$, the number of possible sets P is bounded above by

$$2^{n^{\frac{1}{3}}} \prod_{j=0}^{\alpha} \binom{\frac{n}{(2^j+1)(2^j+2)/2}}{2^j} \ll \exp\left(n^{\frac{1}{3}}\right) \exp\left(\sum_{j=1}^{\alpha} 2^j \log\left(\frac{n}{2^{3j-1}} - 1\right) - \frac{n}{2^{2j-1}} \log\left(1 - \frac{2^{3j-1}}{n}\right)\right)$$

$$\ll \exp\left(n^{\frac{1}{3}} + \sum_{j \geq 1} n^{\frac{1}{3}} 2^{-k} \log(2^{3k})\right) = \exp\left(O\left(n^{\frac{1}{3}}\right)\right).$$

By symmetry the number of sets of M is the same. Thus the number of pairs of set is also of this size. □

By Proposition 3.2 the size of $\log(f^*(n))$ is bounded by $Cn^{\frac{1}{3}} \log(n)$. Thus to determine the asymptotic of $\log(f^*(n))$ it suffices to determine the largest possible number of permutations for a fixed pair of sets P and M. It is clear that the maximal number of permutations will be achieved with $P = M = \{1, 2, .., m\}$ for some m.

Remark. The average size of $\sum_{k=1}^{m} k\pi(k)$ over all permutations in S_m is $\frac{m(m+1)^2}{4}$. Moreover, if we choose m close to $(4n)^{\frac{1}{3}}$, then at least 1/2 of the permutations will satisfy this inequality and we have

$$\log(f^*(n)) \geq \log\left(\frac{1}{2} \lfloor (4n)^{\frac{1}{3}} \rfloor !\right) = \frac{1}{3}(4n)^{\frac{1}{3}} \log(n) + O(n^{\frac{1}{3}}).$$

To see that half of the permutations satisfy the desired inequality note that if π satisfies $\sum_{k=1}^{m} k\pi(k) > \frac{m(m+1)^2}{4}$ then we may replace $\pi(k)$ by the permutation that sends k to $m+$

$1-\pi(k)$. Call that permutation π' then

$$\sum_{k=1}^{m} k\pi'(k) = \sum_{k=1}^{m} k(m+1-\pi(k)) = \frac{m(m+1)^2}{2} - \sum_{k=1}^{m} k\pi(k) < \frac{m(m+1)^2}{4}.$$

Perhaps surprising is that we can do slightly better. For any $\pi \in S_m$, the rearrangement inequalities imply

$$\frac{m(m+1)(m+2)}{6} = \sum_{k=1}^{m} k(m+1-k) \leq \sum_{k=1}^{m} k\pi(k) \leq \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

The idea is that the distribution of the sums $\sum_{k=1}^{m} k\pi(k)$ as π varies over the symmetric group is heavy enough in the tails that there are many permutations close to the minimum. The distribution of this sum was considered by Hotelling and Pabst [4]. They showed that for large n it is approximately normal.

The following proposition gives the lower bound for the asymptotic in Theorem 1.1.

Proposition 3.5. *Let $\epsilon > 0$ and $m = \lfloor (6n)^{\frac{1}{3}}(1-\frac{1}{\log(n)})^{\frac{1}{3}} \rfloor$. The number of permutations $\pi \in S_m$ with $\sum_{k=1}^{m} k\pi(k) \leq n$ is greater than or equal to*

$$\exp\left(\frac{1}{3}(6n)^{\frac{1}{3}}\left(\log(n) - \frac{3}{2}\log\log(n) + \left(\log(6) - \frac{4}{3}\right) + O\left(\frac{\log\log(n)}{\log(n)}\right)\right)\right)$$

The idea is that the minimum sum is achieved by the permutation Π which maps k to $m+1-k$. If we perturb the permutation locally a small amount then the sum will not be altered by much. For instance, switching any two consecutive values of Π will increase the sum by exactly 1. For example consider π defined by

$$\pi(k) = \begin{cases} m-1 & k=1 \\ m & k=2 \\ \Pi(k) & \text{else} \end{cases}.$$

There are $n-1$ such permutations. Switching any two non-adjacent consecutive pairs will increase the sum by exactly 2. There are $\frac{(n-2)(n-3)}{2}$ such pairs.

