ON RAMANUJAN’S DEFINITION OF “MOCK THETA FUNCTION”

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Abstract. In his famous deathbed letter, Ramanujan “defined” the notion of a mock theta function, and offered some examples of functions he believed satisfied this definition. Very recently, Michael Griffin, Ken Ono, and Larry Rolen established for the first time that Ramanujan’s mock theta functions actually satisfy his own definition. On the other hand, Sander Zwegers’s 2001 Ph.D. thesis showed that all of Ramanujan’s examples fit nicely into the theory of harmonic Maass forms. This has lead to an alternate definition of mock theta function. In this note we prove that Ramanujan’s definition of mock theta function is not equivalent to the modern definition.

1. RAMANUJAN’S DEFINITION OF A MOCK THETA FUNCTION

In his famous “deathbed” letter [23], Ramanujan introduced the notion of a mock theta function. Following Andrews and Hickerson [5] and Zwegers [26] we give the following version of Ramanujan’s definition.

Ramanujan’s Definition. A mock theta function is a function \( f \) of the complex variable \( q \), defined by a \( q \)-series of a particular type (Ramanujan calls this the Eulerian form), which converges for \( |q| < 1 \) and satisfies the following conditions:

1. infinitely many roots of unity are exponential singularities,
2. for every root of unity \( \xi \) there is a theta function \( \vartheta_\xi(q) \) such that the difference \( f(q) - \vartheta_\xi(q) \) is bounded as \( q \to \xi \) radially,
3. \( f \) is not the sum of two functions, one of which is a theta function and the other a function which is bounded radially toward all roots of unity.

Remark. Ramanujan’s notion of a theta function is: sums, products, and quotients of series of the form \( \sum_{n \in \mathbb{Z}} \epsilon^n q^{an^2 + bn} \) with \( a, b \in \mathbb{Q} \) and \( \epsilon = \pm 1 \). Up to a power of \( q \), these are modular forms.

Ramanujan gave 17 examples of functions he believed satisfied these properties. The most famous is

\[
 f(q) := 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \frac{q^9}{(1 + q)^2(1 + q^2)^2(1 + q^3)^2} + \cdots
\]

Clearly \( \lim_{q \to \xi} f(q) = O(1) \) when \( \xi \) is an odd order root of unity. In his letter, Ramanujan claimed that at all primitive, even \( 2k \) roots of unity, say \( \xi \),

\[
 \lim_{q \to \xi} \left( f(q) - (-1)^k b(q) \right) = O(1)
\]

where \( b(q) = \frac{(-q)_\infty}{(q)_\infty} \) is a theta-function. Watson [23] proved this claim. Very recently, [20] Griffin, Ono, and Rolen proved that there is no modular form \( M \) such that \( f(q) - M(q) \) is bounded radially toward all roots of unity. Until the results of that paper were announced, it was not known that any of Ramanujan’s mock theta functions satisfied his own definition.

Despite the lack of a definition, Ramanujan’s mock theta functions were shown to possess many striking properties. For example, Ramanujan himself related certain sums of mock theta functions to modular forms. As an example, he claimed that

\[
 2\phi(-q) - f(q) = b(q)
\]
where \( f(q) \) and \( b(q) \) are as above, and \( \phi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^n)(1+q^{2n})} \) is one of his third order mock theta functions. Identities of this flavor are often referred to as “mock theta conjectures” (see the survey of Gordon and McIntosh [19] or the work of Andrews and Garvan [4]). Many examples of such identities proved themselves very difficult to establish. The most significant were proven in Hickerson’s works [21, 22].

Other striking properties are the Hecke-type series found by Andrews [2]. As an example, consider Ramanujan’s fifth order mock theta function

\[
f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)}.
\]

Andrews proved that

\[
f_0(q) = \prod_{n=1}^{\infty} (1-q^n)^{1/2} \sum_{n=0}^{\infty} (-1)^n q^{2n^2} (1-q^{2n+1}).
\]

Notice the resemblance to the series for

\[
\prod_{n=1}^{\infty} (1-q^n)^{2} (1-q^{2n-1}) = \sum_{n=0}^{\infty} \sum_{|j| \leq n} (-1)^n q^{\frac{1}{2} n (3n+1) - j^2} (1-q^{2n+1}),
\]

see (5.15) of [2], for example.

These hints of structure and many others led to Dyson’s [16] statement in 1987:

Somehow it should be possible to build them (the mock theta functions) into a coherent group-theoretic structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future.

This group-theoretic structure was discovered by Sander Zwegers. Zwegers’s 2001 Ph.D. thesis [27] was a break through in the study of the mock theta functions. As a result of his thesis it is known that all of Ramanujan’s examples are essentially the holomorphic part of weight 1/2 weak harmonic Maass forms whose nonholomorphic parts are period integrals of weight 3/2 unary theta functions.

