SOFT ASYMPTOTICS FOR GENERALIZED SPT-FUNCTIONS

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Abstract. We derive a leading order asymptotic for the difference between the second moments of Garvan’s generalized rank statistics. The approach uses Dixit and Yee’s combinatorial interpretation of the difference and soft probabilistic techniques to deal with random partitions.

1. Introduction

Andrews [2] introduced the study of the spt-function. For each partition $\lambda$ of $n$ let $w_{\text{spt}}(\lambda)$ be the number of occurrences of the smallest part in $\lambda$ and

$$\text{spt}(n) = \sum_{\lambda} w_{\text{spt}}(\lambda)$$

where the sum is over all partitions of $n$. Andrews proved that

$$\text{spt}(n) = \frac{1}{2} (M_2(n) - N_2(n))$$

where $M_2$ is the second moment of the crank statistic and $N_2$ is the second moment of the rank statistic. Later, Ji [17] gave a combinatorial proof. This yields a striking relation between the difference of the moments and proves that it is always positive. Garvan [15] later generalized the spt-function to prove that

$$M_{2\ell}(n) - N_{2\ell}(n) > 0$$

for each $n > 0$ and $\ell > 0$ where $N_{2\ell}$ and $M_{2\ell}$ are the $2\ell$th moment of the rank and crank statistic, respectively. The study of rank and crank moments, initiated by Andrews [1], has attracted a lot of further study. See, for instance, the works of the Bringmann, Mahlburg, and the author [7] for detailed discussion of the asymptotics for the moments.

The rank and crank were generalized by Garvan [13] to a family of partition statistics called the generalized ranks. For each $j \in \mathbb{N}$ Garvan defined a statistic $\text{rank}_j(\lambda)$, referred to as the $j$-rank. $\text{rank}_1$ is the crank statistic and $\text{rank}_2$ is the rank statistic. The moments of the $j$-rank statistic are defined by

$$jN_{2k}(n) := \sum_{\lambda} (\text{rank}_j(\lambda))^{2k}$$

where the sum is over all partitions of $n$. The odd moments, by symmetry, are 0. Recently Dixit and Yee [8] generalized Garvan’s higher order spt construction to prove the inequality

$$jN_{2k}(n) > j+1N_{2k}(n)$$

for all $j \geq 1$, $k > 0$, and $n \geq 1$.

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Following the works of Andrews [2] and Garvan [15], Dixit and Yee [8] provide an elegant combinatorial interpretation of the value

$$S_{\text{pt}}^j(n) := \frac{1}{2} (M_2(n) - j_{+1}N_2(n))$$

called the generalized spt-functions. It is striking that modular techniques can be used to provide nearly exact formulas for the $S_{\text{pt}}^j(n)$ and in fact for any $jN_2k(n)$ as $n \to \infty$. Such results exploit the “mock” Jacobi form structure of the $j$-rank generating functions and circle method techniques developed by Bringmann and Mahlburg [5, 6]. This program has been carried out by Waldherr [19].

While modular techniques provide very strong theorems, they can become unwieldy for large $j$ (see [18]) and ignore the combinatorial interpretation of the statistics being considered. The purpose of this note is to use soft combinatorial and probabilistic tools to provide leading order asymptotics for $S_{\text{pt}}^j(n)$ for each $j$. This approach has the advantage that it uses the combinatorial description of this functions and exploits the typical shape of a partition. For instance, the proof gives some insight into why $S_{\text{pt}}^{j+1}(n) > S_{\text{pt}}^j(n)$ or equivalently why

$$jN_2(n) > j_{+1}N_2(n)$$

for each $n$.

**Theorem 1.1.** For each $j > 0$, as $n \to \infty$

$$S_{\text{pt}}^j(n) = \frac{j}{2\pi \sqrt{2n}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \left( 1 + o(j) \right).$$

The argument presented here gives $S_{\text{pt}}^j(n) \sim j\text{pt}(n)$. Moreover, our proof shows that the contribution to $S_{\text{pt}}^j(n)$ from each partition of $n$ is almost always equal to $j$ times the number of 1s in $\lambda$. Unwinding definitions, this tells us that $jN_2(n) > j_{+1}N_2(n)$ for each $j$ and large enough $n$ because the number of 1s is almost surely larger than $j$.

