More measure theory problems

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1 Measure and integration

Problem 1. Let \( f \in L^1(\mathbb{R}), f \neq 0 \). Prove that there exists \( \phi \in C_c^\infty(\mathbb{R}) \) with \( \int_{-\infty}^{\infty} f(t)\phi(t)dt > 0 \).

Problem 2. a. Suppose \( f : X \to [-\infty, \infty] \) and \( g : X \to [-\infty, \infty] \) are measurable. Prove that the sets
\[
\{ x : f(x) < g(x) \} \quad \text{and} \quad \{ x : f(x) = g(x) \}
\]
are measurable.

b. Prove that the set of points at which a sequence of measurable real-valued functions converge (to a finite limit) is measurable.

Problem 3. Suppose that \( f \in L^1(\mu) \). Prove that for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( E \) is measurable with \( \mu(E) < \delta \) then \( \int_E |f|d\mu < \epsilon \).

Problem 4. Suppose that \( \mu \) is a positive measure on \( X \), \( f : X \to [0, \infty] \) is measurable, \( \int_X f d\mu = c \) with \( 0 < c < \infty \) and \( \alpha \) is a constant. Prove
\[
\lim_{n \to \infty} \int n \log[1 + (f/n)^\alpha]d\mu = \begin{cases} 
\infty, & 0 < \alpha < 1 \\
c, & \alpha = 1 \\
0, & 1 < \alpha < \infty
\end{cases}
\]

Problem 5. Construct a Borel set \( E \subset \mathbb{R} \) such that
\[
0 < m(E \cap I) < m(I)
\]
for every non-empty interval \( I \). Is it possible that \( m(E) < \infty \)?

Problem 6. If \( X \) is compact and \( f : X \to (-\infty, \infty) \) is upper semicontinuous, prove that \( f \) attains its maximum at some point of \( X \).

Problem 7. Prove Egoroff’s theorem: If \( \mu(X) < \infty \), if \( \{f_n\} \) is a sequence of complex measurable functions which converge pointwise at every point of \( X \), and if \( \epsilon > 0 \), there is a measurable set \( E \subset X \), with \( \mu(X \setminus E) < \epsilon \), such that \( \{f_n\} \) converges uniformly on \( E \).
Problem 8. Let $\mu$ be a positive measure on $X$ with $\mu(X) < \infty$. A sequence of complex measurable functions $\{f_n\}$ is said to converge in measure to the measurable function $f$ if for every $\epsilon > 0$ there is an $N$ such that for all $n > N$,

$$\mu(\{x : |f_n(x) - f(x)| < \epsilon\}) < \epsilon.$$ 

Prove the following statements:

a. If $f_n(x) \to f(x)$ a.e. then $f_n \to f$ in measure.

b. Let $1 \leq p \leq \infty$. If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure.

c. If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence converging to $f$ a.e.

Problem 9. Suppose that $K \in L^p([0,1] \times [0,1]), 1 < p < \infty$. Let $q$ be the dual exponent, $p^{-1} + q^{-1} = 1$.

a. For $f \in L^q([0,1])$, let $(Af)(x) = \int K(x,y)f(y)dy$. Show that $(Af)(x)$ exists for a.e. $x$ and $A \in L(L^q([0,1]), L^p([0,1]))$.

b. Suppose that for every $f \in L^1([0,1]), (Af)(x) = 0$ for a.e. $x$. Show that $K = 0$ a.e.

2 Differentiation

Problem 10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing, and is differentiable almost everywhere with respect to Lebesgue measure. Show that $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

Problem 11. Suppose that $f \in L^p(\mathbb{T}), 1 < p < \infty$, and $\sup_{h \neq 0, |h| < 1} \int_\mathbb{T} \left| \frac{f(x+h) - f(x)}{h} \right|^p < \infty$. Show that the distributional derivative of $f$, $f'$ satisfies $f' \in L^p(\mathbb{T})$.

Problem 12. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, and for Lebesgue a.e. $y \in \mathbb{R}$, the partial derivative $\frac{\partial f(x,y)}{\partial x}$ exists (as a real number) for Lebesgue a.e. $x \in \mathbb{R}$. Show that $\frac{\partial f(x,y)}{\partial x}$ exists for Lebesgue a.e. $(x,y) \in \mathbb{R}^2$.

(Hint: for each fixed $(x,y) \in \mathbb{R}^2$ consider $D_1f(x,y) = \limsup_{h \to 0, h \neq 0} \frac{f(x+h,y) - f(x,y)}{h}$ and $D_1f(x,y) = \liminf_{h \to 0, h \neq 0} \frac{f(x+h,y) - f(x,y)}{h}$ where $\mathbb{Q}$ is the set of rationals.)

Problem 13. Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $[0,1], \mu$ is a positive Borel measure on $[0,1]$ and that (i) $\lim g_n(x) = 0$ a.e., (ii) $\int_0^1 g_n dx = 1$ for all $n$ and

$$(iii) \quad \lim \int_0^1 f g_n dx = \int_0^1 f d\mu$$

for every continuous $f \in C[0,1]$. Does it follow that $\mu$ is mutually singular with respect to the Lebesgue measure? Prove this or give a counterexample.