Calabi-Yau categories, the Floer theory of a cotangent bundle, and the string topology of the base

Preliminary version

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Abstract

The goal of this paper is to compare two 2-dimensional topological field theories: the string topology of a closed, oriented manifold M, and the (possibly twisted) symplectic cohomology of its cotangent bundle, T^*M. The isomorphism between homology groups is a classical theorem of Viterbo [41, 1, 35], and moreoever it has been established that this isomorphism intertwines various homological field theoretic structures. Our main result implies the existence of full chain-level topological field theory structures on both sides in characteristic zero (coinciding with most previously known chain-level operations), and moreoever that the two resulting chain-level field theories are naturally quasi-equivalent. In particular, we recover as special cases the compatibilities of homological operations in string topology and symplectic cohomology. Our approach uses recent work [28] of Kontsevich and Vlassopolous, who show that a "smooth, Calabi-Yau" (sCY) algebra naturally gives rise to a chain complex valued 2-dimensional ('right positive boundary') topological field theory on its Hochschild chain complex. Using extensions of the notions of smooth and compact Calabi-Yau from algebras to A_∞ categories enriched over chain complexes, we show that the the string topology category $\mathcal{S}_M$ defined in [7] has a natural, geometrically defined sCY structure; its Hochschild complex is equivalent to string topology. The second-named author [21, 20] has has shown that the wrapped Fukaya category of the cotangent bundle $\mathcal{W}_{T^*M}$ similarly has geometrically defined sCY structure; its Hochschild complex is equivalent to symplectic cohomology. We then show that $\mathcal{S}_M$ and $\mathcal{W}_{T^*M}$ are equivalent as sCY-categories, extending a result of Abouzaid.

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We conjecture that this notion of smooth Calabi-Yau structure is precisely the data required to reconstruct an *extended*, or *open-closed*, (chain-level) field theory, in any characteristic. If so, our main results (which are true in any characteristic) would imply the existence and equivalence of *open-closed* theories coming from string topology and wrapped Floer theory.

There is a different, better-studied notion of a "compact Calabi-Yau" algebra, due to Konstevich-Soibelman [29] and related to Costello's work [15]; such algebras give rise to ('left positive boundary') field theories theory on their Hochschild chain complex. In an appendix, we show that Koszul duality interchanges the role of sCY and cCY-categories, and that Koszul dual CY algebras or categories represent dual topological field theories.

Contents

[section]

1 Introduction

Throughout this paper M will denote a fixed closed, orientable smooth manifold of dimension n. There are two 2-dimensional topological field theories associated to it that play an important role in the interface between algebraic topology and symplectic topology. These are the string topology field theory, $\mathcal{S}_M$, and the Floer field theory of its cotangent bundle, $\mathcal{W}_{T^*M}$. These theories are homological in nature (though various at least partial chain-level refinements exist), and they take values in graded vector spaces. The goal of this paper is to lift the values of these theories to the category of chain complexes over a and to then compare these chain complex valued theories.

The string topology field theory, $\mathcal{S}_M$, has as its state space, i.e its value on the circle, $\mathcal{S}_M(S^1) = H_*(LM)$, the homology of the free loop space. Its value on the "pair-of-pants" cobordism P (i.e the surface of genus zero with two incoming boundary circles and one outgoing boundary circle) is given by the Chas-Sullivan string topology product:

$$\mathcal{S}_M(P) : H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(LM).$$

See [10]. The field theoretic structure of this theory was studied by Cohen and Godin [14], Godin [22], and Kupers [31].

The "wrapped Floer field theory" of the cotangent bundle, which we denote by $\mathcal{W}_{T^*M}$, has as its state space, $\mathcal{W}_{T^*M}(S^1) = SH^*(T^*M)$, the symplectic cohomology of T^*M with respect to its canonical exact symplectic (Liouville) structure. A well known theorem of Viterbo, [41, 1, 35] states that if the second Stiefel-Whitney class $w_2(M) = 0$, then

$$SH^*(T^*M) \cong H_{n-*}(LM) \tag{1.1}$$

(the graded isomorphism (1.1) fixes our conventions for symplectic cohomology, which is sometimes also called 'symplectic homology'). If $w_2(M) \neq 0$ then a similar result holds, but one needs to take twisted coefficients for $SH^*(T^*M)$ with respect to a *background class* [30]. The value of the Floer field theory on the pair of pants cobordism

$$\mathcal{W}_{T^*M}(P) : SH^*(T^*M) \otimes SH^*(T^*M) \to SH^*(T^*M).$$

is the "pair of pants product" product defined by counting certain (inhomogeneous) J-holomorphic curves $P \to T^*M$ [1].

The goal of this paper is to show that the open-closed field theories $\mathcal{S}_M$ and $\mathcal{W}_{T^*M}$ can be lifted to chain complex valued theories (which by abuse of notation we also call $\mathcal{S}_M$ and $\mathcal{W}_{T^*M}$) and as such, these theories are naturally quasi-equivalent,

$$\mathcal{W}_{T^*M} \xrightarrow{\simeq} \mathcal{S}_M.$$

Our approach is to use the various classification theorems for topological field theories proved in the last several years. Such theorems were proved by Costello [15], Kontsevich-Soibelman [29], Lurie [32], and most recently, Kontsevich-Vlassopolous [28]. Roughly, these theorems say that "Calabi-Yau categories" give rise to, in a natural way, $2D$ open-closed, noncompact, topological field theories. Furthermore, in certain settings, these field theories are classified by these Calabi-Yau categories. What we will prove in this paper is that the string topology category, which by abuse of notation we call $\mathcal{S}_M$, defined by Blumberg-Cohen-Teleman [7], and the wrapped Fukaya category of the cotangent bundle, $\mathcal{W}_{T^*M}$, defined in [6], both have the structure of Calabi-Yau categories, and are equivalent as such. These categories are enriched over chain complexes, so this implies that they each represent chain complex valued field theories that are in turn equivalent.

Remark 1.1. *As of now, the existence of and quasi-equivalence between these two field theories holds in characteristic zero, where the classification result we use [28] has been established. However, the notion of a Calabi-Yau structure, and our proof of existence and equivalence of Calabi-Yau structures, makes sense (and seems to be the 'correct notion') in any characteristic. We expect corresponding implications for field theories, see the discussion above §1.1.*

We then show that when one passes to homology these field theories are known string topology field theory of the manifold M and the Floer field theory of T^*M.

We now state these results in a more precise fashion.

In [29], Kontsevich and Soibelman defined the notion of a Calabi-Yau structure for a compact, A_∞-algebra. We refer to such a compact, Calabi-Yau structure as cCY. This structure is similar to that defined by Costello [15], whose Calabi-Yau A_∞ algebras A come equipped with nondegenerate pairings $A \times A \to k$, where k is the ground field. The cCY structure of Kontsevich and Soibelman comes equipped with an element of the dual of the cyclic chains, $\tau : \mathrm{CC}_*(A) \to k$ with the property that the induced composition

$$A \otimes A \to A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A \simeq \mathrm{CH}_*(A) \to \mathrm{CC}_*(A) \xrightarrow{\tau} k$$

is "homotopy nondegenerate", in the sense that the adjoint $A \to A^*$ is a chain homotopy equivalence of $A \otimes A^{op}$ A_∞-modules (i.e A-bimodules). Here $CH_*(A)$ are the Hochschild chains, which is a model for the derived tensor product, $A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A$. So a cCY-algebra is "self-dual up to homotopy".

There is a related notion of a Calabi-Yau category due to Kontsevich and Vlassopoulous [28] that is applied to *smooth* differential graded (and A_∞) algebras and categories. (Recall that an algebra A is smooth if it is perfect as a bimodule over itself. That is, if A is a perfect $A \otimes A^{op}$-module.) Roughly the notion of a "smooth Calabi-Yau" (sCY) structure of on an smooth algebra A is choice of a class in the negative cyclic chains, $\sigma \in \mathrm{CC}^-(A)$ that induces an equivalence $\cap\sigma : A^! \to A$, where $A^!$ is the bimodule dual of A:

$$A^! = \mathrm{RHom}_{A \otimes A^{op}}(A, A \otimes A^{op}).$$

We observe that both the notions of cCY and sCY-structures extend to the setting of differential graded categories and A_∞-categories.

The relevance of these Calabi-Yau structures on algebras (and categories) is that by theorems in [29] and [28], both cCY and sCY algebras give rise to $2D$ noncompact (or "positive boundary") topological field theories that take values in chain complexes. This motivated the main result of this paper, which we now state.

Let $\mathcal{S}_M$ denote the string topology category defined by the first author, Blumberg, and Teleman in [7]. This is a category, enriched over chain complexes, whose objects are connected, closed, oriented submanifolds $N \subset M$, and the space of morphisms, $Mor_{\mathcal{S}_M}(N_1, N_2)$ is equivalent to $C_*(\mathcal{P}_M(N_1, N_2))$, the singular chains of the space of paths from N_1 to N_2: $\mathcal{P}_M(N_1, N_2) = \{\gamma : [0,1] \to M, \text{such that}\, \gamma(0) \in N_1,\, \gamma(1) \in N_2\}$. Now let $\mathcal{W}^{conor}_{T^*M}$ be the subcategory wrapped Fukaya category of the cotangent bundle, whose objects are conormal bundles $\nu^*_N \subset T^*M$ of connected, closed, oriented submanifold $N \subset M$ (For $N \neq M$, ν^*_N are noncompact Lagrangian submanifolds of T^*M). The wrapped Fukaya category was defined by Abouzaid and Seidel in [6] and studied by the second author, who proved

Theorem ([20, 21])**.** *Under suitable non-degeneracy hypotheses on a symplectic manifold X, the wrapped Fukaya category of X has a geometrically defined smooth Calabi-Yau structure.*

The non-degeneracy hypotheses are known to hold for cotangent bundles by work of Abouzaid (who also proposed the non-degeneracy criterion, sometimes known as a 'generation criterion' in the symplectic literature) [3, 4]. In particular, $\mathcal{W}^{conor}_{T^*M}$ has a geometrically defined smooth Calabi-Yau structure, which is reviewed in §3.3.

Our main theorem is the following.

Theorem 1. *The A_∞-category $\mathcal{S}_M$ also has a geometrically defined, smooth Calabi-Yau structures (sCY). Furthermore there is a quasi-equivalence $\alpha : \mathcal{W}^{conor}_{T^*M} \xrightarrow{\simeq} \mathcal{S}_M$ preserving sCY structures.*

Note: The string topology category $\mathcal{S}_M$ was constructed as a full subcategory of the category $Perf(C_*(\Omega M))$, of perfect, A_∞-modules over $C_*(\Omega M)$, the singular chains of the based loop space. Furthermore since the endomorphisms of a point $End_{\mathcal{S}_M}(pt) = C_*(\Omega M)$, $\mathcal{S}_M$ clearly generates $Perf(C_*(\Omega M))$. In [5, 3], Abouzaid showed that $\mathcal{W}^{conor}_{T^*M}$ is also quasi-isomorphic to a full subcategory of $Perf(C_*(\Omega M))$ which generates it (in particular, a cotangent fibre $T^*_q M$ generates $\mathcal{W}^{conor}_{T^*M}$, and $End\mathcal{W}^{conor}_{T^*M}(T^*_q M) \cong C_*(\Omega M)$. Thus it was known that $\mathcal{S}_M$ and $\mathcal{W}^{conor}_{T^*M}$ both have isomorphic categorical completions. In this paper we refine Abouzaid's analysis to show that $\mathcal{S}_M$ and $\mathcal{W}^{conor}_{T^*M}$ are in fact equivalent, and this equivalence preserves the geometric sCY-structures.

After we prove Theorem 1, we show that, using the constructions of Kontsevich and Vlassopolous [28] that the chain complex valued TFT's that $\mathcal{S}_M$ and $\mathcal{W}^{conor}_{T^*M}$ represent as sCY-categories, realize, when one passes to homology, the known string topology and Floer field theoretic operations.

Next we prove the following Koszul duality relationship between cCY-categories and sCY-categories, and the field theories they represent.

Theorem 2. *Let A be a smooth differential graded algebra or category. (This notion will be defined carefully in §3.2.1. Suppose B is a differential graded algebra or category that is Koszul dual to A. That is,*

$$B \simeq \mathrm{RHom}_A(k,k) \quad \textit{and} \quad A \simeq \mathrm{RHom}_B(k,k).$$

Then A is sCY if and only if B is cCY-algebra.

If A is an sCY-algebra, and B is its Koszul dual, then by this theorem and the work of Kontsevich and his collaborators [29], [28], they represent field theories $\mathcal{F}_A$ and $\mathcal{F}_B$ respectively. We then examine the relationship between these field theories, and prove that their corresponding closed sectors are linearly dual to each other. Algebraically this generalizes an the old theorem of Jones and McClearly [25] saying that if A and B are Koszul dual, then their Hochschild homologies are linearly dual,

$$HH_*(A) \cong HH_*(B)^*.$$

This establishes a duality between the state spaces of the TFT's corresponding Koszul dual Calabi-Yau algebras, $\mathcal{F}_A(S^1) \simeq \mathcal{F}_B(S^1)^*$. We extend this by showing that the operations on $\mathcal{F}_A(S^1)$ in the field theory $\mathcal{F}_A$ are dual to the corresponding operations in $\mathcal{F}_B$.

Finally we give a discussion relating the notions of Calabi-Yau algebras and categories used here (due to Kontsevich and his coauthors) to the similar notion of a "Calabi-Yau object in a symmetric monodical $(\infty,2)$-category described by Lurie [32]. We make a specific conjecture relating these notions. If our conjecture is true, the results in this paper, together with Lurie's cobordism hypothesis, would imply that $\mathcal{S}_m$ and $\mathcal{W}^{conor}_{T^*M}$ each define equivalent extended $2D$ TFT's. This conjecture and its consequence will be pursued in upcoming work that will be in collaboration with A. Blumberg.

1.1 Algebraic notation and conventions

1.1.1 A_∞ categories and differential graded categories

We work over a field $\mathbb{K}$ of arbitrary characteristic unless otherwise specified. For us, an A_∞ *category* is always enriched over $\mathbb{K}$-chain complexes, and concretely can be specified by a collection of objects ob $\mathcal{C}$, $\mathbb{Z}$-graded $\mathbb{K}$ vector spaces as morphism spaces $\hom^\bullet_{\mathcal{C}}(X,Y)$, and operations $\mu^d : \hom_{\mathcal{C}}(X_{d-1},X_d)\otimes\cdots\hom_{\mathcal{C}}(X_0,X_1)\to\hom_{\mathcal{C}}(X_0,X_d)$ of degree $2-d$ satisfying the A_∞ relations

$$\sum(-1)^{\maltese_1^j}\mu^k(x_d,\ldots,x_{i+j+1},\mu^i(x_{i+j},\ldots,x_{j+1}),x_j,\ldots,x_1)=0$$

where $|x|$ denotes the degree of an element, $||x|| := |x|-1$ denotes the reduced degree, and

$$\maltese_r^s := \sum_{n=r}^{s} ||x_n||. \tag{1.2}$$

(note that in particular, our Koszul sign conventions are as in [37]). A special case is the notion of an A_∞ *algebra*, which is a single graded vector space (thought of as the endomorphisms of a single object) equipped with operations μ^k as before. We say that $\mathcal{C}$ is *cohomologically unital* if the cohomology of morphism spaces $H^*(\hom_{\mathcal{C}}(X,X))$ with respect to the differential μ^1 admit identity morphisms for the cohomological composition $[\mu^2]$. We draw upon [37] for the notions of A_∞ *functors* (and categories of functors), and *split-generation*.

