0.1 Measure theory

Problem 1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $\mu$ be the Lebesgue measure on it. For $A \subset \mathbb{T}$ and $y \in \mathbb{T}$ write $A + y = \{x + y : x \in A\}$. Suppose $A \subset \mathbb{T}$ is measurable, and for each $n \in \mathbb{N}^+$, $A + 2^{-n} = A$. Show that either $\mu(A) = 0$ or $\mu(A) = 1$.

Problem 2. If $\mu$ is a $\sigma$-finite measure on a measurable space $(X, \mathcal{A})$ (i.e. on $\mathcal{A}$, where $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$), then there is a finite measure $\nu$ on $(X, \mathcal{A})$ with $\nu \ll \mu$ and $\mu \ll \nu$.

Problem 3. Let $s > 2$, and let $A_s$ be the set of $x \in [0, 1]$ with the property that there exist infinitely many positive integers $q$ such that for some integer $p \in [0, q]$, $|x - \frac{p}{q}| < \frac{1}{q^s}$. Show that $\mu(A_s) = 0$, where $\mu$ is the Lebesgue measure.

Problem 4. Show that if $g \in L^1(\mathbb{T})$, $\mu$ is a finite measure on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $\mu(x + \alpha\pi) - \mu(x) = g(x)dx$ for some irrational $\alpha$ (where $dx$ denotes the Lebesgue measure), then $\mu$ is absolutely continuous with respect to Lebesgue measure.

0.2 Hilbert space

Problem 5. Suppose $f \in L^1([0, 1])$ but $f \not\in L^2([0, 1])$. Find a complete orthonormal basis $\{\phi_n\}$ for $L^2([0, 1])$ such that each $\phi_n \in C^0([0, 1])$ and such that

$$\int_0^1 f(x)\phi_n(x)dx = 0, \quad \forall \ n.$$

Problem 6. Show that $L^2([0, 1])$ has an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ such that each $\phi_n \in C^1([0, 1])$ and $\phi_n'(1/4) = \phi_n'(1/2)$ for each $n$.

0.3 Uniformly convex Banach spaces

Problem 7. A Banach space $X$ is uniformly convex if for every $\epsilon \in (0, 1)$ there exists $\eta < 1$ such that if $x, y \in X, \|x\| = \|y\| = 1$ and $\|x - y\| > 2\epsilon$ then $\|\frac{1}{2}(x + y)\| < \eta$.

a. Show that every Hilbert space is uniformly convex, and one may take $\eta = (1 - \epsilon^2)^{1/2}$.
b. Let $X$ be a uniformly convex Banach space. Suppose that $C$ is a closed convex subset of $X$, and $z \in X$. Show that there exists a unique $x \in C$ such that $\|x - z\| = \inf\{\|y - z\| : y \in C\}$.

c. Give an example showing that the property in (b) can fail if $X$ is not uniformly convex.

**Problem 8.** For $1 < p < \infty$, show that $L^p$ is uniformly convex. (Hint: Write $\int |f + g|^p d\mu = \int |\phi + \psi|^p d\nu$ where $\phi = \frac{|f|^p + |g|^p}{{(\int |f|^p + |g|^p)^{1/p}}}$, $\psi = \frac{g}{{(\int |f|^p + |g|^p)^{1/p}}}$, and $\nu = (|f|^p + |g|^p)\mu$. This reduces the general case to that where the functions are bounded by 1 and the measure space has total mass 2. Use the convexity of the function $x^p$ in $[-1, 1]$.

**Problem 9.** Let $B$ be a uniformly convex Banach space.

a. Assume that $x_n \in B$ for $n = 1, 2, \ldots$ and $x_n \to x_0$ in the weak topology, and $\|x_n\| \to \|x_0\|$. Prove that $x_n \to x_0$.

Hint: Assume that $\{x_n\}$ is not a Cauchy sequence, and for suitable pairs $n_j, m_j, n_j < m_j, n_j \to \infty$, consider $y_j = (x_{n_j} + x_{m_j})/2$.

b. Give an example that the statement in (a) is false for general Banach spaces.

### 0.4 Topological

**Problem 10.** Let $G$ be an unbounded open set in $(0, \infty)$. Define $D = \{x \in (0, \infty) : nx \in G$ for infinitely many $n\}$. Prove that $D$ is dense in $(0, \infty)$.

**Problem 11.** Let $\{f_k\}$ be a sequence of real-valued functions defined on $[-1, 1]$ such that

$$|f_k(x) - f_k(y)| \leq \sqrt{|x - y|} + \frac{1}{k}$$

for all $k \geq 0$ and $x, y \in [-1, 1]$. Suppose also that each $f_k(0) = 0$. Prove that some subsequence of the $f_k$ converges uniformly to a continuous function $f$ on $[-1, 1]$.

### 0.5 Fourier transform

**Problem 12.** Let $c_0$ denote the closed subspace of $\ell^\infty(\mathbb{Z})$ consisting of all bilateral sequences $x = (x_j)$ such that $x_j \to 0$ when $|j| \to \infty$. The sequence of Fourier coefficients $a_n$ of any function $f \in L^1(\mathbb{T})$ lies in $c_0$. Denoting this map by $\mathcal{F}$, prove that the image of $\mathcal{F}$ is not all of $c_0$. 