Chain rule exercises

1.) If $f : \mathbb{R} \to \mathbb{R}$ is $C^1$ and $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by $g(x) = f(2x_1 - x_2 + 3x_3)$, prove that $g$ is $C^1$ and show $D_1g(x) = -2D_2g(x) = \frac{2}{3}D_3g(x)$ at each point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

2.) If $\phi : \mathbb{R} \to \mathbb{R}$ is $C^1$ and $g(x) = \phi(\|x\|)$ for $x \in \mathbb{R}^n$, then $g$ is $C^1$ on $\mathbb{R}^n \setminus \{0\}$ and $D_jg(x) = \frac{x_j}{\|x\|}\phi'(\|x\|)$ for all $x \neq 0$.

3.) If $\gamma : [0, 1] \to \mathbb{R}^n$ is a $C^1$ curve and if $f : \mathbb{R}^n \to \mathbb{R}$ is $C^1$, then $f \circ \gamma$ is $C^1$ on $[0, 1]$ and $(f \circ \gamma)'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = (D_{\gamma'(t)}f)(\gamma(t))$ for all $t \in [0, 1]$, where $\nabla f$ is the gradient of $f$ and $D_vf$ means the directional derivative of $f$ in direction $v$.

Maximization problems

4.) Let $g : U \to \mathbb{R}$ be a differentiable function, where $U$ is an open set of $\mathbb{R}^n$. Suppose that $K \subset U$ is a closed and bounded subset and that $Dg(x) \neq 0$ for each $x \in K$. Prove that $g$ attains its maximum on $K$ at a point of the boundary of $K$. (The boundary of $K$ is $\{x \in K : \forall \delta > 0, B_{\delta}(x) \not\subset K\}$.)

5.) Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \sum_{i=1}^{n} |x_i|$. Let $\lambda_1, \lambda_2, ..., \lambda_n > 0$. Prove that $f$ achieves its maximum value on the set

$$S = \{x \in \mathbb{R}^n : \lambda_1x_1^2 + ... + \lambda_nx_n^2 \leq 1\}$$

and determine that maximum. What happens if some $\lambda_i \leq 0$?

Open and closed sets

Define a sequence of sets as follows. Let $C_0 = [0, 1]$ and delete the middle-third open interval $(\frac{1}{3}, \frac{2}{3})$ to obtain $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. From each of the intervals in $C_1$ delete the middle-third...
open interval to obtain

\[ C_2 = [0, \frac{1}{9}] \cup \left( \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right) \cup \left[ \frac{8}{9}, 1 \right]. \]

In general, \( C_n \) is the disjoint union of \( 2^n \) closed intervals, each of length \( \frac{1}{3^n} \), and we form \( C_{n+1} \) by deleting the middle open third from each interval in \( C_n \). Let \( \mathcal{C} = \bigcap_{n=0}^{\infty} C_n \). This is the Cantor set.

6.) Prove that \( \mathcal{C} \) is closed and non-empty.

7.) Prove that \( \mathcal{C} \) does not contain any interval \((a, b)\) with \( a < b \).

We now perform a similar operation with open sets. Let \( O_0 = (0, 1) \) and form \( O_1 \) by deleting the middle-third closed interval from \( O_0 \), that is, \( O_1 = (0, \frac{1}{3}) \cup (\frac{2}{3}, 1) \). Now \( O_n \) is the disjoint union of \( 2^n \) open intervals, each of length \( \frac{1}{3^n} \), and \( O_{n+1} \) is formed from \( O_n \) by deleting the middle-third closed interval from each interval in \( O_n \). Let \( \mathcal{O} = \bigcap_{n=0}^{\infty} O_n \).

8.) Prove that \( \mathcal{O} \) is non-empty and not open.

9.) Prove that \( \mathcal{O} \) is not closed. ¹

¹Hint for 8 and 9: first show that \( \mathcal{C} \) contains irrational numbers.