We view all differential graded (*dg*) categories $\mathcal{C}$ as special cases of A_∞ categories via the convention $\mu^1 = d$, $\mu^2(a,b) = (-1)^{|a|} a \circ b$, and the change of grading from homological to cohomological (by reversing signs of gradings).

1.1.2 Modules and bimodules

If $\mathcal{C}$ and $\mathcal{D}$ are A_∞ categories, there are well known associated categories

$$\mathcal{C}\text{−mod},\ \text{mod−}\mathcal{C},\ \mathcal{C}\text{−mod−}\mathcal{D}$$

of A_∞ *left modules* over $\mathcal{C}$, *right modules* over $\mathcal{C}$, and $\mathcal{C}-\mathcal{D}$ *bimodules*. Abstractly an A_∞ left/right module over $\mathcal{C}$ is an A_∞ functor from $\mathcal{C}^{op}/\mathcal{C}$ to chain complexes, and an A_∞ $\mathcal{C}-\mathcal{C}$ bimodule is an A_∞ *bilinear functor* from $\mathcal{C}^{op} \times \mathcal{D}$ to chain complexes (in the sense of @@CITE:Lyubashenko), all over $\mathbb{K}$. Concretely, an A_∞ bimodule $\mathbf{B}$ is an assignment of, for every pair of objects $(X,Y) \in \text{ob}\ \mathcal{C} \times \text{ob}\ \mathcal{D}$, a chain complex $(\mathbf{B}(X,Y), \mu_{\mathbf{B}}^{0|1|0})$, and for any pair of tuples $(K_0,\dots,K_r), (L_0,\dots,L_s)$, higher multiplication maps

$$\begin{aligned}\mu_{\mathbf{B}}^{r|1|s} : \hom_{\mathcal{C}}(K_{r-1},K_r) \otimes \cdots \otimes \hom_{\mathcal{C}}(K_0,K_1) \otimes \mathbf{B}(K_0,L_s) \otimes \\ \hom_{\mathcal{D}}(L_{s-1},L_s) \otimes \cdots \otimes \hom_{\mathcal{D}}(L_0,L_1) \to \mathbf{B}(K_k,L_0)\end{aligned}$$

of degree $1-r-s$ satisfying the A_∞ *bimodule relations* which to first order say that $\mu^{0|1|0}$ is a differential and $\mu^{1|1|0}$ and $\mu^{0|1|1}$ descend to cohomology level multiplications (see e.g., [38] but note its different Koszul sign convention, [37] for modules, [**?**, 20]). A left module can be thought of as a $\mathcal{C}-\mathbb{K}$ bimodule equipped with maps $\mu^{k|1} := \mu^{k|1|0}$, and similarly a right module is a $\mathbb{K}-\mathcal{D}$ bimodule with maps $\mu^{1|s} := \mu^{0|1|s}$—here we are using $\mathbb{K}$ to refer to the category with one object with endomorphisms $\mathbb{K}$.

A *differential graded* (dg) module or bimodule over a dg category corresponds to a differential graded functor into chain complexes, where concretely $\mu^{r|1|s} = 0$ unless $(k,l) = (0,0),(1,0),(0,1)$.

We use the notation

$$\text{RHom}_{\mathcal{C}}(\mathcal{P},\mathcal{Q}), \text{RHom}_{\mathcal{C}^{op}}(\mathcal{P},\mathcal{Q}), \text{RHom}_{\mathcal{C}-\mathcal{C}}(\mathcal{P},\mathcal{Q})$$

to refer to particular chain-level models for the (necessarily derived) models for morphism complexes of modules and bimodules built out of the *bar construction* (in a sense, this is built into the construction of these categories if $\mathcal{C}$ is an A_∞, not just a dg category).

Important canonical examples of bimodules include

- the *diagonal* $\mathcal{C}-\mathcal{C}$ *bimodule*
$$\mathcal{C}_\Delta(X, Y) := \hom_{\mathcal{C}}(Y, X) \tag{1.3}$$
with multiplication maps $\mu^{r|1|s} = \mu^{r+1+s}$, up to sign–see e.g., @@CITE)

- *Yoneda bimodules*: for any pair of objects $(K, L) \in \text{ob } \mathcal{C} \times \text{ob } \mathcal{D}$, define
$$Y_{K,L}(A, B) := \hom_{\mathcal{C}}(K, A) \otimes \hom_{\mathcal{D}}(B, L) \tag{1.4}$$

Similarly, we fix explicit chain-level models for the module and bimodule (derived) tensor products coming from the bar resolution:

$$\mathcal{P} \otimes^{\mathbb{L}}_{\mathcal{C}} \mathcal{Q}, \mathcal{P} \otimes^{\mathbb{L}}_{\mathcal{C}-\mathcal{C}} \mathcal{Q}$$

1.1.3 Hochschild chain and co-chain complexes

If $\mathcal{C}$ is an A_∞ (or dg) category and $\mathcal{B}$ is an A_∞ (or dg) bimodule, we use the notation

$$\mathrm{CH}_*(\mathcal{C}, \mathcal{B})$$

for the Hochschild chain complex of $\mathcal{C}$ with coefficients in $\mathcal{B}$, and

$$\mathrm{CH}^*(\mathcal{C}, \mathcal{B})$$

for the *Hochschild co-chain complex of* $\mathcal{C}$ *with coefficients in* $\mathcal{B}$. When $\mathcal{B}$ is the diagonal bimodule $\mathcal{C}_\Delta$, we often use the simplified notation

$$\begin{aligned} \mathrm{CH}_*(\mathcal{C}, \mathcal{C}) &:= \mathrm{CH}_*(\mathcal{C}, \mathcal{C}_\Delta) \\ \mathrm{CH}^*(\mathcal{C}, \mathcal{C}) &:= \mathrm{CH}^*(\mathcal{C}, \mathcal{C}_\Delta). \end{aligned} \tag{1.5}$$

In the former case, there is also an alternate complex which we will make use of called the *non-unital Hochschild chain complex*

$$\mathrm{CH}^{nu}_*(\mathcal{C}, \mathcal{C});$$

A well-understood deficiency of this explicit perspective is that... @@INSERT: FACT that hom from the diagonal bimodule doesn't equal the chosen Hochschild co-chain complex on the nose, but instead there are standard equivalences.

$$\mathrm{RHom}_{\mathcal{C}-\mathcal{C}}(\mathcal{C}_\Delta, \mathcal{B}) \cong \mathrm{CH}^*(\mathcal{C}, \mathcal{B}). \tag{1.6}$$

$$\mathcal{C}_\Delta \otimes_{\mathcal{C}-\mathcal{C}} \mathcal{B} \cong \mathrm{CH}_*(\mathcal{C}, \mathcal{B}). \tag{1.7}$$

1.1.4 Products of singular chains

It is convenient for many technical reasons to use singular chains instead of the cubical chains which have more commonly appeared in the Floer theory/string topology comparison literature @@CITE. To remedy the deficiency that products of singular simplices are not singular simplices we make use of natural associative *Eilenberg-Zilber maps*

$$C_i(X) \otimes C_j(Y) \to C_{i+j}(X \times Y),$$

generalizing the familier *prism operator*, which allow one to cut apart a product of singular simplices into a sum of singular simplices @@CITE-FELIX-Halperin-Thomas. Following @@CITE-Brown?, we will refer to the Eilenberg-Zilber map applied to an element $\alpha \otimes \beta$ simply as

$$\alpha \times \beta.$$

Using Eilenberg-Zilber maps, one can also associate an element of $C_{i_1+\cdots+i_k}(X)$, which we denote

$$sing(\sigma)$$

to any map of the form $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to X$.

More details on both constructions are provided in Appendix A.1.1.

2 The equivalence of the wrapped Fukaya category and the string topology category

The purpose of this chapter is to define the string topology category $\mathcal{S}_M$ and (twisted) wrapped Fukaya category $\mathcal{W}_b$, to construct a functor, extending Abouzaid's work [5], from the wrapped Fukaya category of conormal bundles of a cotangent bundle to the string topology category, $\mathcal{S}_M$, and to show that it is a quasi-equivalence.

2.1 The string topology category

In this section we introduce the string topology category $\mathcal{S}_M$, where M is a connected, closed, oriented n-manifold. We summarize constructions from [7] and we refer the reader to that paper for details.

The string topology category $\mathcal{S}_M$ was motivated by the construction of Sullivan [40] of "open string pairings" on the homology of path spaces. More specifically, let $N_1, N_2, N_3 \subset M$ be connected, closed, oriented submanifolds. Let $\mathcal{P}_{N_i,N_j} = \{\gamma \in Map([0,r], M)$ for some $r > 0$, so that $\gamma(0) \in N_i$, $\gamma(r) \in N_j\}$. The open string pairings Sullivan constructed were maps in homology,

$$\mu_* : H_*(\mathcal{P}_{N_0,N_1}) \otimes H_*(\mathcal{P}_{N_1,N_2}) \longrightarrow H_*(\mathcal{P}_{N_0,N_2})$$

These pairings have total degree shift of $n_1 = dim\, N_1$ and are associative in the obvious sense. In particular given a fixed submanifold $N \subset M$, the homology $H_*(\mathcal{P}_{N,N})$ inherits the structure of a graded algebra, once we account for the dimension shift.

There are three important examples of of these pairings that we would like to point out.

- Choose a basepoint $x_0 \in M$, Let $N = x_0$. Then $\mathcal{P}_{x_0 x_0}$ is the based loop space $\Omega_{x_0} M$, and the open string product in this case is the Pontrjagin product.
- Let $N = M$. Then the path space $\mathcal{P}_{M,M}$ is naturally homotopy equivalent to M, and the open string product in this case is the intersection product in $H_*(M)$.
- Start with a closed manifold N, and let $M = N \times N$. Consider the diagonal embedding, $\Delta : N \hookrightarrow N \times N$. In this case there is a natural homotopy equivalence between the path space and the free loop space, $\mathcal{P}_{\Delta(N),\Delta(N)} \simeq LN$, and the open string product on $H_*(\mathcal{P}_{\Delta(N),\Delta(N)}) \cong H_*(LN)$ is the original Chas-Sullivan (closed) string product [10].

In [7] these constructions were studied at the chain complex level, and the appropriate coherence issues were resolved. That is, the following theorem was proved, and the goal of this section is to review the constructions in that proof.

Theorem 3. *Let k be a field. There exists a DG-category (over k) $\mathcal{S}_M$ with the following properties:*

1. *The objects are $Ob_{\mathcal{S}_M} = \{$connected, closed, oriented submanifolds $N \subset M\}$*
2. *The morphism complex, $Mor_{\mathcal{S}_M}(N_1, N_2)$, is chain homotopy equivalent to the singular chains on the path space $C_*(\mathcal{P}_{N_1,N_2})$.*
3. *The compositions in $\mathcal{S}_M$ realize the open string topology operations on the level of homology.*
4. *The Hochschild homology of $\mathcal{S}_M$ is equivalent to the homology of the free loop space,*

$$HH_*(\mathcal{S}_M) \cong H_*(LM; k).$$

Note. In this theorem we construct a DG-category with strict compositions rather than an A_∞ category.

As a first observation, notice that there is a homotopy cartesian (pullback) square,

$$\begin{tikzcd} \mathcal{P}_{N_0,N_1} \arrow[r, "e_0"] \arrow[d, "e_r"'] & N_0 \arrow[d, "\cap"] \\ N_1 \arrow[r, "\subset"'] & M \end{tikzcd}$$

where e_0 and e_r are evaluation of a path $\gamma \in \mathcal{P}_{N_0,N_1}$ at 0 and r respectively.

Now given a connected subspace N of M, let F_N be the homotopy fiber of the inclusion $N \hookrightarrow M$. A model for F_N is $\mathcal{P}_{x_0,A}$ where $x_0 \in M$ is a fixed basepoint. F_A has an action of the based loop space ΩM given by juxtaposing the loop at x_0 with the path. In particular the chain complex $C_*(F_{N_i})$ can be given the structure of a DG-module over the DGA $C_*(\Omega M)$, and we then have chain homotopy equivalences,

$$\begin{aligned} k \otimes^L_{C_*(\Omega M)} C_*(F_N) &\simeq C_*(N) \\ k \otimes^L_{C_*(\Omega N)} C_*(\Omega M) &\simeq C_*(F_N). \end{aligned} \tag{2.1}$$

Here $\otimes^L$ denotes derived tensor product. A two-sided bar construction is always used as our model of this product.

Notice we then have a resulting homotopy cartesian square:

$$\begin{array}{ccc} \Omega M & \longrightarrow & F_{N_0} \\ \downarrow & & \downarrow \\ F_{N_1} & \longrightarrow & \mathcal{P}_{N_1,N_2}. \end{array}$$

An *Eilenberg-Moore* argument then yields a chain homotopy equivalence

$$C_*(\mathcal{P}_{N_1,N_2}) \simeq C_*(F_{N_1}) \otimes^L_{C_*(\Omega M)} C_*(F_{N_2}). \tag{2.2}$$

Note. In the above discussion nowhere was it used that M is a manifold or the $N_i's$ are submanifolds. Everything that was described so far applies to arbitrary connected spaces and subspaces. Next, however, we will need to use these manifold structures as we wish to apply a derived form of Poincaré duality.

More specifically, recall that for any connected space X, one has natural Eilenberg-Moore equivalences,

$$\begin{aligned} Tor_{C_*(\Omega X)}(k,k) &\cong H_*(X;k) \\ Ext_{C_*(\Omega X)}(k,k) &\cong H^*(X;k). \end{aligned}$$

(Here and below we will suppress grading.) Written on the level of chain complexes there are equivalences,

$$\begin{aligned} k \otimes^L_{C_*(\Omega X)} k &\simeq C_*(X;k) \\ Rhom_{C_*(\Omega X)}(k,k) &\simeq C^*(X;k) \end{aligned}$$

the later equivalence being one of DGA's. Now since M is a connected, closed, oriented manifold, then the fundamental class, $[M] \in Tor_{C_*(\Omega M)}(k,k) \cong H_*(M;k)$ is represented by a fundamental cycle $[M] \in k \otimes^L_{C_*(\Omega M)} k$. Classical Poincaré duality can be viewed as saying that the cap product gives a chain homotopy equivalence,

$$\cap[M] : Rhom_{C_*(\Omega M)}(k,k) \xrightarrow{\simeq} k \otimes^L_{C_*(\Omega M)} k.$$

It follows from the work of Dwyer-Greenlees-Iyengar [17] and Klein [26] [27] that indeed one has a chain equivalence,

$$\cap\, [M] : Rhom_{C_*(\Omega M)}(k,P) \xrightarrow{\simeq} k \otimes^L_{C_*(\Omega M)} P, \tag{2.3}$$

where P is *any* differential graded module over $C_*(\Omega M)$, that is bounded below. This is the "derived" form of Poincaré duality that we need.

