Hilbert space problems

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Problem 1. If $M$ is a closed subspace of Hilbert space $H$, prove that $(M^\perp)^\perp = M$. What if $M$ is not closed?

Problem 2. Prove that Hilbert space $H$ is separable if and only if it contains a maximal orthonormal set which is at most countable.

Problem 3. Let $u_1, u_2, \ldots$ be an orthonormal set in Hilbert space $H$, and let $\delta_n$ be a sequence of positive numbers. Prove that the set

$$\left\{ \sum_n c_n u_n : |c_n| < \delta_n \right\}$$

is compact if and only if $\sum_n \delta_n^2 < \infty$.

Problem 4. Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n$ is a sequence of positive numbers with $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

Problem 5. If $H_1$ and $H_2$ are two Hilbert spaces, prove that one of them is isomorphic to a subspace of the other.

Problem 6. Find a closed subset $E \subset L^2(\mathbb{T})$ that contains no element of smallest norm.

Problem 7. Suppose $f$ is a continuous function on $\mathbb{R}$ with period 1. Prove that

$$\lim_N \frac{1}{N} \sum_{j=1}^N f(\alpha j) = \int_0^1 f(t) dt$$

for all irrational numbers $\alpha$.

Problem 8. Let $C$ be a closed convex set in a Hilbert space $H$. Prove that $C$ contains a unique element of minimal norm.

Problem 9. Let $v_1, \ldots, v_N$ be a finite sequence of unit vectors in a Hilbert space $H$. Suppose that there exists a number $a \in (0, 1)$ such that

$$\langle v_i, v_j \rangle \leq -a, \quad \forall \ i \neq j.$$

Find an upper bound for $N$ in terms of $a$. 
Problem 10. Let \( x_1, x_2, \ldots \) be a sequence of positive numbers and set \( s_n = \frac{1}{n} \sum_{j \leq n} x_j \). Prove that for some fixed constant \( C \) (independent of \( x \) and \( N \))

\[
\sum_{n \leq N} s_n^2 \leq C \sum_{n \leq N} x_n^2.
\]

Problem 11. Let \( f \in C^0([\alpha, \beta]) \), where \( 0 < \alpha < \beta < a \). For each \( n = 1, 2, \ldots \), define

\[
P_n(x) = \frac{\int_{\alpha}^{\beta} f(u)[1 - (u - x)^2]^n \, du}{\int_{-1}^{1}(1-u^2)^n \, du}.
\]

Show that \( P_n(x) \) is a polynomial of degree at most \( 2n \) and that for any closed subinterval \([a, b] \subset (\alpha, \beta)\), \( P_n \to f \) uniformly.

Problem 12. Let \( H \) be a separable Hilbert space with an orthonormal basis \( \{x_n\} \). Let \( \{y_n\} \) be a sequence in \( H \) and prove that the following two statements are equivalent.

1. \( \lim_{n \to \infty} (x, y_n) = 0 \) for all \( x \in H \).
2. \( \lim_{n \to \infty} (x_m, y_n) = 0 \) for each \( m = 1, 2, \ldots \), and \( \{\|y_n\|\} \) is bounded.

Problem 13. Let \( H \) be a separable infinite dimensional Hilbert space, and suppose that \( e_1, e_2, \ldots \) is an orthonormal system in \( H \). Let \( f_1, f_2, \ldots \) be another orthonormal system which is complete, i.e. such that the closure of the span of the \( f_j \) is all of \( H \).

a. Prove that if \( \sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1 \), then \( \{e_n\} \) is also a complete orthonormal system.

b. Suppose only that \( \sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty \). Prove that it is still true that \( \{e_n\} \) is a complete orthonormal system. (Hint: choose \( N \) so that \( \sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < 1 \); consider the subspace \( S \) spanned by the vectors \( \tilde{f}_n = f_n - \sum_{k=n+1}^{\infty} (f_n, e_k)e_k \), for \( n \leq N \).)