Proof of Proposition 3.5. Let $\Pi \in S_m$ be the permutation that sends k to $m+1-k$. This Π minimizes the sum $\sum_{k=1}^{m} k\pi(k)$. We can perturb this permutation by considering intervals of size $m\delta$ for some small δ. In each interval of integers $[k\lfloor m\delta \rfloor + 1, (k+1)\lfloor m\delta \rfloor)$ we choose a permutation that maps this interval onto $(m+1-(k+1)\lfloor m\delta \rfloor, m+1-k\lfloor m\delta \rfloor + 1]$ we assemble each of these smaller permutations into a large permutation π then by the rearrangement inequality on each interval of length $\lfloor m\delta \rfloor$, for each of these permutations we have

$$\sum_{s=1}^{m} s\pi(s) = \frac{m^3 + 3m^2 + m + m^3\delta^2 + O(m^3\delta^3)}{6}.$$

If we choose δ so that the right hand side is $\leq n$ then we have constructed

$$(\lfloor m\delta \rfloor !)^{\lfloor \frac{1}{\delta} \rfloor} \sim (2\pi m\delta)^{\frac{1}{2\delta}} \left(\frac{m\delta}{e}\right)^{m}$$

permutations satisfying our desired bound.

The choice of $m = \lfloor (6n\,(1+\delta^2))^{-\frac{1}{3}} \rfloor$ and $\delta = \log(n)^{-\frac{1}{2}}$ results in

$$\begin{aligned}
&\exp\left(m\log(m\delta) - m + O\left(\log(n)^2\right)\right)\\
&\quad= \exp\left[(6n)^{\frac{1}{3}}\left(1-\frac{1}{3}\log(n)^{-1}+O\left(\log(n)^{-2}\right)\right)\left(\frac{1}{3}\log(6n)-\frac{1}{2}\log(\log(n))+O\left(\log(n)^{-1}\right)\right)\right.\\
&\qquad\qquad\left.-(6n)^{\frac{1}{3}}+O\left(n^{\frac{1}{3}}\log(n)^{-1}\right)\right]\\
&\quad= \exp\left(\frac{1}{3}(6n)^{\frac{1}{3}}\left(\log(n)-\frac{3}{2}\log\log(n)+\left(\log(6)-\frac{4}{3}\right)+O\left(\frac{\log\log(n)}{\log(n)}\right)\right)\right)
\end{aligned}$$

permutations of S_m satisfying the desired inequality. □

This proposition establishes the asymptotic lower bound we desire in the proof of Theorem 1.1. To prove the upper bound let Π be the permutation of $\{1, 2, \cdots, m\}$ that sends k to $m+1-k$. Then for any $\pi \in S_m$ we have

$$\sum_k k\pi(k) = \frac{m(m+1)(m+2)}{6} + \frac{1}{2}D(\Pi,\pi)^2 \tag{3.2}$$

where

$$D(\Pi,\pi)^2 := \sum_{k=1}^{m}\left(\Pi(k)-\pi(k)\right)^2$$

is the *discrepancy* between Π and π. The following result readily yields the desired upper bound for the best possible choice of m.

Proposition 3.6. *Let $\epsilon, \delta > 0$ and $m = \lfloor (6n)^{\frac{1}{3}}(1-\epsilon)\rfloor$ and $\frac{m^3}{6}(1+\delta^2) \le n$. Assume that $\epsilon, \delta = O(1)$ as $n \to \infty$. The number of permutations $\pi \in S_m$ with $D(\Pi,\pi)^2 \le 2m^3\delta^2$ is less than or equal to*

$$\exp\left(\frac{1}{3}(6n)^{\frac{1}{3}}\left(\log(n)-\frac{3}{2}\log\log(n)+O\left(n^{\frac{1}{3}}\right)\right)\right)$$

To prove this proposition we will need the following estimate for the number of lattice points in $\mathbb{Z}^m$ in a ball of radius $(\alpha m)^{\frac{3}{2}}$. For this radius size the asymptotic count is expected to be close to the volume of the ball in The following result is stated without proof in the work of Mazo and Odlyzko [5].