We will then easily be able to conclude the following.

Lemma 4. *Let M be a closed, connected oriented manifold and N_0 aand N_1 closed, connected oriented submanifolds. There are natural "Poincaré duality equivalences",*

$$C_*(\mathcal{P}_{N_0,N_1}) \xrightarrow{\simeq} C_*(F_{N_1}) \otimes^L_{C_*(\Omega M)} C_*(F_{N_2}) \xrightarrow{\simeq} Rhom_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2}).$$

Proof.

$$\begin{aligned} C_*(\mathcal{P}_{N_1,N_2}) &\simeq C_*(F_{N_1}) \otimes^L_{C_*(\Omega M)} C_*(F_{N_2}) \quad \text{by (2.2)} \\ &\simeq k \otimes^L_{C_*(\Omega N_1)} C_*(F_{N_2}) \quad \text{by (2.1) and change of rings,} \\ &\simeq Rhom_{C_*(\Omega N_1)}(k, C_*(F_{N_2})) \quad \text{by (2.3) for } N_1, \\ &\simeq Rhom_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2})) \quad \text{by (2.1) and change of rings.} \end{aligned}$$

We remind the reader that in the above equivalences all gradings (and grading shifts) are suppressed. Further, it is worth emphasizing that it is Poincaré duality for N_1 that is used in these equivalences. □

By using cofibrant-fibrant replacements of $C_*(F_{N_i})$, e.g., [23, §7], [36, 39]), we can regard the derived homomorphism complexes as possessing a strict composition pairing. This observation gives rise to the definition of the string topology category.

Definition 1. *Let M be a connected, closed, oriented manifold with fixed basepoint $x_0 \in M$. The string topology category $\mathcal{S}_M$ has as*

1. *Objects the pairs (N, F_N), where N is a connected, closed, oriented submanifold $N \subset M$ and F_N is a specific choice of model for the homotopy fiber of $N \to M$ with an action of ΩM.*

2. *Morphisms from N_1 to N_2 the derived homomorphism complex*

$$Rhom_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2})),$$

computed via functorial cofibrant-fibrant replacement of $C_*(F_{N_i})$.

In other words, $\mathcal{S}_M$ is the full subcategory of the DG-category of differential graded modules over $C_*(\Omega M)$ with objects cofibrant-fibrant replacements of $C_*(F_N)$ for $N \subset M$ a submanifold as above.

Notice that if $N = x_0 \in M$ is the basepoint, then the endomorphism algebra $End_{\mathcal{S}_M}(x_0) = C_*(\Omega M)$. The inclusion of categories $End_{\mathcal{S}_M}(x_0) \hookrightarrow \mathcal{S}_M$ defines an induced map in Hochschild homologies,

$$\iota_* : HH_*(\Omega M) \to HH_*(\mathcal{S}_M).$$

Goodwillie's result [24] states that the left hand side is naturally equivalent to $H_*(LM)$. Thus the Hochschild homology statement in Theorem 3 then would follow from knowing that ι_* is an isomorphism. This is a Morita equivalence result, which is a straightforward consequence of the general theory developed in [8] and the fact that the thick closure of $\mathcal{S}_M$ inside the category of $C_*(\Omega M)$-modules is the entire category of finite $C_*(\Omega M)$-modules.

2.2 The wrapped Fukaya category

The wrapped Fukaya category of an exact symplectic manifold was first constructed in [6], following a cohomology level construction and proposal of [16] (this in turn was based on a construction of wrapped Floer homology in the special case of a cotangent fibre [1]). The technical details of several variant constructions are by now well known in the symplectic literature. We give a brief overview of the construction described in [4].

Is it worth stating that this is needed for the later theorem, or is this already implicit?

2.2.1 Liouville manifolds

The wrapped Fukaya category of [4] is defined for a *Liouville manifold* (X^{2n}, λ), that is a manifold X with one form λ such that $d\lambda = \omega$ is a symplectic form, such that X is modeled outside a compact region $\bar{X}$ on the semi-infinite symplectization of a contact manifold

$$X \backslash \bar{X} = (\partial \bar{X} \times [1, +\infty)_r, r\lambda_{\partial \bar{X}}) \tag{2.4}$$

where the *Liouville vector field* Z, (the symplectic dual to λ, defined by $i_Z \omega = \lambda$), is transverse to $\partial \bar{X} \times \{1\}$ and acts on (2.4) by translation proportional to the coordinate r: $Z|_{X\backslash \bar{X}} = r\partial_r$.

The time $\log(\rho)$ flow of Z is called the *Liouville flow* and denoted ψ^ρ (note that explictly it sends an element $(r, m) \in X\backslash \bar{X}$ to $(\rho \cdot r, m)$). We fix a representation of X of the form (2.4), also called a *conical structure*. The contact manifold $(\partial \bar{X}, \bar{\lambda} = \lambda|_{\partial X})$ has a canonically associated *Reeb vector field* R, defined by the requirements that $d\bar{\lambda}(R, \cdot) = 0$, and $\bar{\lambda}(R) = 1$. We assume λ has been chosen generically so that all Reeb orbits of $\bar{\lambda}$ are non-degenerate. In order to obtain $\mathbb{Z}$-graded theories (though this is not strictly necessary for our discussion), we fix some additional data:

$$\text{fix a trivialization of } (\Lambda^n_{\mathbb{C}} T^* X)^{\otimes 2}. \tag{2.5}$$

Following [5] we will consider *sign-twisted theories*, which require an extra choice:

$$\text{fix an element } b \in H^2(X, \mathbb{Z}_2), \text{ referred to as a } \textbf{background class}, \tag{2.6}$$

(the choice $b = 0$ recovers the untwisted versions of these theories). For simplicity, we can assume b is the second Stiefel-Whitney class of a vector bundle $b = w_2(E_b)$ (such an E_b can at least be chosen on the 3-skeleton of X, as in [**?**, Ch. 9]—which is sufficient for defining signed operations).

Consider a class of *quadratic Hamiltonians*, those which are quadratic in the conical region:

$$\mathcal{H}(X) = \{H \in C^\infty(X, \mathbb{R}), H|_{X \setminus \bar{X}}(r, y) = r^2\}, \tag{2.7}$$

Note: this class of complex structures is a slight generalization of the two slightly modified classes that have appeared; written this way for simplicity.

and a class of almost-complex structures $\mathcal{J}(X)$ that are *contact type* on the conical end, meaning that for a positive function $f(r)$ with non-negative derivative,

$$\lambda \circ J = f(r)dr. \tag{2.8}$$

2.2.2 (wrapped) Lagrangian Floer homology

Definition 2.1. *An* admissible Lagrangian brane *consists of an exact properly embedded Lagrangian submanifold $L \subset X$, satisfying*

$$w_2(L) = b|_L \text{ (if } b = 0,\ L \text{ is Spin);} \tag{2.9}$$

$$2c_1(M, L) = 0, \text{ where } c_1(X, L) \in H^2(X, L) \text{ is the relative first Chern class; and} \tag{2.10}$$

$$\lambda|_L \text{ vanishes away from a compact set,} \tag{2.11}$$

and equipped with the following extra data: a primitive $f_L : L \to \mathbb{R}$ for $\lambda|_L$, *an* orientation on L, *a* Spin structure on $TL \oplus (E_b)|_L$, *and a* grading.

Let ob $\mathcal{W}$ be a finite collection of admissible Lagrangian branes, henceforth simply referred to as Lagrangians. Assume that λ has been chosen further generically so that all Reeb chords between Lagrangians in ob $\mathcal{W}$ are non-degenerate. For any pair $L_0, L_1 \in$ ob $\mathcal{W}$, define $\chi(L_0, L_1)$ to be the set of time 1 Hamiltonian flows of H between L_0 and L_1, where $H \in \mathcal{H}(X)$ is our fixed choice of Hamiltonian from §3.3.2. Using the extra data chosen for X in §3.3.2 and for each L_i above, the *Maslov index*, denoted

$$\deg : \chi(L_0, L_1) \to \mathbb{Z}. \tag{2.12}$$

defines an absolute grading on $\chi(L_0, L_1)$. Given $x \in \chi(L_0, L_1)$, the grading structure gives an associated path Λ_x of Lagrangians subspaces in $x^*(TX)$, unique up to homotopy relative endpoints and from here index theory associates a real one-dimensional vector space, the *orientation line* o_x (see [37]*§11h-l for more details on both constructions). For any coefficient field $\mathbb{K}$ the $\mathbb{K}$-*normalization*

$|o_x|_{\mathbb{K}}$ is the $\mathbb{K}$-vector space generated by the two orientations on o_x, modulo the relation that the sum of the orientations is zero [37]*§12e (for $\mathbb{K} = \mathbb{Z}/2$, that $|o_x|_{\frac{\mathbb{Z}}{2\mathbb{Z}}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$ canonically). Choose a *relative Spin structure* for each such x, which is the data of

$$\begin{aligned}&\text{a Spin structure on } \Lambda_x \oplus x^*(E_b) \text{ restricting at endpoints to the chosen}\\ &\text{Spin structure on } T_{x(i)}L_i \oplus E_b.\end{aligned} \tag{2.13}$$

Given a family of almost complex structures $J_t \in \mathcal{J}_1(M)$ parametrized by $t \in [0,1]$, the the *wrapped Floer co-chain complex* over $\mathbb{K}$ is the graded vector space

$$\hom^i_{\mathcal{W}}(L_0, L_1) := CW^i(L_0, L_1, H, J_t) := \bigoplus_{x \in \chi(L_0,L_1), deg(x)=i} |o_x|_{\mathbb{K}}. \tag{2.14}$$

equipped with a differential $d = \mu^1 : CW^*(L_0, L_1; H, J_t) \longrightarrow CW^*(L_0, L_1; H, J_t)$ whose elementary linear pieces

$$\begin{aligned} d &:= \bigoplus_{x_0,x_1} N_{x_1,x_0} \\ N_{x_1,x_0} &: |o_{x_1}|_{\mathbb{K}} \to |o_{x_0}|_{\mathbb{K}}, \end{aligned} \tag{2.15}$$

each involving (signed) counts of solutions of Floer-type PDEs with given asymptotics. Specifically, given a pair of time 1 chords $x_0, x_1 \in \chi(L_0, L_1)$, define $\tilde{\mathcal{R}}^1(x_0; x_1)$ to be the set of finite-energy maps $u : (-\infty, \infty) \times [0,1] \to M$ satisfying Floer's equation for J_t

$$(du - X \otimes dt)^{0,1} = 0, \tag{2.16}$$

with boundary conditions and asymptotics

$$\begin{cases} u(s,i) \in L_i, \ i = 0,1 \\ \lim_{s \to -\infty} u(s,t) = x_1 \\ \lim_{s \to +\infty} u(s,t) = x_0. \end{cases}$$

For generic J_t, the moduli space $\tilde{\mathcal{R}}^1(x_0; x_1)$ a transversely cut out manifold of dimension $\deg(x_0) - \deg(x_1)$. The induced $\mathbb{R}$ action by translation in the s direction is free unless the dimension is zero, so one passes to the quotient:

$$\mathcal{R}^1(x_0; x_1) := \begin{cases} \tilde{\mathcal{R}}^1(x_0; x_1)/\mathbb{R} & \deg(x_0) > \deg(x_1) \\ \emptyset & \deg(x_0) = \deg(x_1) \end{cases} \tag{2.17}$$

The Morse-Floer type compactification $\overline{\mathcal{R}}^1(x_0; x_1)$ enlarges $\mathcal{R}^1(x_0; x_1)$ by adding in *broken strips* $\overline{\mathcal{R}}^1(x_0; x_1) = \coprod \mathcal{R}^1(x_0; y_1) \times \mathcal{R}^1(y_1; y_2) \times \cdots \times \mathcal{R}(y_k; x_1)$. An important *no escape Lemma* verified in [4]*§B (following [6]*Lem. 7.2) establishes, using the specific profile of the Hamiltonian, almost complex structure, and Lagrangians near ∞, an a priori bound on the images of holomorphic curves in the target only depending on x_1 (in particular $\mathcal{R}^1(x_0; x_1)$ is empty for all but finitely many x_0). From here, standard compactness and gluing results establish that the moduli space $\overline{\mathcal{R}}^1(x_0, x_1)$ is a compact manifold with boundary of dimension $\deg(x_0) - \deg(x_1) - 1$.

For rigid elements $u \in \mathcal{R}^1(x_0; x_1)$, if $\deg(x_0) = \deg(x_1) + 1$, gluing theory for determinant lines gives an induced isomorphism (depending on b):

$$\mu_u : o_{x_1} \longrightarrow o_{x_0}. \tag{2.18}$$

(see [37]*§12f, [43], or [5]*§A.1 for an exposition of the case with non-trivial b): Define the (x_1, x_0) component of the differential (2.15) to be 0 unless $\deg(x_0) = \deg(x_1) + 1$, in which case

$$N_{x_1,x_0}([x_1]) = \sum_{u \in \overline{\mathcal{R}}^1(x_0;x_1)} (-1)^{\deg(x_1)} \cdot \mu_u([x_1]) \tag{2.19}$$

where above μ_u is the induced morphism on $\mathbb{K}$-normalizations. A standard argument codimension 1 boundary argument implies that $d^2 = 0$. The resulting group is called the *wrapped Floer cohomology*

$$HW^*(L_0, L_1) = H^*(\hom(L_0, L_1)). \tag{2.20}$$

2.2.3 The A_∞ category

For $d \geq 2$, denote by

$$\mathcal{R}^d \tag{2.21}$$

the moduli space of discs with $d+1$ marked points modulo reparametrization, with one point z_0^- marked as negative and the remainder $z_1^+, \ldots, z_d^+$ (labeled counterclockwise from z_0^-) marked as positive.

(2.21), a quotient, can be identified with the space of unit discs with z_0^-, z_1^+, and z_2^+ in fixed position; the positions of the remaining ordered points identify $\mathcal{R}^d$ with an open subset of $\mathbb{R}^{d-2}$. Following [37]*12g, we orient $\mathcal{R}^d$ by pulling back the $\mathbb{R}^{d-2}$ orientation $dz_2 \wedge \cdots \wedge dz_{d-2}$. The (Deligne-Mumford) compactification

$$\overline{\mathcal{R}}^d, \tag{2.22}$$

a manifold with corners, consists of trees of stable discs with a a total of d exterior positive marked points and 1 exterior negative marked point, modulo compatible reparametrization of each disc in the tree; a stratum with s discs has codimension $s - 1$.