Following Dixit and Yee [8], to interpret $S_{\text{pt}}^j(n)$ we associated to each $\lambda$ a partition of $n$ a weight $W_j(\lambda)$. The weight counts the multiplicity numbers of the parts that appear in the lower Durfee squares of the partition. See Section 2.1 for details. As a result, the weight is determined by the smallest parts of the partition. In particular, if $\lambda$ has at least $j$ parts of size 1, then the weight is determined by the number of 1s. Fristedt’s results [12] or classical q-series results and Tauberian theorems give that the probability that a partition of size $n$ has less than $j$ parts of size 1 is asymptotic to $\frac{j^2}{\sqrt{6n}}$. Our proof is devoted to showing that the contribution from such partitions is negligible.

The following result for the second generalized rank moments is straightforward from Theorem 1.1.

**Corollary 1.2.** For each $j \geq 0$, as $n \to \infty$

$$j_{+1}N_2(n) = \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \left( \frac{1}{2\sqrt{3}} - \frac{2j + 1}{2\pi \sqrt{2n}} + o(j) \right).$$

In particular, Dyson proved $M_2(n) = 1N_2(n) = 2np(n)$ where $p(n)$ is the number of partitions of $n$. Therefore, Rademacher’s formula for $p(n)$ gives an exact formula for $M_2(n)$. Consequentially, it gives the above corollary for $j = 0$.

It is an interesting problem to use Garvan’s combinatorial interpretation of the higher order Spt functions and techniques from random partitions to deduce asymptotics for differences of other
rank moments. It seems this approach would lead to identities for special values of the Riemann zeta-function.

In Section 2 we give background about the combinatorial description of the Spt-functions. We also give some basic results about random partitions. In Section 3 we give the proof of the main theorem.

2. Probability and Combinatorics Background

In this section we give some combinatorial background on the generalized ranks and probabilistic background on random partitions. Throughout we let \( X_j(\lambda) \) be the number of times \( j \) occurs as a part in the partition \( \lambda \).

2.1. Combinatorics Background. We briefly recall the definition of \( \text{Spt}_j(n) = \sum_{\lambda} W_j(\lambda) \) where the sum is over all partitions of \( n \). To define the weight \( W_j \) Dixit and Yee used the lower-Durfee squares. The largest square that fits inside the Ferrers diagram of \( \lambda \) starting from the lower left corner is the lower-Durfee square. The second lower-Durfee square is the largest square that fits inside of \( \lambda \) right above the first lower-Durfee square. Successive lower-Durfee squares are defined similarly.

We mark each part of a partition with a ‘multiplicity number’. For example, if \( \lambda = (7, 6, 5, 5, 2, 1, 1, 1, 1) \) then the marked version of \( \lambda \) is

\[
(7_1, 6_1, 5_1, 5_2, 2_1, 1_1, 1_2, 1_3, 1_4).
\]

That is, if a positive integer occurs as a part in \( \lambda \), we mark all of its occurrences with positive integers in an increasing order from the left to the right.

If \( \lambda \) is a partition, we define \( W_j(\lambda) \) as the the sum of the marks of each part in the first \((j - 1)\) successive lower-Durfee squares and the mark of the part directly above the \((j - 1)\)st lower-Durfee square. If \( \lambda \) has fewer than \( j - 1 \) successive lower-Durfee squares, \( W_j(\lambda) \) is defined to be the sum over all of the marks in the partition.

For example, when \( \lambda = (7, 6, 5, 5, 2, 1, 1, 1, 1) \) we have \( W_1(\lambda) = 4, W_2(\lambda) = 4 + 3 = 7, W_3(\lambda) = 4 + 3 + 2 = 9, W_4(\lambda) = 4 + 3 + 2 + 1 = 10, \) and \( W_5(\lambda) = 4 + 3 + 2 + 1 + 1 = 11 \). Also, \( W_6(\lambda) = 4 + 3 + 2 + 1 + (1 + 2) + 1 = 14 \), since the first four lower-Durfee squares are of size 1 and the fifth lower-Durfee square is \( 2 \times 2 \) and contains the part of size 2 and the second part of size 5. Likewise, \( W_7(\lambda) = 4 + 3 + 2 + 1 + (1 + 2) + (1 + 1 + 1) = 16 \) and \( W_8(\lambda) = 16 \) for all \( j > 7 \) because there are exactly six lower-Durfee squares.