Lemma 3.7. *The number of $(x_1, \cdots, x_m) \in \mathbb{Z}^m$ satisfying $x_1^2 + \cdots + x_m^2 \le (\alpha m)^3$ is asymptotic to*

$$\frac{(\pi(\alpha m)^3)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}\left(1+\exp\left(O(m)\right)\right)$$

as $m \to \infty$.

Proof of Proposition 3.6. For any vector of numbers $\mathbf{x} = (x_1, \cdots, x_m)$ we let $D\left(\Pi, \mathbf{x}\right)^2 = \sum_{k=1}^{m}(\Pi(k)-x_k)^2$. Then the number of permutations in $\pi \in S_m$ with $D(\Pi,\pi)^2 \le 2m^3\delta$ is less than

$$\sum_{\mathbf{x}=(x_1,\cdots,x_m)\in\{1,2,\cdots,m\}^m} \mathbf{1}_{D(\Pi,\mathbf{x})^2\le 2m^3\delta^2} \le \sum_{\mathbf{x}=(x_1,\cdots,x_m)\in\mathbb{Z}^m} \mathbf{1}_{D(\Pi,\mathbf{x})^2\le 2m^3\delta^2}$$
$$=\frac{(\sqrt{2\pi}\delta m^{\frac{3}{2}})^m}{\Gamma\left(\frac{m}{2}+1\right)}(1+\exp(O(m)))$$
$$=\delta^m m^m \sqrt{4\pi e}^m (1+\exp(O(m)))$$
$$=\exp\left(m\log(m)+m\log(\delta)+O(m)\right),$$

where the first equality follows from Lemma 3.7. Substituting our value of m we see that the term inside the exponential is

$$\frac{(6n)^{\frac{1}{3}}}{3}\log(n) + (6n)^{\frac{1}{3}}\left(-\frac{\epsilon}{3}\log(n)+\log(\delta)\right) + O\left(n^{\frac{1}{3}}\right).$$

We want to maximize this bound with respect to the constraint $1+6\delta \le \frac{1}{(1-\epsilon)^3}$ we see that $\delta = \sqrt{\epsilon}(1+O(\epsilon))$. Therefore, to maximize the upper bound we take $\epsilon = \frac{1}{\log(n)}(1+o(1))$ and we obtain the result. □

Proof of Theorem 1.1. By Lemmas 3.3 and 3.4 it is enough to establish an asymptotic for the $m < (6n)^{\frac{1}{3}}$ with the maximum number of $\pi \in S_m$ satisfying $\sum_{k=1}^m k\pi(k) \le n$. By Proposition 3.5 we see that taking $m = \lfloor (6n)^{\frac{1}{3}}\left(1-\frac{1}{\log(n)}\right)^{\frac{1}{3}}\rfloor$ gives as least

$$\exp\left(\frac{(6n)^{\frac{1}{3}}}{3}\left(\log(n) - \frac{3}{2}\log\left(\log(n)\right)+O(1)\right)\right)$$

such permutations.

Let m be optimal such that there is a maximum number of $\pi \in S_m$ such that $\sum k\pi(k) \le n$. By the remark after the proof of Lemma 3.4 we know that $(4n)^{\frac{1}{3}} \le m \le (6n)^{\frac{1}{3}}$, so we write $m = \lfloor (6n)^{\frac{1}{3}}(1-\epsilon)\rfloor$ with $\epsilon = O(1)$. Then we have a maximum number of π satisfying

$$\frac{1}{2}D(\Pi,\pi)^2 \le \left(n - \frac{m(m+1)(m+2)}{6}\right) = O\left(m^3\epsilon^2\right).$$

Applying Proposition 3.6 we obtain an upper bound of the correct order of magnitude. □

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