In order to define operations on wrapped Floer co-chain complexes using such domains, there is a helpful *rescaling trick* proposed in [16] and applied in [4]: one notes that pullback of solutions to (2.16) by the Liouville flow for time $\log(\rho)$ defines a canonical identification

$$CW^*(L_0, L_1; H, J_t) \simeq CW^*\left(\psi^\rho L_0, \psi^\rho L_1; \frac{H}{\rho} \circ \psi^\rho, (\psi^\rho)^* J_t\right). \tag{2.23}$$

The right hand object is equivalently the Floer complex for $(\psi^\rho L_0, \psi^\rho L_1)$ for a strip with one form ρdt using Hamiltonian $\frac{H}{\rho^2} \circ \psi^\rho$ and $(\psi^\rho)^* J_t$. Then one observes that (c.f., [4]*Lemma 3.1) for any ρ, the function $\frac{H}{\rho^2} \circ \psi^\rho$ lies in $\mathcal{H}(M)$, i.e., equals r^2 at ∞.

Need to finish a sentence here.

Recall that we have fixed a single Hamiltonian H and time-dependent almost complex structure J_t such that $CW^*(L_i, L_j, H, J_t)$ is defined for every $L_i, L_j \in \text{ob}\, \mathcal{W}$. We use the following symbols to refer to the (positive and negative) semi-infinite strips:

$$Z_+ := [0, \infty) \times [0, 1] \tag{2.24}$$

$$Z_- := (-\infty, 0] \times [0, 1] \tag{2.25}$$

Definition 2.2. *A* **Floer datum** $\mathbf{F}_S$ *on a stable disc* $S \in \overline{\mathcal{R}}^d$ *consists of the following choices on each component:*

1. *A collection of* **strip-like ends** $\mathfrak{S}$*; that is maps* $\epsilon_k^\pm : Z_+ \to S$ *all with disjoint image in* S*. These should be chosen so that positive strips map to neighborhoods of positively-labeled boundary marked points, and similarly for negative marked points.*
2. *For each strip-like end* $\epsilon_k^\pm$*, a real number, called a* **weight** w_k.
3. **closed 1-form***: a one-form* α_S *satisfying* $d\alpha_S = 0$*, and* $(\alpha_S)|_{\partial S} = 0$*, restricting to* $w_k dt$ *on a strip-like end with weight* w_k*:* $(\epsilon_k^\pm)^* \alpha_S = w_k dt$.
4. *A* **Hamiltonian** $H_S : S \to \mathcal{H}(M)$*, restricting to* $\frac{H}{w_k^2} \psi^{w_k}$ *on a strip-like end with weight* w_k*:* $(\epsilon_k^\pm)^* H_S = \frac{H}{w_k^2} \circ \psi^{w_k}$.
5. *A* **boundary-shifting map** $\rho : \partial \bar{S} \backslash \{\textit{marked points}\} \to (0, \infty)$ *satisfying* $(\epsilon_k^\pm)^* \rho = w_k$.
6. *An* **almost-complex structure** J_S *satisfying* $(\epsilon_k^\pm)^* J_S = (\psi^{w_k})^* J_t$.

Definition 2.3. *A pair of Floer data* $\mathbf{F}_S^1$*,* $\mathbf{F}_S^2$ *on* $S \in \overline{\mathcal{R}}^d$ *are said to be* **conformally equivalent** *if for some constant* K*,*

- *the strip-like ends coincide,*
- $\alpha_1 = K \alpha_2$*,*
- $\rho_1 = K \rho_2$
- $H_1 = \frac{H_2}{K} \circ \psi^K$*, and*
- $J_1 = (\psi^K)^* J_2$.

Definition 2.4. *A* **consistent choice of Floer data for the** A_∞ **structure** *is a(n inductive) choice of Floer data, for each* $d \geq 2$ *and for each representative* S *of* $\overline{\mathcal{R}}^d$*, smoothly varying in* S*, whose restriction to each boundary stratum is conformally equivalent to the product of Floer data coming from lower-dimensional spaces. With respect to the boundary gluing charts, the Floer data should agree to infinite order at boundary strata with Floer data obtained via gluing.*

Consistent choices of Floer data for the A_∞ structure exist by an inductive construction, as at each level there is a contractible space of choices consistent with choices on lower strata. Fixing one such choice, given objects $L_0, \dots, L_d$ of $\mathcal{W}$, and a sequence of input chords $\vec{x} = \{x_k \in \chi(L_{k-1}, L_k)\}$ as well as an output chord $x_0 \in \chi(L_0, L_d)$, write

$$\mathcal{R}^d(x_0; \vec{x}) \tag{2.26}$$

for the space of maps $u : S \to M$ with source an arbitrary element $S \in \mathcal{R}^d$, satisfying moving boundary conditions and asymptotics

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_k & \text{if } z \in \partial S \text{ lies between } z^k \text{ and } z^{k+1} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k \end{cases} \tag{2.27}$$

and solving a version of Floer's equation

$$(du - X_S \otimes \alpha_S)^{0,1} = 0 \tag{2.28}$$

with respect to the choice of complex structure J_S and Hamiltonian H_S.

The consistency of our Floer data with respect to the codimension one boundary of the abstract moduli spaces $\overline{\mathcal{R}}^d$ implies that the (Gromov-type) compactification $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is obtained by adding the images of the natural inclusions

$$\overline{\mathcal{R}}^{d_1}(x_0; \vec{x}_1) \times \overline{\mathcal{R}}^{d_2}(y; \vec{x}_2) \to \overline{\mathcal{R}}^d(x_0; \vec{x}) \tag{2.29}$$

where y agrees with one of the elements of $\vec{x}_1$ and $\vec{x}$ is obtained by removing y from $\vec{x}_1$ and replacing it with the sequence $\vec{x}_2$. Here, we let d_1 range from 1 to d, with $d_2 = d - d_1 + 1$, with the stipulation that $d_1 =$ or $d_2 = 1$ is the semistable case $\overline{\mathcal{R}}^1(x_0; x_1)$ constructed previously.

Lemma 2.1. *For a generically chosen Floer data $\mathbf{D}_\mu$, the moduli space $\overline{\mathcal{R}}^d(x_0; \vec{x})$ is a smooth compact manifold of dimension* $\deg(x_0) + d - 2 - \sum_{1 \le k \le d} \deg(x_k)$, *and for fixed $\vec{x}$ is empty for all but finitely many x_0.*

As before, the most non-trivial part of this Lemma is establishing a *no escape Lemma* as described for $\mathcal{R}^1(x_0; x_1)$ (using the form H and J near infinity and sub-closedness of α); after which one applies standard methods e.g., [37]*(9k), (11h), Prop. 11.13.

If $\deg(x_0) = 2 - d + \sum_1^d \deg(x_k)$ the elements of $\overline{\mathcal{R}}^d(x_0; \vec{x})$ are rigid. By [37]*(11h), (12b),(12d), having fixed an orientation of $\mathcal{R}^d$, any rigid element $u \in \overline{\mathcal{R}}^d(x_0; \vec{x})$ determines an isomorphism of orientation lines (again using the background class b)

$$\mathcal{R}^d_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \longrightarrow o_{x_0}. \tag{2.30}$$

One defines the dth A_∞ operation

$$\mu^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_d) \tag{2.31}$$

as a sum

$$\mu^d([x_d], \dots, [x_1]) := \sum_{\deg(x_0)=2-d+\sum \deg(x_k)} \sum_{u \in \overline{\mathcal{R}}^d(x_0;\vec{x})} (-1)^{\bigstar_d} \mathcal{R}_u^d([x_d], \dots, [x_1]) \tag{2.32}$$

where the sign is given by

$$\bigstar_d = \sum_{i=1}^{d} i \cdot \deg(x_i) \tag{2.33}$$

(note that this sum is finite by Corollary 2.1). An analysis of the codimension 1 boundary of 1-dimensional moduli spaces along with their induced orientations as in [37]*Prop. 12.3 establishes the maps μ^d satisfy the A_∞ relations.

2.2.4 Cotangent bundles and their conormal bundles

Let M be a smooth oriented manifold of dimension n, and let

$$X = (T^*M, -qdp)$$

denote its cotangent bundle, with canonical one form $\lambda = -qdp$ (shorthand for $\sum_{i=1}^n -q_i dp_i$ in local coordinates). In a coordinate free fashion, this is the one form which, at a point $s = (q, v^*)$ in T^*M, is $v^* \circ d\pi : T_{q,v^*}T^*M \to T_qM \to \mathbb{R}$. The associated Liouville vector field is $p\partial_p$. Picking a metric on M, we obtain a conical structure by setting $\bar{X}$ to be the unit disc bundle, and conical coordinate $r = |p|$. The contact form $\bar{\lambda} = -\frac{q}{|q|}dp$. We fix on X a holomorphic volume form giving the data (2.5) by complexifying a volume form from the base. We also once and for all set background class b equal to the pullback of the second Stiefel-Whitney class from the base:

$$b = \pi^* w_2(M). \tag{2.34}$$

Note that b is w_2 of the vector bundle E_b on T^*M given by pulling back TM from the base:

$$E_b = \pi^*(TM) \tag{2.35}$$

(note that if M is Spin, $b = 0$).

The cotangent bundle $X = T^*M$ contains many natural Lagrangians, for instance the zero section M and cotangent fibres T_q^*M. More generally, for any (oriented) submanifold $N \subset M$ the *conormal bundle*

$$L_N := T_N^*M := \{(q,p) \in T^*M | q \in N, p(v) = 0 \text{ for } v \in T_qN\} \tag{2.36}$$

is a Lagrangian submanifold (the cases $N = M, q$ recover M and T_q^*M respectively). We will recall how L_N can be canonically equipped with structures to make it into an *admissible Lagrangian brane* in the sense of Definition 2.1. First, note that for a conormal bundle the Liouville form satisfies $\lambda|_{L_N} = 0$, so there is a canonical choice of primtive $f_{L_N} = 0$. L_N also admits a canonical grading.

Lemma 2.2 (See e.g., [34] Prop. 5.3.6)**.** *$L_N = T^*_N M$ admits a canonical orientation and Spin structure relative the background class $b = \pi^* w_2(M)$.*

Proof. The inclusion $N \hookrightarrow L_N$ is a homotopy equivalence, so suffices to show that $TL_N|_N$ admits a canonical orientation and Spin structure relative to $b = \pi^* w_2(M)$. The main claim is that using a fixed ambient metric, there is an identification

$$(TL_N)|_N \cong (E_b)|_N = (TM)|_N; \tag{2.37}$$

hence $(TL_N)|_N \oplus (E_b)|_N \cong (E_b)|_N \oplus (E_b)|_N$, so one can choose the orientation on L_N coming from the symplectic form ω, and $w_2(V|_N) = 2w_2((TL_N)_N) = 0$. Finally, we note that the any vector bundle of the form $W \oplus W$, where W is oriented, admits a *canonical Spin structure*, coming from the canonical lift of the diagonal embedding $SO(W) \to SO(W \oplus W)$ to $\mathrm{Spin}(W \oplus W)$ (see e.g., [**?**]*§6.3 or [43]*(32)).

Proof of main claim was removed, because it seems straightforward at this point, and was space inefficient.

□

Going forward, we use L_N to refer to (2.36) equipped with its canonical brane structure.

2.3 The comparison functor

For the remainder of the paper, we specialize to $X = T^*M$, and fix background class $b \in H^2(X, \mathbb{Z}_2)$ equal to the pullback of the second Stiefel-Whitney class $w_2(M)$. We will denote the associated wrapped Fukaya category with any finite collection of objects by $\mathcal{W}_b$. Let $\mathcal{N} = \{N_i\}$ be a (finite) collection of oriented submanifolds of M including the basepoint q, and let $\mathcal{S}_M$ the string topology category with objects $\mathcal{N}$. Correspondingly, let $\mathcal{W}_b^{conor} := \mathcal{W}_b^{conor}(T^*M)$ be the sub-category of $\mathcal{W}_b$ consisting of conormal bundles of elements of $\mathcal{N}$, equipped with their canonical brane structures (see §2.2.4). Since $q \in \mathcal{N}$, $T^*_q M \in \mathrm{ob}\ \mathcal{W}_b^{conor}(T^*M)$.

Using the work of Abouzaid [5], modified with some additional Floer theory, Morse theory, and homological algebra, we construct the equivalence in Theorem 1 (we do not yet compare the smooth Calabi-Yau structures):

Theorem 2.1. *There exists an A_∞ quasi-isomorphism*

$$\mathbf{F} : \mathcal{W}_b^{conor}(T^*M) \xrightarrow{\sim} \mathcal{S}_M \tag{2.38}$$

We give an overview here, using the material developed in the following subsections. First, we will repeatedly make use of the following Lemma in A_∞-homological algebra in order to construct A_∞ functors.

Lemma 2.3. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are A_∞ categories each admitting functors into a third category $\mathcal{E}$*

$$F : \mathcal{C} \to \mathcal{E} \tag{2.39}$$

$$G : \mathcal{D} \to \mathcal{E} \tag{2.40}$$

with equivalent cohomological essential images (meaning every object in the image of F is isomorphic in $H^0(\mathcal{E})$ to an object in the image of G and vice versa). Moreover suppose that G is cohomologically full and faithful. Then there is an A_∞ functor $H : \mathcal{C} \to \mathcal{D}$ such that $G \circ H \simeq F$, which is a quasi-equivalence if and only if F is cohomologically full and faithful.

Proof. Denoting by $\tilde{\mathcal{E}}$ the full subcategory of $\mathcal{E}$ with objects in the image of F or G, $G : \mathcal{D} \to \tilde{\mathcal{E}}$ is a quasi-equivalence. By [37]*Thm. 2.9, there is a (homotopy-)inverse quasi-isomorphism to G, $G^{-1} : \tilde{\mathcal{E}} \to \mathcal{D}$. The desired A_∞ functor is the composition of G^{-1} with F. $\square$

The relevance of Lemma 2.3 is as follows. Let $\mathcal{W}'$ the wrapped Fukaya category of a any collection of Lagrangians which are all transverse to the zero section. Abouzaid's work [5] essentially constructs a cohomologically full and faithful embedding

$$\mathbf{P} : \mathcal{W}' \to perf(C_{-*}(\Omega_q M)) \tag{2.41}$$

from any such category $\mathcal{W}'$ to perfect $C_{-*}(\Omega_q M)$ modules (strictly speaking, the target of Abouzaid's functor naturally embeds in into perfect $C_{-*}(\Omega_q M)$ modules). Since the string topology category $\mathcal{S}_M$ is naturally a full and faithful subcategory of $perf(C_{-*}(\Omega_q M))$, by Lemma 2.3 Theorem 2.1 requires showing that $\mathbf{P}$ has the modules $C_{-*}(\mathcal{P}_{N,q})$ in its essential image. More precisely, we prove Theorem 2.1 by applying Lemma 2.3 twice to the following Proposition:

Proposition 2.1. *For every conormal bundle $L_N \in \mathcal{W}_b^{conor}$, there is an object $\tilde{L}_N$ of the wrapped Fukaya category which is transverse to the zero section, such that*

(i) $\tilde{L}_N$ differs from L_N by a compactly supported Hamiltonian perturbation.