**Lemma 2.1.** Let \( X_j(\lambda) \) be the number of parts of size \( j \) in the partition \( \lambda \). Fix \( j \geq 1 \). If \( X_1(\lambda) \geq j \) then \( W_j(\lambda) = jX_1(\lambda) - \frac{j(j-1)}{2} \). If \( X_1(\lambda) = j - r \) with \( r \in \{1, \ldots, j\} \) and \( X_2(\lambda) \geq 2r \), then

\[
W_j(\lambda) = 2rX_2(\lambda) + \frac{j(j+1) - r(r-1) - 2jr}{2}.
\]

**Proof.** Assume that \( X_1(\lambda) \geq j \). Then the first \((j - 1)\) lower-Durfee squares are of size 1. They contain the parts of size one with marks \( X_1(\lambda), X_1(\lambda) - 1, \ldots, X_1(\lambda) - (j - 2) \). Moreover, the first part after the \((j - 1)\)st lower-Durfee square is of size 1 and has mark \( X_1(\lambda) - (j - 1) \). This gives the first claim. The second claim follows in a similar way, by guaranteeing that the first \((j - 1)\) successive lower-Durfee squares are all of size 1 or 2. Additionally, we have

\[
W_j(\lambda) = X_1(\lambda) + \cdots + (X_1(\lambda) - (j - r - 1))
+ [(X_2(\lambda) + (X_2(\lambda) - 1)) + \cdots ((X_2(\lambda) - (2r - 2)) + (X_2(\lambda) - (2r - 1)))]
+ (X_2(\lambda) - (2r))
\]
2.2. Probability Background. In this subsection we give some results dealing with statistics of random partitions of \( n \) for large \( n \).

Recall the generating function for the number of partitions of \( n \) is

\[
P(q) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \sum_{n=0}^{\infty} p(n)q^n.
\]

Moreover, the modularity of the generating function gives \( P(e^{-\epsilon}) \sim \sqrt{\frac{\pi}{2\epsilon}} e^{\frac{\pi^2}{8\epsilon}} \) and

\[
p(n) = \frac{1}{4\sqrt{3n}} \exp \left( \frac{\pi}{3} \left( \frac{2n}{3} \right) \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \right).
\]

We will make use of the following Tauberian theorem of Ingham [16]. In the version given here it can be found in [4].

**Theorem 2.2** (Ingham). Let \( f(z) = \sum_{n=0}^{\infty} a(n)z^n \) be a power series with real nonnegative coefficients and radius of convergence equal to 1. If there exists \( A > 0, \lambda, \alpha \in \mathbb{R} \) such that

\[f(z) \sim \lambda(-\log(z))^{\alpha} \exp \left( -\frac{A}{\log(z)} \right)\]
as \( z \to 1^- \), then

\[
\sum_{n=0}^{n} a(m) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2} - \frac{1}{4}}}{n^{\frac{\alpha}{2} + \frac{1}{4}}} \exp \left( 2\sqrt{An} \right)
\]
as \( n \to \infty \).

Fristedt [12] proved that for large \( n \) a typical partition of \( n \) has about \( \sqrt{\frac{6n}{\pi}} \) partitions. In fact, he proved a distributional result for the number of parts of any small size.

**Theorem 2.3** (Fristedt). Let \( P_n \) be the uniform measure on partitions of size \( n \). We have

\[
P_n \{-\frac{\pi}{\sqrt{6n}} X_1(\lambda) \leq y\} \to 1 - e^{-y}
\]
as \( n \to \infty \) uniformly in \( y \).

We use a \( q \)-series approach to establish the following result on the number of partitions with a constrained number of 1s or 2s. Let \( f_a(n) \) be the number of partitions of \( n \) that contain \( < a \) parts of size 1. So that \( F_a(q) = \sum_{n=0}^{\infty} f_a(n)q^n = (1 - q^a)P(q) \). Similarly let \( f_{a,b}(n) \) be the number of partitions of \( n \) with \( < a \) 1s and \( < b \) 2s. Then \( \sum_{n=0}^{\infty} f_{a,b}(n)q^n = (1- q^a)(1 - q^{2b})P(q) \). Straightforward applications of Theorem 2.2 with \( (1 - q) \sum_{n=0}^{\infty} f_{a,b}(n)q^n \) and \( (1 - q^a) \sum_{n=0}^{\infty} f_{a,b}(n)q^n \) give the following.

**Proposition 2.4.** For any \( a, b \geq 0 \), as \( n \to \infty \)

\[
\frac{f_a(n)}{p(n)} \sim \frac{\pi a}{\sqrt{6n}} \quad \text{and} \quad \frac{f_{a,b}(n)}{p(n)} \sim \frac{abn^2}{n}
\]

We give the following result which follows easily from Fristedt’s results on random partitions.