This has led to a huge number of results which are out of reach otherwise. Perhaps the most astonishing break throughs are congruence properties and exact formulas for the coefficients of the mock theta functions (see the works of Bringmann and Ono [12, 13]). These results have also lead to simpler and more theoretical proofs of some older results, such as the mock theta conjectures, see Folsom’s work [17].

Zwegers’s construction has given us a new definition of “mock theta function”. However, that definition has seemingly nothing to do with Ramanujan’s definition. In this note, we explain how these definitions cannot be equivalent. We also raise a number of questions that will hopefully lead to some reconciliation of the two definitions.

2. The Modern Definition of Mock Theta Function

Following Zagier [25, 15], we offer the following definition of a mock theta function.

**Modern Definition.** A mock theta function is a \( q \)-series \( H(q) = \sum_{n=0}^{\infty} a_n q^n \) such that there exists a rational number \( \lambda \in \mathbb{Q} \) and a unary theta function \( g(z) = \sum_{n \in \mathbb{Q}^+} b_n z^n \) of weight \( k \), such that \( h(z) = q^\lambda H(q) + g^*(z) \) is a non-holomorphic modular form of weight \( 2 - k \), where

\[
g^*(z) = c_k \sum_{n \in \mathbb{Q}^+} n^{2-k} b_n \Gamma(k-1, 4\pi ny) q^{-n}
\]
with \( \Gamma(w, t) = \int_t^\infty u^{w-1}e^{-u}du \), the incomplete Gamma function and \( c_k \) a constant that depends only on \( k \). The function \( g \) is called the "shadow."

As remarked in [15], the condition that the shadow be a unary theta function forces the weight to be either 1/2 or 3/2. All of Ramanujan’s examples have \( k = 3/2 \).

There are two particularly elegant examples of mock theta functions with shadow proportional to weight 1/2 unary theta functions. The work of Bringmann and Lovejoy [10, 11] studies the series

\[
\bar{f}(q) = 1 + 2 \sum_{n=1}^\infty \frac{q^{n(n+1)}}{(1+q)\cdots(1+q^{n-1})(1+q^n)^2}.
\]

The coefficients of this series are related to the rank of an overpartition. They prove that this is a mock theta function with shadow proportional to the unary theta function \( \Theta(q) = \sum_{n \in \mathbb{Z}} q^n \). This series is particularly interesting because of its relation with the class number generating function of Zagier [24] which is also a mock theta function with the same shadow.

The second elegant example of a mock theta function with shadow proportional to a weight 1/2 unary theta functions is the generating functions for the number of smallest parts of a partition, denoted spt. The value spt\((n)\) is defined as the number of appearances of the smallest parts in the partitions of \( n \). Andrews [3] showed that

\[
S(q) := \sum_{n=0}^\infty \text{spt}(n)q^n = \sum_{n=1}^\infty \frac{q^n}{(1-q^n)^2(1-q^{n+1})(1-q^{n+2})}\cdots.
\]

Moreover, it was proved by Bringmann [7] that \( S(q) \) is essentially a mock modular form with shadow

\[
\eta(q) = q^{12} \prod_{n=1}^\infty (1-q^n) = \sum_{n \in \mathbb{Z}} \left( \frac{12}{n} \right) q^{\frac{n^2}{24}},
\]

where \( (\cdot) \) is the Kronecker symbol.

**Theorem 2.1.** Ramanujan’s definition of a mock theta function is not equivalent to the modern definition of a mock theta function.

**Proof.** We introduce the two functions

\[
V_1(q) := \frac{1}{(q)_\infty} \left( \frac{1}{12} - \sum_{n \neq 0} \frac{q^n}{(1-q^n)^2} \left( 3 + (-1)^n q^{3n^2-n} (1+q^n) \right) \right),
\]

\[
V_2(q) := \frac{1}{(q)_\infty} \left( \frac{1}{12} - 2 \sum_{n=1}^\infty \frac{nq^n}{1-q^n} \left( 1 + (-1)^n q^{1/2 n(n-1)} \right) \right),
\]

where \( (q)_\infty = \prod_{n=1}^\infty (1-q^n) \).

**Remark.** In the notation of [6], \( V_1(q) = (q)_\infty^2 v_1(q) \) and \( V_2(q) = -(q)_\infty^2 (v_2(q) + v_3(q)) \).

We will show that either \( V_1(q) \) is a mock theta function according to the modern definition, but not Ramanujan’s definition or \( V_2(q) \) is a mock theta function according to Ramanujan’s definition, but not according to the modern definition.

First, it follows from Theorem 1.2 (1) of [6] that \( V_1(q) \) is a mock theta functions according to the modern definition with shadow proportional to \( \eta(q) \).