(ii) The object $\mathbf{P}(\tilde{L}_N)$, a perfect $C_{-*}(\Omega_q M)$ module, is isomorphic in $H^0(perf(C_{-*}(\Omega_q N)))$ to the dg module $C_{-*}(\mathcal{P}_{N,q})$ (which is the corresponding object of $\mathcal{S}_M$).

The relevance of finding $\tilde{L}_N$ is that Abouzaid's functor only applies to Lagrangians transverse to the zero section. Assuming Proposition 2.1 for now, we give a proof of Theorem 2.1:

Proof of Theorem 2.1. Let $\tilde{\mathcal{W}}_b^{conor}$ the sub-category of the wrapped Fukaya category with objects a choice of $\tilde{L}_N$ satisfying Prop. 2.1 for each $N \in \mathcal{N}$. By standard results about Fukaya categories property (i) of Proposition 2.1 implies that L_N and $\tilde{L}_N$ are isomorphic in $H^0(\mathcal{W})$ (see e.g., [37]*(8c)). Applying Lemma 2.3 to the pair of inclusions

$$\begin{tikzcd} \mathcal{W}_b^{conor} \arrow[dr] & \\ & \mathcal{W} \\ \tilde{\mathcal{W}}_b^{conor} \arrow[ur] & \end{tikzcd} \tag{2.42}$$

therefore produces an A_∞ quasi-equivalence $f : \mathcal{W}^{conor}_b \simeq \tilde{\mathcal{W}}^{conor}_b$ sending L_N to $\tilde{L}_N$.

Next, property (ii) of Proposition 2.1 implies that the pair

$$\begin{tikzcd}
\tilde{\mathcal{W}}^{conor}_b \arrow[dr, "\mathbf{P}"] & \\
& perf(C_{-*}(\Omega_q M)) \\
\mathcal{S}_M \arrow[ur] &
\end{tikzcd} \tag{2.43}$$

satisfy the hypotheses of Lemma 2.3, namely that $\mathbf{P}$ on $\tilde{\mathcal{W}}^{conor}_b$ has the modules $C_{-*}(\mathcal{P}_{N,q})$ in its essential image. Applying Lemma 2.3 produces a functor $g : \tilde{\mathcal{W}}^{conor}_b \to \mathcal{S}_M$ sending $\tilde{L}_N$ to N. The desired functor $\mathbf{F}$ is the composition $g \circ f$.

Finally, we show $\mathbf{F}$ is a quasi-equivalence. Since f is a quasi-equivalence, by Lemma 2.3 this amounts to $\mathbf{P}$ being cohomologically full and faithful, which was shown by Abouzaid [3] [5]. A summary of the argument is the following: First, $\mathbf{P}$ restricted to the subcategory with object $T^*_q M$ is cohomologically full and faithful by [5]*Lem. 5.1 (which shows the induced map on cohomology groups is one-sided inverse to Abbondandolo-Schwarz's isomorphism [1]). Next, [3]*Thm 1.1 proves that $T^*_q M$ generates $\mathcal{W}$ (and hence $\mathcal{W}^{conor}_b$). By general properties of A_∞ functors CITE, $\mathbf{P}$ must be cohomologically full on $\mathcal{W}^{conor}_b$. □

The next two sections will be devoted to the proof of Proposition 2.1.

2.3.1 Morse models in the string topology category

In order to establish (ii) of Proposition 2.1, for any oriented submanifold N, we use Morse theory to construct a $C_{-*}(\Omega_q M)$ module $\mathcal{M}'_N \in \mathrm{ob}\, perf(C_{-*}(\Omega_q))$, which is naturally quasi-isomorphic as $C_{-*}(\Omega_q M)$ modules to $C_{-*}(\mathcal{P}_{N,q})$. In the subsequent section, we choose a perturbation $\tilde{L}_N$ as in Proposition 2.1 (i) whose image under the functor $\mathbf{P}$ coincides with $\mathcal{M}'_N$, therby reducing (ii) to the following Proposition 2.2 established here:

Proposition 2.2. *For $N \subset M$, the data of a Morse pair (f, g) of a metric g on N and a Morse-Smale (with respect to g) function $f : N \to \mathbb{R}$ determines a a $C_{-*}(\Omega_q M)$ module $\mathcal{M}'_N$ to (N, f, g) built out of $C_{-*}(\Omega_{q^i,q} M)$, for $q^i \in \mathrm{crit}(f)$, as well as a quasi-isomorphism*

$$\rho_{f,N} : \mathcal{M}'_N \xrightarrow{\sim} C_{-*}(\mathcal{P}_{N,q}) \tag{2.44}$$

of $C_{-*}(\Omega_q M)$ modules.

For a pair of points p, q, denote by

$$\Omega_{p,q}(M) \tag{2.45}$$

the space of Moore paths in M from p to q. Singular chains (cohomologically graded) on the spaces (2.45) are morphism spaces from p to q in the *path category* of M

$$\mathcal{P}(M); \tag{2.46}$$

the allowable objects are arbitrary points in M. Via the Pontryagin product induced by composition of paths, $\mathcal{P}(M)$ is a dg category. We think of $\mathcal{P}(M)$ as a special case of an A_∞ category via the convention $\mu^1 = d$, $\mu^2(a,b) = (-1)^{|a|} a \cdot b$. Observe that $\hom(q,q) = C_{-*}(\Omega_q M)$, and that compositions in $\mathcal{P}(M)$ extend the usual product on based loops. As a special case of the full category structure, we notice that $C_{-*}(\Omega_{q,p}(M))$ is naturally a *left dg module* over $C_{-*}(\Omega_q M)$; we once more view this as a special case of a *left A_∞ module* by using the structure map

$$\mu^2 : C_{-*}(\Omega_{q,q}(M)) \otimes C_{-*}(\Omega_{q,p}(M)) \to C_{-*}(\Omega_{q,p}(M)) \tag{2.47}$$

(with its associated sign twist) to define the composition.

Let $\{p_i\}_{i=1}^m$ denote the critical points of f, ordered by *action* (value of f), and define

$$\overline{\mathcal{M}}(q^i, q^j) \tag{2.48}$$

to be the compactified space of gradient (Morse) trajectories of $-f$ from p^i to p^j with respect to the metric g . For generic (f,g) this is a compact manifold with boundary of dimension $|q^i| - |q^j|$. need to quotient by $\mathbb{R}$
Pick a family of chains for (2.48) whose existence is guaranteed by the following Lemma:

Lemma 2.4. *There is an inductive choice fundamental chains for* (2.48) *in singular homology satisfying*

$$\partial[\overline{\mathcal{M}}(p^i, p^k)] = \sum_j (-1)^* [\overline{\mathcal{M}}(p^i, p^j)] \times [\overline{\mathcal{M}}(p^j, p^k)]. \tag{2.49}$$

insert signs

By associating to any gradient trajectory the corresponding (compactified) path from p^i to p^j, using arc length with respect to the background metric g to fix parametrisations, there is an evaluation map

$$\mathrm{ev} : \mathcal{M}(p^i, p^k) \to \Omega_{p^i, p^k} N. \tag{2.50}$$

Since the arc length parametrizations of a sequence of Morse trajectories converge to the arc length parametrization of any limit broken trajectory,

Lemma 2.5. ev *extends continuously to compactifications, in a manner compatible with boundary strata; that is, there is a commutative diagram*

$$\begin{tikzcd} \overline{\mathcal{M}}(p^i, p^j) \times \overline{\mathcal{M}}(p^j, p^k) \arrow[r] \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] & \overline{\mathcal{M}}(p^i, p^k) \arrow[d, "\mathrm{ev}"] \\ \Omega_{p^i, p^j} N \times \Omega_{p^j, p^k} N \arrow[r] & \Omega_{p^i, p^k} N \end{tikzcd} \tag{2.51}$$

where the top horizontal arrow is the inclusion of boundary strata and the bottom one is concatenation of Moore paths.

Using this, define the following dg module over $C_{-*}(\Omega_q M)$:

$$\mathcal{M}'_N := \left(\bigoplus_i C_{-*+|p^i|}(\Omega_{q,p^i} M), \bigoplus_i \mu^1_i \oplus \bigoplus_{i,j} \mu^2(\mathrm{ev}_*([\mathcal{M}(p^i,p^j)]), -) \right), \tag{2.52}$$

change μ^2 to Pontryagin composition

where $\mu^1_i : C_{-*}(\Omega_{q,p^i} M) \to C_{-*+1}(\Omega_{q,p^i} M)$ is just the boundary operator ∂. The module structure is simply induced by Pontryagin product on the left in each factor.

The pair (f,g) also induces the canonical morphism of modules (2.44) defined as follows: denote by

$$W^u(p^i) \tag{2.53}$$

the *unstable manifold* associated to the critical point p^i; this is the space of all points converging via positive gradient flow to p^i. It is more suggestive to define $W^u(p^i)$ as the moduli space of semi-infinite trajectories:

$$W^u(p^i) := \{\gamma : [0,\infty) \to M | \dot{\gamma}(t) = -\nabla f(\gamma(t)), \lim_{t\to+\infty} \gamma = p^i\}; \tag{2.54}$$

evaluation at 0 identifies this moduli space with the earlier definition. It is a standard fact in Morse theory that (2.54) is a smooth manifold of dimension $|p^i|$. The perspective (2.54) allows us to observe that (2.54) admits a Morse-like compactification to a topological manifold with corners by *broken semi-infinite flowlines*, which are a sequence of (broken) infinite flowlines, followed by a semi-infinite flowline:

$$\overline{W}^u(p^i) = \coprod_{k;x_1,\dots,x_k\in\mathrm{crit}(f)} \mathcal{M}(p^i,x_0) \times \mathcal{M}(x_0,x_1) \times \cdots \times \mathcal{M}(x_{k-1},x_k) \times W^u(x_k) \tag{2.55}$$

In particular, the boundary of $\overline{W}^u(p^i)$ is covered by the images of the natural inclusions of the moduli spaces

$$\overline{\mathcal{M}}(p^i,p^j) \times \overline{W}^u(p^j) \tag{2.56}$$

A straightforward inductive argument, starting with critical points of lowest index, establishes that

Lemma 2.6. *There exists a choice of fundamental chains for the spaces $\{\overline{W}^u(p^i)\}$ in singular homology such that*

$$\partial[\overline{W}^u(p)] = \sum_q (-1)^* [\overline{\mathcal{M}}(p,q)] \times [\overline{W}^u(q)] \tag{2.57}$$

insert signs

The definition (2.54) allows us to also note that there is a natural embedding

$$\mathrm{ev} : W^u(p^i) \hookrightarrow \Omega_{N,p^i}(N) \tag{2.58}$$

to the space of Moore paths starting anywhere in N and ending at p^i; ev is defined exactly as (2.50) by using arc length to fix parametrizations. Compactness in Morse theory guarantees that the Riemannian lengths of gradient flow trajectories extends continuously to the space of broken trajectories, and hence the extension of ev to compactifications is compatible with the concatenation of the product of evaluation maps defined on higher strata.

elaborateonthisco

In particular, a fundamental chain for $\overline{W}^u(p^i)$ induces an element $\mathrm{ev}_*([\overline{W}^u(p^i)]) \in C_{-|p^i|}(\Omega_{N,p^i}N)$. Define $\rho_{f,N}$ as the linear extension of the map sending element $\alpha_i \in C_{-*+|p^i|}(\Omega_{q,p^i})$ to

$$\rho_{f,N} : \alpha_i \mapsto \mu^2(\alpha_i, \mathrm{ev}_*([\overline{W}^u(p^i)])). \tag{2.59}$$

Change μ^2 to Pontryagin product of loops.

Insert signs

Proof of Proposition 2.2. We verify first that $\rho_{f,N}$ is a chain map, computing that:

$$\begin{aligned}
\partial\rho_{f,N}(\alpha_i) &= \mu^2(\partial\alpha_i, \mathrm{ev}_*([\overline{W}^u(p^i)])) + \mu^2(\alpha_i, \partial\mathrm{ev}_*([\overline{W}^u(p^i)])) \\
&= \rho_{f,N}(\partial\alpha_i) + \sum_j \mu^2(\alpha_i, \mu^2(\mathrm{ev}_*[\overline{\mathcal{M}}(p^i,p^j)], \mathrm{ev}_*[\overline{W}^u(p^j)])) \\
&= \rho_{f,N}(\partial\alpha_i) + \sum_j \mu^2(\mu^2(\alpha_i, \mathrm{ev}_*[\overline{\mathcal{M}}(p^i,p^j)]), \mathrm{ev}_*[\overline{W}^u(p^j)]) \\
&= \rho_{f,N}(\partial\alpha_i + \sum_j D_{ij}\alpha_i) \\
&= \rho_{f,N}(\mu^1_{\mathcal{M}'_N}\alpha_i),
\end{aligned} \tag{2.60}$$

Might have to say something about associativity of/ compatibility with Eilenberg-Zilber maps somehow

as desired. Here, the first equality used the fact that the Pontryagin product is a chain map, the second used the relationship (2.57), and the third used associativity of composition. It is straightforward to see that $\rho_{f,N}$, which is built out of (Pontryagin) multiplication maps on the left, is compatible with the $C_{-*}(\Omega_q M)$ module structure (of multiplying on the right).

To now prove that this map induces a homology isomorphism, we note that both sides admit a *cellular filtration* induced by f compatible with the map $\rho_{f,N}$. Namely, for each $k \geq 0$, restricting to the generators corresponding to critical points with $i \leq k$ produces a sub-module

$$F^k\mathcal{M}'_N := \left(\oplus_{i\leq k} C_{-*+|p^i|}(\Omega_{q,p^i}M), D_{F^k\mathcal{M}'_N} := D_{\mathcal{M}'_N}|_{F^k\mathcal{M}'_N}\right). \tag{2.61}$$

There is a corresponding submodule of $\mathcal{M}_N$: we define

$$N^k := \cup_{i\leq k}\overline{W}^u(p^i) \subset N \tag{2.62}$$

Revise equation: is this the union of compactifications or a single compactification?

where the map to N, is given by evaluating a broken semi-infinite flowline at zero. The set-theoretic image of the inclusion to N is $\oplus_{i\leq k} W^u(p^i)$. Set

$$F^k \mathcal{M}_N := C_{-*}(\mathcal{P}_{N^k,q}) \tag{2.63}$$

The map $\rho_{f,N}$ evidently restricts to maps $\rho^k_{f,N}$ between submodules, which is an isomorphism for $k = 1$ (as $|p^1|$ must equal 0, in which case $N^1 = p^1$ and $\rho^1_{f,N}$ is the identity morphism $C_{-*}(\Omega_{p_1,q}(M)) \to C_{-*}(\Omega_{p_1,q}(M))$). It will follow that $\rho^k_{f,N}$ is a homology isomorphism for all k (and hence $\rho_{f,N}$ is) if the map on the associated graded piece at each level k is:

$$\begin{tikzcd}
F^{k-1}\mathcal{M}'_N \arrow[r] \arrow[d, "\rho^{k-1}_{f,N}"] & F^k\mathcal{M}'_N \arrow[r] \arrow[d, "\rho^k_{f,N}"] & C_{-*+|p^k|}(\Omega_{p^k,q}(M)) \arrow[d, "?"] \\
F^{k-1}\mathcal{M}'_N \arrow[r] & F^k\mathcal{M}'_N \arrow[r] & C_{-*}(\Omega_{\overline{W}^u(p^k),q}, \Omega_{\partial\overline{W}^u(p^k),q})
\end{tikzcd} \tag{2.64}$$

This associated graded map, by definition, sends $\alpha \in C_{-*+|p^k|}(\Omega_{p^k,q}(M))$ to $\mu^2(\alpha, \mathrm{ev}_*([\overline{W}^u(p^k)])) = \pm\alpha \cdot \mathrm{ev}_*([\overline{W}^u(p^k)])$. We will verify this is an isomorphism, completing the proof.

First, we note that $\overline{W}^u(p^k)$ is naturally contractible in N and M (as is well known **@@CITE**, it has the homotopy type rel. its boundary of D^r rel S^{r-1} where $r = |p^k|$). Morse theory provides us with a natural deformation retraction to p^k; given an element $p \in \overline{W}^u(p^k)$,

$$\gamma_p := \mathrm{ev}(p) \in \Omega_{p,p^k}(N) \tag{2.65}$$

gives a (Moore) path from p to p^k and an associated normalized length 1 path $\bar{\gamma}_p$; the deformation retraction is given by flowing along these paths. Along the same vein, we see that for any $b \in M$ there is a natural homotopy equivalence

Remember these are Moore paths, so elements are actually pairs $(t, \gamma : [0,t] \to M)$; maybe we should be more precise here

$$f^b : \Omega_{\overline{W}^u(p^k),b}(M) \xrightarrow{\sim} \overline{W}^u(p^k) \times \Omega_{p^k,b}(M) \tag{2.66}$$

which sends a path γ to the pair $(\gamma(0), \gamma \cdot \overline{\mathrm{ev}(\gamma(0))})$ where $\overline{\beta}$ denotes the reversed path. In words, we associate to an element γ the pair consisting of its start point, and the path γ composed with the canonical Moore path p^k to $\gamma(0)$ given by the gradient trajectory of f. We note that f^b is naturally compatible with the multiplication on the right by elements of $\Omega_{b,b'}(M)$, i.e., there is a strictly commutative diagram

$$\begin{tikzcd}
\Omega_{\overline{W}^u(p^k),b} \times \Omega_{b,b'} \arrow[r, "comp"] \arrow[d, "f^b \times id", "\sim"'] & \Omega_{\overline{W}^u(p^k),b'} \arrow[d, "f^{b'}", "\sim"'] \\
\overline{W}^u(p^k) \times \Omega_{p^k,b} \times \Omega_{b,b'} \arrow[r, "id \times comp"] & \overline{W}^u(p^k) \times \Omega_{p^k,b'}
\end{tikzcd} \tag{2.67}$$

One notes next that the evaluation map ev fits into a homotopy commutative diagram with respect to f:

$$
\begin{tikzcd}
\overline{W}^u(p^k) \arrow[r, "\mathrm{ev}"] \arrow[dr, "{(id,*)}"'] & \Omega_{\overline{W}^u(p^k),p^k}(M) \arrow[d, "f", "\sim"'] \\
& \overline{W}^u(p^k) \times \Omega_{p^k,p^k}(M)
\end{tikzcd}
\quad . \tag{2.68}
$$

Here $*$ indicates the identity element in Ω_{p^k,p^k}. Putting these two together, we see that there is a homotopy commutative diagram

$$
\begin{tikzcd}
\overline{W}^u(p^k) \times \Omega_{p^k,q} \arrow[r, "comp\circ(ev{,}id)"] \arrow[dr, "id"'] & \Omega_{\overline{W}^u(p^k),q} \arrow[d, "f^q", "\sim"'] \\
& \overline{W}^u(p^k) \times \Omega_{p^k,q}
\end{tikzcd}
\quad . \tag{2.69}
$$

This entire discussion was compatible with the boundary $\partial\overline{W}^u(p^k)$, in the sense that restriction of (2.69) to an element in $\partial\overline{W}^u(p^k) \times \Omega_{p^k,q}$ induces a homotopy commutative diagram

$$
\begin{tikzcd}
\partial\overline{W}^u(p^k) \times \Omega_{p^k,q} \arrow[r, "comp\circ(ev{,}id)"] \arrow[dr, "id"'] & \Omega_{\partial\overline{W}^u(p^k),q} \arrow[d, "f^q", "\sim"'] \\
& \partial\overline{W}^u(p^k) \times \Omega_{p^k,q}
\end{tikzcd}
\quad . \tag{2.70}
$$

To do: streamline proof, and somewhere in this section, probably the beginning, make the observation that $\mathcal{M}'_N$ is the Morse chains with coefficients in the local system of chains on the based loop space, coming from the Serre fibration for $\Omega_{N,q}$.

The diagram of pairs (2.69) and (2.70) imply that, the image under the quasi-isomorphism f^q_* : $C_{-*}(\Omega_{\overline{W}^u(p^k),q}, \Omega_{\partial\overline{W}^u(p^k),q}) \cong C_{-*}(\overline{W}^u(p^k)\times\Omega_{p^k,q}, \partial\overline{W}^u(p^k)\times\Omega_{p^k,q})$ of the element $\alpha\cdot\mathrm{ev}_*([\overline{W}^u(p^k)])$ is homologous to $\alpha \times [\overline{W}^u(p^k)]$. But the map

$$
\begin{aligned}
C_{-*+|p^k|}(\Omega_{p^k,q}(M)) &\to C_{-*}(\overline{W}^u(p^k) \times \Omega_{p^k,q}, \partial\overline{W}^u(p^k) \times \Omega_{p^k,q}) \\
\alpha &\mapsto [\overline{W}^u(p^k)] \times \alpha
\end{aligned} \tag{2.71}
$$

is a homology isomorphism by Künneth and the fact **@@CITE** that $(\overline{W}^u(p^k), \partial\overline{W}^u(p^k)) \simeq (D^{|p^k|}, S^{|p^k|-{}}$

□

2.3.2 The functor to the path category of M

Recall that the main result of [5] implies the following theorem:

Theorem 2.2 ([5]). *Let $\mathcal{W}_b(T^*M)'$ denote any finite subcategory of the wrapped Fukaya category of T^*M with objects transverse to the zero section, and let $\mathcal{P}(M)$ denote the path category of M of* (2.46)*. Then, there is an A_∞ functor*

$$
\bar{\mathbf{F}} : \mathcal{W}_b(T^*M)' \to \mathrm{Tw}(\mathcal{P}(M)), \tag{2.72}
$$

where $\mathrm{Tw}\mathcal{C}$ *denotes the category of* twisted complexes *in $\mathcal{C}$, defined below.*

Remark 2.1. *In fact, Abouzaid's methods allow one to replace $\mathcal{W}_b(T^*M)'$ by $\mathcal{W}_{b'}(X)'$ for any Liouville X containing M as an exact Lagrangian and admitting a background class b' restricting to $b \in T^*M$.*

Remark 2.2. *On the level of objects, for any Lagrangian $M \subset X$ satisfying suitable technical hypotheses (such as exactness, etc.), a version of this twisted complex (or module) over $\mathcal{P}(M)$ was first associated to a Lagrangian $L \subset X$ by Barraud and Cornea [**?**].*

Since a cotangent fibre generates the wrapped Fukaya category [3], and $\bar{\mathbf{F}}$ restricted to the subcategory consisting of a cotangent fibre is a quasi-isomorphism essentially by earlier work of Abbondandolo-Schwarz [**?**], one has:

Corollary 2.1. *The functor* (2.72) *is full and faithful.*

Recall that a *a twisted complex in a dg category* $\mathcal{C}$ consists of the data of

- a finite collection of objects $\{q_i\}_{i=1}^r$ and integers m_i
- morphisms $D = \oplus_{i \leq j} D_{i,j}$ of degree 1 in $\mathrm{Hom}_*(q^i[m_i], q^j[m_j]) := \mathrm{Hom}_*(q^i, q^j)[m_j - m_i]$ satisfying
$$\mu^1(D) + \mu^2(D, D) = 0 \tag{2.73}$$
where (2.73) encodes a matrix of equations whose entries are, for each i, j:
$$0 = \mu^1 D_{i,j} + \sum_k \mu^2(D_{k,j}, D_{i,k}). \tag{2.74}$$

Twisted complexes of $\mathcal{C}$ are themselves the objects of a pre-triangulated category $\mathrm{Tw}(\mathcal{C})$, which gives a particularly efficient model for the pre-triangulated hull of $\mathcal{C}$. We will omit the description of morphisms.

To describe $\bar{\mathbf{F}}$ at the level of objects, we assume first that a given object L intersects the zero section M transversely in a collection of points $\{q^i\}_{i=1}^m$, ordered by action $f_M - f_L$ (we will lift this hypothesis later). Assuming the zero section and M are both graded, each q^i inherits a degree $|q^i|$, which coincides with the degree of q^i thought of as a generator of the Floer complex $CF^*(L, M)$.

With respect to some interval dependent almost complex structure J_t, $t \in [0, 1]$, the (Gromov-)compactified moduli space of pseudo-holomorphic strips from q^i to q^j
$$\overline{\mathcal{H}}(q^i, q^j) \tag{2.75}$$
is (the compactification of) the quotient by $\mathbb{R}$ translation of the space of maps $u : \mathbb{R} \times [0, 1] \to T^*M$ solving the $\overline{\partial}$-equation
$$du \circ j - J_t \circ du = 0. \tag{2.76}$$

with boundary conditions and asymptotics

$$\begin{cases} u(s,1) \in L \\ u(s,0) \in M \\ \lim_{s\to+\infty} u(s,t) = q^j \\ \lim_{s\to-\infty} u(s,t) = q^i. \end{cases} \tag{2.77}$$

This space of maps also appears in the Floer differential for $CF^*(L,M)$, but here we will want to use higher dimensional components as well:

Lemma 2.7 ([5] Prop 2.5, see also [**?**]). *For generic J_t, $\overline{\mathcal{H}}(q^i,q^j)$ is a topological manifold (with boundary) of dimension $|q^i|-|q^j|$, with codimension 1 boundary covered by the images of natural embeddings*

$$\overline{\mathcal{H}}(q^i,q^{i_1}) \times \overline{\mathcal{H}}(q^{i_1},q^j) \to \overline{\mathcal{H}}(q^i,q^j) \tag{2.78}$$

By restricting an element u to $\mathbb{R}\times\{0\}$ and using the the arc length parametrization induced by our fixed metric on M, one obtains a family of continuous evaluation maps

$$\mathrm{ev} : \overline{\mathcal{H}}(q^i,q^j) \to \Omega_{q^i,q^j} \tag{2.79}$$

which restrict to the composition of the evaluations along various boundary strata [5]*Lemma 2.7. A sign comparison shows that

Lemma 2.8 ([5], (2.25)). *There exists a choice of fundamental chains for $\{\overline{\mathcal{H}}(q^i,q^j)\}_{i,j}$ in singular homology such that*

$$\partial[\overline{\mathcal{H}}(q^i,q^j)] = \sum_k (-1)^{|q^i|+|q^k|}[\overline{\mathcal{H}}(q^i,q^k)] \times [\overline{\mathcal{H}}(q^k,q^j)] \tag{2.80}$$

Finally, one can define the functor $\bar{\mathbf{F}}$ on the level of objects as:

$$\bar{\mathbf{F}}(L) := \left(\bigoplus_{q^i \in M\cap L} q^i[-|q^i|], D = \sum_{i,j} (-1)^{|q^i|(|q^j|+1)} \mathrm{ev}_*([\overline{\mathcal{H}}(q^i,q^j)]) \right) \tag{2.81}$$

It is straightforward to see over $\mathbb{Z}_2$ that the relations (2.80) imply the entries of the twisted complex equation (2.73); see [**?**]*Lemma 2.9 for verification with signs. Next, there is a Yoneda embedding

$$Yoneda : \mathrm{Tw}(\mathcal{P}(M)) \to perf(C_{-*}(\Omega_q M)) \tag{2.82}$$

which associates to any object $p \in M$ the module $C_{-*}(\Omega_{p,q}M)$; $Yoneda$ extends to Tw because $perf(C_{-*}(\Omega_q M))$ is pre-triangulated, and in this case one can write down a specific image

$$\left(\bigoplus_i q_i[m_i], \bigoplus_{i,j} D_{i,j} \right) \longmapsto \left(\bigoplus_i C_{-*+m_i}(\Omega_{q_i,q}(M)), \sum_i \mu_i^1 + \sum_{i,j} \mu^2(D_{i,j},-) \right). \tag{2.83}$$

Composing $\bar{\mathbf{F}}$ with $Yoneda$, we obtain a full and faithful functor

$$\mathbf{F}' : \mathcal{W}_b(T^*M)' \to perf(C_{-*}(\Omega_q M)) \tag{2.84}$$

with

$$\mathbf{F}'(L) = \left(\bigoplus_{q^i \in M \cap L} C_{-*+|q^i|}(\Omega_{q,q^i}(M)), D = \sum_i \partial_i + \sum_{i,j} (-1)^{|q^i|(|q^j|+1)} \mu^2(-, \mathrm{ev}_*([\overline{\mathcal{H}}(q^i, q^j)])) \right). \tag{2.85}$$

We now identify a perturbation L'_N of the conormal bundle L_N for which the image twisted complex $\mathbf{F}'(L'_N)$ can be described in terms of Morse-theoretic data.

Let $f : N \to \mathbb{R}$ be a self-indexing C^2-small Morse function, and denote by $\tilde{f}$ some extension $M \to \mathbb{R}$ that agrees with projection to N followed by f in a small tubular neighborhood and is 0 outside a larger tubular neighborhood of N. Pick a decreasing cutoff function $\beta : [0, \infty) \to \mathbb{R}$ that is 1 near 0 and 0 near ∞ and define

$$\begin{aligned} H : T^*M &\to \mathbb{R} \\ (q, p) &\mapsto \beta(|p|) \cdot \tilde{f}(q). \end{aligned} \tag{2.86}$$

As H is zero outside of a compact set, the time-1 Hamiltonian flow ϕ^1_H of the associated Hamiltonian vector field is compactly supported. Define

$$L'_N := \phi^1_H(L_N). \tag{2.87}$$

Because L'_N differs from L_N by a compactly supported Hamiltonian isotopy, standard results about Fukaya categories guarantee that

Need to equip L'_N with the relevant brane structure...

Lemma 2.9. *$L'_N \sim L_N$ in the wrapped Fukaya category.*

We equip L'_N with the brane structure induced by the one on L_N. The perturbed Lagrangian L'_N is now transverse to M so we can define

$$\mathbf{F}(L'_N) \tag{2.88}$$

as in (2.81).

Note that the intersection points of L'_N with M, which lie along N, are in bijection with critical points of the function f. More is true: since f was chosen to be C^2 small, it is a fact due to Pozniak, generalizing Floer, that

Insert relevant citation

Proposition 2.3. *There is an (oriented) homeomorphism of moduli spaces:*

$$\overline{\mathcal{H}}(p^i, p^j) \cong \overline{\mathcal{M}}(p^i, p^j) \tag{2.89}$$

Choosing fundamental chains for each of these moduli spaces compatibly with the homeomorphism, we obtain

To insert: Orientations. In particular, how the background class factors in to give exactly the right orientation in this diffeomorphism of moduli spaces

Corollary 2.2. *For perturbation L'_N chosen as above, the dg $C_{-*}(\Omega_q M)$ modules $\mathbf{F}'(L'_N)$ and $\mathcal{M}'_N$ (constructed in* (2.52)*) are equal.*

2.3.3 Functor for a point

This overview needs to be revised, and we need to justify in a couple words that $\mathbf{F}$ restricted to a point coincides with Abouzaid's $\bar{\mathbf{F}}$ restricted to a point, at least up to homotopy.

So far, we have not explicitly described the functor $\mathbf{F}$ on morphism spaces. Let $\mathcal{W}_q \subset \mathcal{W}(T^*M)$ denote the full subcategory consisting of a cotangent fibre $T^*_q M$, and $\mathcal{S}_q \subset \mathcal{S}_M$ the full sub-category with the single point object q. We begin by recalling the construction involved to build $\mathbf{P} := \mathbf{F}|_{\mathcal{W}_q}$:

Theorem 2.3 ([**?**], [1])**.** *There is an A_∞ quasi-isomorphism*

$$\mathbf{P} : \mathcal{W}_q \to \mathcal{S}_q. \tag{2.90}$$

That is, there is a morphism of A_∞ algebras.

$$\mathbf{P} : CW^*(T^*_q M, T^*_q M) \to C_{-*}(\Omega_q M), \tag{2.91}$$

inducing a homology isomorphism.

The **moduli space of half-discs** with d inputs, denoted $\mathcal{P}_d$ is the space of disks with $d+2$ boundary punctures modulo automorphism, where d consecutive punctures $z_1, \ldots, z_d$ labeled as *incoming* and the opposite boundary segment $\{z_{-1}, z_0\}$ called the *outgoing boundary segment.* (strictly speaking $\mathcal{P}_d$ is abstractly isomorphic to $\mathcal{R}^{d+1}$, though we use it to construct operations in a different fashion). We can formally extend this to the case $d = 0$, in which case we think of $\mathcal{P}_0$ as $pt/\mathbb{R}$, as in the moduli space of strips.

The compactification $\overline{\mathcal{P}}_d$ is diffeomorphic to $\overline{\mathcal{R}}^{d+1}$, except we view components containing part of the outgoing edge of $\mathcal{P}_d$ as belonging to a $\mathcal{P}_{d_1}$, and components in which incoming points bubble off are viewed as belonging to $\mathcal{R}^{d_1}$. Hence in codimension 1, the boundary of $\overline{\mathcal{P}}_d$ is covered by the natural inclusions of the following strata:

$$\overline{\mathcal{P}}_{d_1} \times \overline{\mathcal{P}}_{d_2} \to \partial\overline{\mathcal{P}}_d,\ d_1 + d_2 = d. \tag{2.92}$$

$$\overline{\mathcal{P}}_{d_1} \times \overline{\mathcal{R}}^{d_2} \to \partial\overline{\mathcal{P}}_d,\ d_1 + d_2 = d + 1. \tag{2.93}$$

Fix a diffeomorphism $\overline{\mathcal{P}}_d \cong \mathcal{R}^{d+1}$ sending $z_1, \ldots, z_d$ and z_{-1} to the incoming boundary points of $\mathcal{R}^{d+1}$, in that order; this equips $\overline{\mathcal{P}}_d$ with a set of strip-like ends, and hence a manifold with corners structure (note that although our convention is for z_{-1} to be an outgoing point, it is equipped with a positive strip-like end, as in [5] [3].

A *universal and consistent Floer datum for the open-string comparison map* consists of an inductive choice for each $d \geq 1$ and each $T \in \overline{\mathcal{P}}_d$, of a *Floer datum* for $\overline{\mathcal{P}}_d$ in the sense of Definition 2.2 (i.e., the existing strip-like ends and a choice of Hamiltonian, one-form, boundary shifting map, almost complex structure) satisfying the following additional properties/exceptions:

- The **closed 1-form** α_S need only vanish on ∂T minus the outgoing segment, and should also be 0 in a neighbordhood of z^{-1} and z^0.
- The **Hamiltonian term** $H_S : S \to \mathcal{H}(M)$ satisfies an additional property: on the outgoing segment, $H_S = 0$ in a neighborhood of the zero section.
- The **boundary-shifting map** $\rho : \partial\bar{S} \backslash \{\text{marked points}\} \to (0, \infty)$ should be equal to 1 on the outgoing segment; in particular, the weights associated to z_{-1} and z_0 should both be 1.

Remark 2.3. *The additional freedom of α_S to be non-zero along the outgoing boundary segment Γ ensures that the weights associated to puncture $z_1, \ldots, z_d$ are free to be arbitrarily positive while preserving closedness of α.*

For any collection $\{x_1, \ldots, x_d\}$ of chords in $\chi(T_q^* M, T_q^* M)$, we get a parametrized moduli space

$$\mathcal{P}(q, \vec{x}, q) \tag{2.94}$$

of maps from a domain in $\mathcal{P}_d$ to T^*M of finite energy, such that the *outgoing segment* maps to the zero section Q, every other boundary component maps to T^*M, satisfying Floer's equation

$$(du - X \otimes \alpha)^{0,1} = 0 \tag{2.95}$$

with respect to the universal and consistent Floer datum chosen above. For generically chosen data, the (2.94) is a smooth manifold of dimension $d - 1 - \sum_{i=1}^{d} |x_i|$, with Gromov bordification $\overline{\mathcal{P}}(q, \vec{x}, q)$ a compact manifold with boundary covered by the closures of the natural inclusions

$$\mathcal{P}(q, \vec{x}^1, q) \times \mathcal{P}(q, \vec{x}^2, q) \tag{2.96}$$

$$\mathcal{P}(q, \vec{x}^2, q) \times \mathcal{R}(x; \vec{x}^2) \tag{2.97}$$

Lemma 2.10 ([5], Lemma 4.14)**.** *There exists a family of fundamental chains*

$$[\overline{\mathcal{P}}(q, \vec{x}, q)] \in C_*(\overline{\mathcal{P}}(q, \vec{x}, q)) \tag{2.98}$$

in the normalized singular chain complex whose boundary is given by

$$\sum_{\vec{x}^1 \cup \vec{x}^2 = \vec{x}} (-1)^{\flat} [\overline{\mathcal{P}}(q, \vec{x}^1, q)] \times [\overline{\mathcal{P}}(q, \vec{x}^2, q)] + \sum_{\vec{x}^1 - \{x\} \cup \vec{x}^2 = \vec{x}} (-1)^{\sharp} [\overline{\mathcal{P}}(q, \vec{x}^1, q)] \times [\overline{\mathcal{R}}(x; \vec{x}^2)]. \tag{2.99}$$

with signs

$$\flat = (d_2+1)\left(\sum_{i=1}^{d_1} |x_i|\right) + d_1 + 1 \tag{2.100}$$

$$\sharp = d_2\left(\sum_{j=1}^{k+d_2} |x_j|\right) + d_2(d-k) + k + 1, \tag{2.101}$$

where $\times$ *denotes the Eilenberg-Zilber map, as discussed in the Introduction and §A.1.1.*

Next, note that there is an evaluation map

$$\mathrm{ev} : \overline{\mathcal{P}}(q, \vec{x}, q) \to \Omega_q M \tag{2.102}$$

which takes every half disc u to the based Moore loop in M obtained by restricting u to the outgoing segment, using the length parametrization in the target with respect to a fixed metric (as in [5]) to remove any ambiguity.

Thus, we can define

$$\mathbf{P}^d : (CW_b^*(T_q^*M))^{\otimes d} \to C_{-*}(\Omega_q M) \tag{2.103}$$

$$x_d \otimes \cdots \otimes x_1 \mapsto (-1)^{\dagger + d|\vec{x}|}\mathrm{ev}_*([\overline{\mathcal{P}}(q, \vec{x}, q)]) \tag{2.104}$$

where the sign is given by

$$\dagger = \sum_{k=1}^{d} k|x_k| \tag{2.105}$$

and $|\vec{x}|$ denotes the sum of the degrees of the inputs.

Lemma 2.11 ([5], Lemma 4.15)**.** *The collection of maps* $\mathbf{P}^d$ *satisfy the* A_∞ *equation for functors; that is,*

$$\begin{aligned}\sum_{d_1+d_2=d+1} &(-1)^{\maltese_1^i}\mathbf{P}^d(x_d, \ldots, x_{i+d_2+1}, \mu_F^{d_2}(x_{i+d_2}, \ldots, x_{i+1}), x_i, \ldots, x_1) = \\ &\mu_P^1(\mathbf{P}^d(x_d, \ldots, x_1)) + \sum_{d_1+d_2=d} \mu_P^2(\mathbf{P}^{d_2}(x_d, \ldots, x_{d_1+1}), \mathbf{P}^{d_1}(x_{d_1}, \ldots, x_1)),\end{aligned} \tag{2.106}$$

where

$$\maltese_1^i = i + \sum_{j=1}^{i} |x_i|. \tag{2.107}$$

3 Calabi-Yau structures in Floer theory and string topology

3.1 $C_{-*}(S^1)$ modules and cyclic chains

The dg algebra $C_{-*}^{sing}(S^1; \mathbb{K})$ of singular chains on S^1, with multiplication given by Pontryagin product, is *formal*, and in particular quasi-isomorphic as an A_∞ algebra to an exterior algebra on one generator Λ of degree -1, $\mathbb{K}|\Lambda|$. Henceforth, when we refer to $C_{-*}(S^1)$ we will mean $\mathbb{K}|\Lambda|$ unless otherwise specified:

$$C_{-*}(S^1) := \mathbb{K}|\Lambda|, \ \deg(\Lambda) = -1.$$

A *chain complex with (weak, or A_∞) circle action*, or succintly an *S^1-complex*, is a strictly unital A_∞ module over $C_{-*}(S^1) = \mathbb{K}|\Lambda|$. Concretely, an S^1-complex is given by the data of a graded vector space M equipped with maps of degree $1-2k$

$$\delta_k^M := \mu^{k|1}(\underbrace{\Lambda, \ldots, \Lambda}_{k}, -) : M \to M, \ k \geq 0,$$

denoted δ_k if M is implicit, satisfying the A_∞ module equations:

$$\sum_{i=0}^{k} \delta_j \delta_{k-j} = 0 \tag{3.1}$$

(*strict unitality* implies that multiplication by sequences including the element $1 \in \mathbb{K}|\Lambda|$ is entirely determined). In particular δ_0 is a differential, and δ_1 is an (anti-)chain map, descending to a homology level operator $[\delta_1]$ which squares to zero. An S^1 complex M with $\delta_j = 0$ for $j > 1$ is exactly the data of a *mixed complex*, taking $d = \delta_0$, $\Delta = \delta_1$ (in this case, we will say the S^1 action is strict, or differential graded).

Letting u be a formal variable of degree $+2$, it is convenient to package the collection of operations δ_k into a formal generating series of total degree 1:

$$\delta_{eq}^M := \sum_{i=0}^{\infty} \delta_i^M u^i.$$

The A_∞ module equations (3.1) are then equivalent to the fact that $\delta_{eq}^2 = 0$.

Chain complexes with A_∞ circle action form a dg category (as do A_∞ modules over any algebra), with morphism spaces again denoted $\mathrm{RHom}_{C_{-*}(S^1)}(\mathcal{P}, \mathcal{Q})$. As we are working with strictly unital modules, one can used the *reduced bar complex* to define chain-level morphism spaces;

$$\mathrm{RHom}^{\bullet}_{C_{-*}(S^1)}(\mathcal{P}, \mathcal{Q}) := \hom^{\bullet}_{Vect}(\oplus_{k\geq 0}\overline{\mathbb{K}|\Lambda|}^{\otimes k} \otimes \mathcal{P}, \mathcal{Q}) \cong \prod_{k \geq 0} \hom^{\bullet - 2k}_{Vect}(\mathcal{P}, \mathcal{Q}).$$

where the augmentation ideal $\overline{\mathbb{K}|\Lambda|}$ the copy of $\mathbb{K}$ in degree -1 spanned by Λ. Once more, we can formally write elements of this morphism space as series in u:

$$\mathrm{RHom}^{\bullet}_{C_{-*}(S^1)}(\mathcal{P}, \mathcal{Q}) = \{f_{eq} := \sum_{k=0}^{\infty} f_k u^k | f_i \in \hom^{\bullet-2k}_{Vect}(\mathcal{P}, \mathcal{Q})\}$$

With respect to this notation, composition of morphisms is the u-linearly extended composition of linear maps:

$$(\sum_{k=0}^{\infty} f_k u^k) \circ (\sum_{r=0}^{\infty} g_m u^m) := \sum_{n=0}^{\infty} \left(\sum_{t=0}^{n} f_t \circ g_{n-t} \right) u^n,$$

and the differential acts as

$$d(f_{eq}) := f_{eq} \circ \delta_{eq}^{\mathcal{P}} - \delta_{eq}^{\mathcal{Q}} \circ f_{eq} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} f_j \circ \delta_{n-j}^{\mathcal{P}} - \delta_{n-j}^{\mathcal{Q}} \circ f_j \right) u^n.$$

If f_{eq} is *closed*, (i.e., $d(f_{eq}) = 0$), then in particular f_0 is a chain map with respect to $\delta_0^{\mathcal{P}}$ and $\delta_0^{\mathcal{Q}}$, inducing a cohomology level morphism $[f_0]$ which intertwines the BV operators ($[\delta_1^{\mathcal{P}}]$ and $[\delta_1^{\mathcal{Q}}]$). We say f_{eq} is a *quasi-isomorphism* if f_{eq} is closed and the resulting cohomology map $[f_0]$ is an isomorphism.

