The standard matrix basis

1.) Let \( A = \begin{pmatrix} - & \cdots & - \\ - & a_1 \top & - \\ \vdots & & \ddots \\ - & a_m \top & - \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \) be an \( m \times n \) matrix with rows \( a_i \top, a_i \in \mathbb{R}^n \) and columns \( \alpha_i \in \mathbb{R}^m \). Let

\[
M_{i,j}^k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 1 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

be the \( k \times k \) square matrix consisting of all 0’s except for a 1 in entry \( i,j \).

Prove that \( M_{i,j}^m A \) has \( i \)th row equal to \( a_j \top \) with all other rows equal to \( 0 \), and \( AM_{i,j}^n \) has \( j \)th column equal to \( \alpha_i \) with all other columns equal to \( 0 \).

\[
M_{i,j}^m A = \begin{pmatrix} - & 0 & - \\ \vdots & a_j \top & - \\ - & \cdots & - \\ - & 0 & - \end{pmatrix}, \quad AM_{i,j}^n = \begin{pmatrix} 0 & \cdots & \alpha_i & \cdots & 0 \end{pmatrix}. \]

2.) Prove that in a matrix product \( AB \), the columns of \( AB \) are a linear combination of the columns in \( A \) and the rows in \( AB \) are a linear combination of the rows in \( B \).

\(^1\)The matrices \( M_{i,j}^k \) form the ‘standard basis’ for all \( k \times k \) matrices thought of as vectors in \( \mathbb{R}^{k^2} \).
Invertible matrices

3.) Let $A$ be an $m \times n$ matrix, so $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $v_1, ..., v_k$ be $k$ vectors in $\mathbb{R}^n$. Show that 
$$\dim(\text{span}(Av_1, ..., Av_k)) \leq \dim(\text{span}(v_1, ..., v_k)).$$

4.) An $m \times n$ matrix $A$ is said to be invertible if there exists an $n \times m$ matrix $B$ such that for all $w \in \mathbb{R}^m$, $ABw = w$ and for all $v \in \mathbb{R}^n$, $BAv = v$. Prove that if $A$ is invertible then for any $k$ vectors $v_1, ..., v_k \in \mathbb{R}^n$ we have
$$\dim(\text{span}(Av_1, ..., Av_k)) = \dim(\text{span}(v_1, ..., v_k)).$$
Hence deduce that if $A$ is invertible then $m = n$.

5.) Suppose $A$ is an $m \times n$ matrix. Let $B$ be an $n \times n$ matrix and $C$ be an $m \times m$ matrix and suppose that both $B$ and $C$ are invertible. Prove that
$$\text{nullity}(A) = \text{nullity}(BA) = \text{nullity}(AC), \quad \text{rank}(A) = \text{rank}(BA) = \text{rank}(AC),$$
$$\text{row rank}(A) = \text{row rank}(BA) = \text{row rank}(AC).$$

Elementary operations

6.) Recall that if $v_1, ..., v_k$ are $k$ vectors in $\mathbb{R}^n$, the three ‘elementary operations’ on $v_1, ..., v_k$ are
1. interchange any pair of vectors
2. multiply one of the vectors by a non-zero scalar
3. replace vector $v_i$ with vector $\tilde{v}_i = v_i + \mu v_j$ where $i \neq j$ and $\mu \in \mathbb{R}$.

Show that each of the elementary operations on the columns of $A$ can be achieved by multiplying on the right by an invertible $n \times n$ matrix ($\tilde{A} = AR$). Show also that each of the elementary operations on the rows of $A$ can be achieved by multiplying on the left by an invertible $m \times m$ matrix ($\tilde{A} = LA$).

7.) Suppose that $A$ is an $m \times n$ matrix. Prove that by a sequence of elementary row and column operations, $A$ may be reduced to the form
$$\tilde{A} = \begin{bmatrix} I_k & 0_{k \times (n-k)} \\ 0_{(m-k) \times k} & 0_{(m-k) \times (n-k)} \end{bmatrix}$$
where $I_k$ is the $k \times k$ identity matrix and $0_{i,j}$ is the $i \times j$ matrix of all 0’s. Combined with problems 6 and 7, this gives a proof of the fact that the row-rank and column-rank of a matrix are equal. We’ll see a different proof of this fact next week.
8.) Let $A$ be an $n \times n$ matrix and suppose that for all $n \times n$ matrices $B$ we have $AB = BA$. Prove that $A = \lambda I_n$ for some scalar $\lambda \in \mathbb{R}$.

Challenge!) There are two matrices $A$ and $B$ of size $m \times n$ each filled only by 0s and 1s. It is given that along any row or column its elements do not decrease (from left to right and from top to bottom). It is also given that the numbers of 1s in both matrices are equal and for any $k = 1, \ldots, m$ the sum of the elements in the top $k$ rows of the matrix $A$ is no less than that of the matrix $B$. Prove for any $l = 1, \ldots, n$ the sum of the elements in left $l$ columns of the matrix $A$ is no greater than that of the matrix $B$. 
