Relatively open and closed sets

Let \( U \subset \mathbb{R}^n \) be any set and let \( V \subset U \). Recall that \( V \) is relatively open in \( U \) if for each \( v \in V \) there exists \( \epsilon > 0 \) such that \( B_\epsilon(v) \cap U \subset V \). We say that \( V \) is relatively closed in \( U \) if for any sequence \( \{v_n\} \subset V \) with \( \lim v_n = u \in U \) we have \( u \in V \).

1.) Let \( U = [0, 1) \subset \mathbb{R} \). Prove that \( V = [0, \frac{1}{2}) \) is relatively open in \( U \) and \( V' = [\frac{1}{2}, 1) \) is relatively closed in \( U \).

2.) Let \( V \subset U \). Prove that \( V \) is relatively open in \( U \) if and only if \( U \setminus V \) is relatively closed in \( U \). The set \( U \setminus V \) is the relative complement of \( V \) in \( U \).

Suppose \( f : U \rightarrow V \) with \( U \subset \mathbb{R}^n \) and \( V \subset \mathbb{R}^m \). If \( W \subset V \) we define the pre-image of \( W \) in \( U \) to be the set\(^1\)

\[ f^{-1}(W) = \{ u \in U : f(u) \in W \} \]

3.) Let \( f : U \rightarrow V \) be a continuous map and let \( W \subset V \). Prove that if \( W \) is relatively open (respectively, rel. closed) in \( V \) then \( f^{-1}(W) \) is relatively open (closed) in \( U \).

Recognizing sub-manifolds

4.) Which of the following is a \( C^1 \) sub-manifold:
   
i. The point \( x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \)?
   
   ii. The closed ball \( \overline{B}_1(0) = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \} \)?
   
   iii. The set \( S \subset \mathbb{R}^2 \), \( S = \{ (x, y) : y > x^2 \} \)?
   
   iv. The figure 8 curve \( C \subset \mathbb{R}^2 \) defined by \( C = \{ (\sin t, \sin 2t) : t \in [0, 2\pi) \} \)?

\(^1\)Note that this definition of the pre-image \( f^{-1}(W) \) is defined even if \( f \) does not have an inverse function \( f^{-1} : W \rightarrow U \), i.e. \( f \) need not be 1-to-1 for this to make sense.
Max-min problem

5.) Find the global maximum and minimum of \( P(x) = x_1^3 + x_2x_3 \) on the ball \( \|x\| \leq 1 \).

Real analysis odds and ends

6.) The intermediate value theorem states: Let \( f : [a, b] \to \mathbb{R} \) continuous with \( f(a) < f(b) \). Then for each \( y \in (f(a), f(b)) \) there exists \( c \in (a, b) \) with \( f(c) = y \). Prove this by the method of bisection.

7.) Let \( a < b \) and suppose \( f : [a, b] \to \mathbb{R} \) is continuous and 1-to-1. Prove that \( f \) is either strictly increasing or strictly decreasing.

8.) Prove that the function

\[
 f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}
\]

is differentiable at each point \( x \in \mathbb{R} \), but that \( f(x) \) is not \( C^1 \) on \( \mathbb{R} \).

Power series

9.) Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series with radius of convergence \( \rho > 0 \) about 0. Prove the term-by-term integration formula:

\[
 F(x) := \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad |x| < \rho.
\]

Show that this power series for \( F \) also has radius of convergence \( \rho \).

10.) (Bonus) A well-known formula states that \( \pi \approx 4 = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \), but this is a relatively slow way of computing \( \pi \). Prove that\(^2\)

\[
 \left| \frac{\sqrt{3}\pi}{6} - \sum_{n=0}^{N-1} \frac{(-1/3)^n}{2n+1} \right| \leq \frac{1}{3N(2N+1)}.
\]

How would you estimate \( \sqrt{3} \) rapidly?

11.) (Double bonus) Prove convergence of the improper integral \( \int_{-\infty}^{\infty} \sin(x) \sin(x^2)dx \).

\(^2\)Hint: you may assume \( \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \).