More Schwarz class problems

September 6, 2011

**Problem 1.** Prove that any separately continuous multilinear form on $F_1 \times \ldots \times F_n$, is jointly continuous if all the $F_i$ are Fréchet spaces.

**Problem 2.** Let $U_a$ be translation by $a$ as an operator on $S'(\mathbb{R})$. Let $d/dx$ be the derivative operation on $S'$. Prove that $(U_a - 1)a^{-1}$ converges pointwise in the $\sigma(S', S)$ topology to $d/dx$.

**Problem 3.** Let $\delta$ be the point mass at zero. Prove directly that the distributional derivative $\delta'$ satisfies $\delta' \in S'$. Prove that $\delta'$ does not come from a measure.

**Problem 4.**

a. Let $\phi \in S$ and let $\phi_y$ be the function in $S$ defined by $\phi_y(x) = \phi(x - y)$. Prove that the map $y \mapsto \phi_y$ is a $C^\infty$ function from $\mathbb{R}^n$ to $S(\mathbb{R}^n)$ with $D^\alpha(\phi_y) = (-1)^\alpha(D^\alpha\phi)_y$. To say $y \mapsto \phi_y$ has derivative $\partial \phi_y / \partial y_j$ as a function with values in $S$ means

$$\lim_{y \to y_0} \frac{|y - y_0|}{N} \left[ \phi_y (\phi_y - \phi_{y_0}) + \sum_{j=1}^{N} \frac{\partial}{\partial y_j} (\phi_y) \cdot (y - y_0)_j \right] = 0$$

in the topology of $S$.

b. Let $T \in S'$. Let $\phi \in S$. Define $T^\phi$ to be the function, $T^\phi(y) = T(\phi_y)$. Prove that $T^\phi \in C^\infty$.

c. Let $\phi_n \in S$ with $\phi_n \to \delta$ in the weak topology on $S'$. Prove that $T^\phi_n \to T$ for all $T \in S'$ in the weak topology on $S'$.

d. Prove that $S$ is dense in $S'$.

**Problem 5.** Prove that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{(x - x_0)^2 + \epsilon^2} = \pi \delta(x - x_0)$$

in the sense of distributions.