**Proposition 2.5.** Let \( \mathcal{F}(n) \) be the the set of partitions of \( n \) with any fixed number of 1s or the set of partitions with any fixed number of 1s or 2s. Then, as \( n \to \infty \)

\[
\frac{1}{|\mathcal{F}(n)|} \sum_{\lambda \in \mathcal{F}(n)} \left( \sum_{j} X_j(\lambda) \right)^{\ell} = O \left( (\sqrt{n}\log(n))^\ell \right)
\]
for \( \ell = 1, 2 \). Moreover, if \( \mathcal{F}(n) \) is the set of partitions of \( n \) with a fixed number of 1s and at least \( b \) 2s, then

\[
\frac{1}{|\mathcal{F}(n)|} \sum_{\lambda \in \mathcal{F}(n)} X_2(\lambda) = O\left(\sqrt{n}\right).
\]

**Proof.** The proof of each result is the same. Fristedt’s results prove that the number of 1s or 2s in a partition is independent from the number of parts of size \( k \) for all \( k = o\left(n^{1/4}\right) \). Therefore, the final claim follows immediately and from Fristedt’s result that as \( n \to \infty \)

\[
P_n\left\{\frac{2\pi}{\sqrt{6n}} X_2 \leq v\right\} \to 1 - e^{-v}.
\]

The proof of the claim for \( \sum_j X_j(\lambda) \) is completely analogous to the proof of Fristedt’s result proving

\[
P_n\left\{\frac{\pi}{\sqrt{6n}} \sum_j X_j - \log\left(\frac{\sqrt{6n}}{\pi}\right) \right\} \to e^{-e^{-v}}
\]

or one could follow Erdös-Lehner’s [11] original proof of this result. In particular, considering partitions without 1s or 2s does not effect the expected size of \( \sum_j X_j(\lambda) \).

\( \square \)

### 3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Throughout this section all sums on \( \lambda \) range over the partitions of \( n \), for some fixed \( n \).

We begin with the proof in the case of \( j = 1 \). Notice that if \( X_1(\lambda) > 0 \) then \( w_{\text{spt}}(\lambda) = X_1(\lambda) \). Therefore,

\[
\sum_{\lambda} w_{\text{spt}}(\lambda) = \sum_{\lambda} X_1(\lambda) + O\left(\sum_{\lambda : X_1(\lambda) = 0} \sum_j X_j(\lambda)\right)
\]

\[
= p(n) \frac{\sqrt{6n}}{\pi} (1 + o(1)) + O\left(\frac{p(n)}{\sqrt{n}} \frac{\sqrt{n} \log(n)}{n}\right)
\]

\[
= \frac{1}{2\pi \sqrt{2n}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right) (1 + o(1)).
\]

where the second equality follows Theorem 2.3 and Propositions 2.4 and 2.5 and the final equality follows from (2.2).

To handle the case of \( \text{Spt}_j(n) \) with \( j > 1 \) we apply Lemma 2.1 and note that

\[
W_j(\lambda) \leq \sum_i \sum_{k=1}^{X_i(\lambda)} i = \sum_i \frac{1}{2} X_i(\lambda)(X_i(\lambda) - 1).
\]
Thus applying Lemma 2.1 there are constants $c_{j,b}$ such that

\[
\sum_{\lambda} W_j(\lambda) = \sum_{\lambda \in \mathcal{F}_{a,b}(n)} \left( jX_1(\lambda) - \frac{(j-1)j}{2} \right)
\]

\[+ \sum_{\lambda \in \mathcal{F}_{a,b}(n)} \sum_{b \in \{1, \ldots, j\}} \sum_{\lambda \in \mathcal{F}_{a,b}(n)} \left( 2bX_2(\lambda) + c_{j,b} \right) + O \left( \sum_{a,b} \sum_{\lambda \in \mathcal{F}_{a,b}(n)} \sum_{i} X_i(\lambda)^2 \right).\]

where $\mathcal{F}_{a,b}(n)$ is the set of all partitions of $n$ with exactly $a$ 1s and $b$ 2s and the sum on $a$, $b$ is over those pairs satisfying $a < j$ and $b < 2a + 1$. Using Theorem 2.3 and Propositions 2.4 and 2.5 gives

\[
\sum_{\lambda} W_j(\lambda) = j \sqrt{\frac{6n}{\pi}} p(n)(1 + o(1)) + O \left( \frac{p(n)}{\sqrt{n}} \cdot \sqrt{n} \right) + O \left( \frac{p(n)}{n} \cdot n \log^2(n) \right).
\]

An application of (2.2) gives the result.

References