Assume that \( V_1(q) \) is mock theta function according to Ramanujan’s definition as well. In this case, we will prove that \( V_2(q) \) is a mock theta function according to Ramanujan’s definition, but that it is
not a mock theta function according to the modern definition. It was proved by Andrews [1] that
\[ V_1(q) - V_2(q) = \sum_{n=0}^{\infty} q^{n+1}(1 - q)^2(1 - q^2)^2 \cdots (1 - q^n)^2. \]

The right hand side of this identity is \( O(1) \) for all roots of unity \( q \), because the sum is finite. Therefore, \( V_1 \) and \( V_2 \) have all of the same singularities at roots of unity. Since \( V_1(q) \) is a mock theta function according to Ramanujan’s definition for every root of unity \( \xi \), there exists a theta function \( \vartheta_{\xi} \) such that \( V_1(q) - \vartheta_{\xi}(q) \) is bounded as \( q \to \xi \) radially. Therefore, \( V_2(q) - \vartheta_{\xi}(q) \) is also bound as \( q \to \xi \) radially. Moreover, since the exponential singularities of \( V_1(q) \) and \( V_2(q) \) are identical and \( V_1(q) \) was assumed to be a mock theta function according to Ramanujan, it follows that there is no single theta function which ‘cuts out’ all of the singularities of \( V_2(q) \).

To complete the proof it suffices to prove that \( V_2(q) \) is not a mock theta function according to the modern definition. However, by Theorem 1.2 of [6], \( \frac{1}{(q)_2} V_2(q) \) is a mock theta function with shadow proportional to \( \eta^3(q) \). Therefore, \( V_2(q) \) is a mixed-mock modular form, that is, it is the product of a mock theta function and a modular form. Since the product of two harmonic Maass forms is only a harmonic Maass form when both forms being multiplied are actually modular forms or one of them is a constant, it is clear that \( V_2(q) \) is not the holomorphic part of a harmonic Maass form and thus not a mock theta function according to the modern definition. \( \square \)

3. Questions and Remarks

The work of Griffin, Ono, and Rolen [20] as well as this text leaves many questions to be answered.

3.1. The example in the proof of our main theorem are mixed-mock theta functions. Are all mixed-mock theta functions actually mock theta functions according to Ramanujan’s definition?

3.2. Does there exist a weight 1/2 counter example to the equivalence of these objects? Our example has weight 3/2.

3.3. Is \( V_1(q) \), given above, a mock theta function according to Ramanujan’s definition? The results of [20] show that there is no modular form \( M \) such that \( V_1(q) - M(q) \) is bounded radially toward all roots of unity. However, it is not clear that there is a collection of modular forms \( \{ \vartheta_{\xi}(q) \} \) such that for each root of unity \( \xi \), we have \( V_1(q) - \vartheta_{\xi}(q) \) is bounded as \( q \to \xi \) radially.

3.4. Let \( F \) be a mock theta function according to Ramanujan’s definition. Let \( S(F) = \{ \vartheta_{\xi} \} \) be the set of non-zero theta functions from condition (1) of Ramanujan’s definition. Define the Gordon-McIntosh [19] universal mock theta functions by
\[
(3.1) \quad g_3(w; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(w; q)_{n+1}(w^{-1}; q)_{n+1}} \quad \text{and} \quad g_2(w; q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)}}{(w; q)_{n+1}(w^{-1}; q)_{n+1}}
\]

where \( (x; q)_n := \prod_{j=0}^{n-1} (1 - xq^j) \).

Since \( g_3(w; q) = -\frac{1}{w} + \frac{1}{w(1-w)} R(w; q) \) (see Theorem 3.1 of [8], for example) where \( R(w; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(w^q q)_n(w^{-1}; q)_n} \). Theorem 1.2 of [18] and its proof show that for \( \zeta \) a root of unity the set \( S(g_3(\zeta q^a; q^A)) \) is finite. The same should be true for \( w = \zeta q^a \) where \( \zeta \) is a root of unity and \( a \) is rational. For \( \zeta \) a root of unity and \( B \) and \( A \) integers, prove that \( S(g_3(\zeta q^B; q^A)) \) and \( S(g_2(\zeta q^B; q^A)) \) are finite. What are these sets?

Remark. It should be noted that for some the theta functions occasionally need to be modified by a power of \( q \) to get the asymptotics to match. See the remark after Theorem 1.2 of [18].
3.5. There exists a modular form \( M(q) \) such that

\[
f_0(q) + 2\psi(q) = M(q)
\]

where \( \psi(q) = \sum_{n=0}^{\infty} q^{(n+1)(n+2)}(1 + q) \cdots (1 + q^n) \) is one of Ramanujan’s fifth order mock theta functions (see page 104 of [19]). Moreover, \( f_0(q) \) and \( 2\psi(q) \) have singularities at disjoint sets of roots of unity. As a result, we see that, in the notation above, the set \( \mathcal{S}(f_0) \) contains only one non-trivial modular form. This appears to be typical of the fifth order mock theta functions of Ramanujan.

References