The *homotopy fixed points* (or *negative cyclic chains*) of an S^1-complex M, denoted M^{hS^1}, is the chain complex of morphisms (in the category of $C_{-*}(S^1)$ modules) from the trivial module $\mathbb{K} = (\mathbb{K}, \delta_{eq} = 0)$ to M. In light of the above discussion, this chain complex can be modeled in degree n by series of elements of the form $\sum_{k=0}^{\infty} m_k u^k$, where $m_k \in M^{n-2k}$:

$$M_\bullet^{hS^1} := \mathrm{RHom}^\bullet_{C_{-*}(S^1)}(\mathbb{K}, M) := ((M[[u]])^\bullet, \delta_{eq} = \sum_{i=0}^{\infty} \delta_i^M u^i). \tag{3.2}$$

This complex comes equipped with a chain map into (M, δ_0) modeling the *inclusion of (homotopy) fixed points*

$$\begin{aligned} \iota : M^{hS^1} &\to M \\ \sum_{i=0}^{\infty} m_k u^k &\mapsto m_0. \end{aligned} \tag{3.3}$$

The *homotopy orbits* (or *cyclic chains*), is any chain complex computing the (derived) tensor product of M with the trivial module $\mathbb{K}$ (this should be compared to the homotopy orbits in the category of spaces, which similarly given by homotopy fiber product over S^1 with pt). Using reduced complexes, this can again be given the following explicit model:

$$M_{hS^1} := \mathbb{K} \otimes^{\mathbb{L}}_{C_{-*}(S^1)} M := (M((u))/uM[[u]], \delta_{eq}). \tag{3.4}$$

This complex comes equipped with a chain map from M, the *projection onto (homotopy) orbits*:

$$\begin{aligned} pr : M &\to M_{hS^1} \\ m &\mapsto m \cdot u^0. \end{aligned} \tag{3.5}$$

There are other equivariant homology theories associated to such an M (such as its *Tate*, or *periodic cyclic* complex, and cohomological variants), which moreoever satisfy various relations, see e.g., @@CITE. An important and well-known homotopy invariance principle for these groups is:

Proposition 3.1. *Any closed morphism $f_{eq} \in \mathrm{RHom}_{C_{-*}(S^1)}(\mathcal{P}, \mathcal{Q})$ induces a chain map, also denoted f_{eq}, between the $\mathcal{P}^{hS^1}$ and $\mathcal{Q}^{hS^1}$ (respectively $\mathcal{P}_{hS^1}$ and $\mathcal{Q}_{hS^1}$). This induced cohomology map is an isomorphism whenever the original map f_{eq} is a quasi-isomorphism of modules (e.g., whenever $[f_0]$ is an isomorphism).*

In either case, the induced map f_{eq} between homotopy fixed point or orbit complexes is exactly the u-linear extension of composition by $f_{eq} = \sum_{i=0}^{\infty} f_k u^k$.

@@Insert/modify: Negative and ordinary (positive?) cyclic chains of a category

Using cohomological grading conventions, the Hochschild chain complex $\mathrm{CH}_*(A)$ of an algebra or category A comes equipped with a differential b of degree $+1$, and a Connes' B operator of degree -1, 'multiplication by $[S^1]$', which together satisfy the equations $b^2 = 0$, $B^2 = 0$, $Bb + bB = 0$; in particular $(\mathrm{CH}_*(A), \delta_0 = b, \delta_1 = B)$ is a *mixed complex*, and hence is an A_∞ (in fact dg) module over $C_{-*}(S^1) = \mathbb{K}|\Lambda|$ with $\delta_{eq}^{\mathrm{CH}_*(A)} = b + uB$. The discussion carries straightforwardly over to the Hochschild complex of an A_∞ category, with one notable caveat: the Connes' B operator involves the insertion of 'units', and the Fukaya category is not geometrically strictly unital, so instead of the usual Hochschild complex, one should consider a variant, the *non-unital Hochschild complex*

$$(\mathrm{CH}_*^{nu}(\mathcal{F}), b^{nu})$$

which always possesses an operator B^{nu}, equivalent to B whenever B is defined, with (b^{nu}, B^{nu}), $b_{eq}^{nu} = b^{nu} + uB^{nu}$ as before

When the constructions of homotopy fixed points and homotopy orbits are applied to the Hochschild complex, the resulting cohomology theories are often known as *(positive) cyclic homology* and *negative cyclic homology*: $\mathrm{HC}^+(\mathcal{C}) := H^*((\mathrm{CH}_*^{nu}(\mathcal{C}))_{hS^1})$, $\mathrm{HC}^-(\mathcal{C}) := H^*((\mathrm{CH}_*^{nu}(\mathcal{C}))^{hS^1})$. In addition to cohomological variants, there is a third construction of interest, often called the *Tate*, or *periodic equivariant construction*, one inverts the action of u on the homotopy fixed point complex. We call this P^{Tate} and the *periodic cyclic homology* is $\mathrm{HC}^\infty(\mathcal{C}) = \mathrm{HP}(\mathcal{C}) = H^*((\mathrm{CH}_*^{nu}(\mathcal{C})^{Tate})$.

3.2 Calabi-Yau structures

3.2.1 Finiteness and duality properties

To begin, let A refer to a dg algebra and B a dg bimodule over it. We use the notation A_Δ and terminology *diagonal bimodule* to refer to A thought of as a bimodule over itself.

Definition 3.1 (Smoothness)**.** *A bimodule B is* perfect *if it is split-generated by the free A bimodule $A \otimes A^{op}$. A is* smooth *if the diagonal bimodule A_Δ is a* perfect *bimodule.*

Definition 3.2. *The* bimodule, (or smooth) dual *$B^!$ of an A bimodule B is, as a chain complex,*

$$\mathrm{RHom}^\bullet_{A\text{–}A}(B, A \otimes A^{op}) \tag{3.6}$$

where we use the outer *bimodule structure on $A \otimes A^{op}$ to compute the (derived) hom complex. The bimodule structure on $A^!$ comes from the inner bimodule structure on $A \otimes A^{op}$. This is a general instance of the following fact: for $B = A \otimes A^{op}$, and $M = A$ a left B-module, $\mathrm{RHom}^\bullet_B(M, B)$ naturally inherits a right B (or left B^{op}) module structure.*

The inverse dualizing bimodule*, or* inverse Serre bimodule*, is the bimodule dual of the diagonal bimodule*

$$A^! := A^!_\Delta. \tag{3.7}$$

Remark 3.1. *If A is smooth, then $A^!$ is perfect.*

As in the case of vector spaces, under suitable finiteness hypotheses one has a natural equivalence between an algebra and its double bimodule dual.

Lemma 3.1. *There is a natural evaluation morphism*

$$\mathrm{Ev} : A_\Delta \to (A^!)^! \tag{3.8}$$

which is a quasi-isomorphism if A is smooth.

Proof. One more generally constructs an evaluation morphism

$$\mathrm{Ev}_B : B \to (B^!)^! \tag{3.9}$$

for any A bimodule B, and verifies that it is functorial, commuting with finite colimits. Then, the fact that $\mathrm{Ev} := \mathrm{Ev}_A$ is a quasi-isomorphism will follow from the verification that $\mathrm{Ev}_{A \otimes_{\mathbb{K}} A}$ is a quasi-isomorphism. $\square$

Corollary 3.1. *If A is smooth, and B is any perfect bimodule, then there are canonical quasi-isomorphisms*

$$\mathrm{CH}_*(A, B) \xrightarrow{\sim} \hom_{A\text{–}A}(A^!, B) \tag{3.10}$$

$$A^! \otimes_{A\text{–}A} B \xrightarrow{\sim} CH^*(A, B) \tag{3.11}$$

Proof. If B is perfect, then the natural inclusion of complexes

$$\phi : \mathrm{RHom}_{A\text{–}A}(A^!, A \otimes A) \otimes_{A\text{–}A} B = (A^!)^! \otimes_{A\text{–}A} B \xrightarrow{\sim} \mathrm{RHom}_{A\text{–}A}(A^!, B) \tag{3.12}$$

is an equivalence. Evaluation (3.8) induces an equivalence

$$\mathrm{Ev}_* : \mathrm{CH}_*(A, B) = A \otimes^{\mathbb{L}}_{A\text{–}A} B \xrightarrow{\sim} (A^!)^! \otimes^{\mathbb{L}}_{A\text{–}A} B \tag{3.13}$$

so $\phi \circ \mathrm{Ev}_*$ provides the desired first equivalence (3.10). The second equivalence (3.11) is simpler, following from the series of equivalences

$$\mathrm{RHom}_{A\text{–}A}(A, A \otimes A) \otimes_{A\text{–}A} B \simeq \mathrm{RHom}_{A\text{–}A}(A, A \otimes_A B \otimes_A A) \simeq \mathrm{RHom}_{A\text{–}A}(A, B), \tag{3.14}$$

the first of which requires perfectness, and the second of which is induced by the natural equivalence of bimodules

$$A \otimes^{\mathbb{L}}_A B \otimes^{\mathbb{L}}_A A \simeq B. \tag{3.15}$$

$\square$

Corollary 3.2. *The quasi-isomorphism*

$$\mathrm{ev} : \mathrm{CH}_*(A, B) \to \mathrm{RHom}_{A\text{–}A}(A^!, B) \tag{3.16}$$

sends a Hochschild cycle σ *to a closed morphism* $\mathrm{ev}(\sigma)$ *representing, on the level of homology, the cap-product*

$$\cap\,[\sigma] : H^\bullet(A^!) = H^\bullet(\mathrm{RHom}(A, A \otimes A^{op})) \longrightarrow H^\bullet((A \otimes_{\mathbb{K}} A) \otimes^{\mathbb{L}}_{A \otimes A^{op}} B)) \cong H^\bullet(B). \tag{3.17}$$

Proof. `@@INSERT PROOF HERE`. □

The definition carries over naturally to the setting of A_∞ algebras and their A_∞ bimodules, and as a chain complex one can also define $B^! := \mathrm{RHom}_{A\text{–}A}(B, A \otimes A^{op})$. However, there is some subtelty in defining the induced bimodule structure on $B^!$, coming from the fact that the left and right bimodule structures on $A \otimes A^{op}$ no longer not commute; as an analogy, note that for A an A_∞ algebra, the left and right module structures on A_Δ do not commute past each other on the chain level; there are instead higher homotopies. See [20, Def. 2.40] for more details; for this paper it is sufficient to understand $B^!$ as a chain complex, or to replace A and B by an equivalent dg algebra and dg bimodule.

If $\mathcal{C}$ is an A_∞ or dg category, we can repeat all of these definitions using the notions of A_∞/dg bimodules recalled in §1.1.2. We say, for instance, that a bimodule $\mathcal{B}$ over $\mathcal{C}$ is *perfect* if it is split-generated by Yoneda bimodules, and that $\mathcal{C}$ is *smooth* if the diagonal bimodule $\mathcal{C}_\Delta$ is perfect. To any bimodule $\mathcal{B}$, one can associate a *bimodule dual*; for a pair of objects K, L, the underlying chain complex is

$$\mathcal{B}^!(K, L) := \mathrm{RHom}_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{B}, Y_{K,L}) \tag{3.18}$$

and the bimodule structure gives natural maps between these chain complexes for varying K, L; see [20, Def. 2.40].We once more abbreviate $\mathcal{C}^! := \mathcal{C}^!_\Delta$, and note that Lemma 3.1, Corollary 3.1 and Corollary 3.2 continue to hold. Most importantly,

Corollary 3.3. *For a smooth category* $\mathcal{C}$*, the identification* $\mathrm{CH}_*(\mathcal{C}, \mathcal{C}) \cong \mathrm{RHom}_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{C}^!, \mathcal{C}_\Delta)$ *associates, to a class* σ*, the morphism of bimodules whose underlying morphism of chain complexes, for any pair of objects* (K, L) *is* cap product *with* σ*:*

$$\cap\,\sigma : \mathrm{RHom}_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{C}_\Delta, Y_{K,L}) \to \mathcal{C}_\Delta \otimes^{\mathbb{L}}_{\mathcal{C}\text{–}\mathcal{C}} Y_{K,L} \simeq \hom_{\mathcal{C}}(K, L). \tag{3.19}$$

3.2.2 Smooth and compact Calabi-Yau structures

Definition 3.3. *Let* A *be a smooth* A_∞ *or dg algebra. An element* $\eta \in \mathrm{CH}_{-d}(A, A)$ *is said to be* non-degenerate *if, under the isomorphism* $\mathrm{CH}_{-d}(A, A) \cong \mathrm{RHom}^{-d}_{A\text{–}A}(A^!, A_\Delta) = \mathrm{RHom}^0_{A\text{–}A}(A^![d], A_\Delta)$*, via Corollary 3.1,* η *induces a bimodule quasi-isomorphism from* $A^![d]$ *to* A_Δ*.*

A weak smooth Calabi-Yau structure*, or* weak sCY structure*, of degree d is a non-degenerate element $\eta \in \mathrm{CH}_{-d}(A, A)$. A* (strong) smooth Calabi-Yau structure*, or* sCY structure*, is an element of negative cyclic chains*

$$\sigma \in \mathrm{CC}^{-}_{-d}(A)$$

such that the induced element $\iota(\sigma) \in \mathrm{CH}_d(A, A)$ is a weak smooth Calabi-Yau structure, where ι is the inclusion of homotopy fixed points (3.3)*.*

A *sCY algebra* is a pair (A, η) of a smooth A_∞ algebra equipped with a sCY structure $\eta \in \mathrm{CC}^{-}_{-d}(A)$.

The exact same definition holds in the case that $\mathcal{C}$ is an A_∞ or dg category. Namely, we say that an element $\eta \in \mathrm{CH}_{-d}(\mathcal{C}, \mathcal{C})$ is *non-degenerate* if the induced bimodule morphism under Corollary 3.3 is a quasi-isomorphism; that is, for each pair of objects (K, L), if the map

$$\cap\, \eta : \mathrm{RHom}_{\mathcal{C}\text{-}\mathcal{C}}(\mathcal{C}_\Delta, Y_{K,L}) \to \mathcal{C}_\Delta \otimes^{\mathbb{L}}_{\mathcal{C}\text{-}\mathcal{C}} Y_{K,L} \simeq \hom_{\mathcal{C}}(K, L). \tag{3.20}$$

is a quasi-isomorphism of chain complexes. Then a weak *sCY* structure and an *sCY* structure are defined as before. An *sCY category* is a pair $(\mathcal{C}, \sigma)$ of a category equipped with an sCY structure.

Finally, there is the notion of isomorphisms between categories equipped with sCY structures, which we will state, without loss of generality, for categories.

Definition 3.4. *Let $(\mathcal{C}, \lambda)$ be an sCY category and $(\mathcal{D}, \eta)$ another sCY category. A* quasi-equivalence of sCY categories

$$\mathbf{F} : (\mathcal{C}, \lambda) \xrightarrow{\sim} (\mathcal{D}, \eta) \tag{3.21}$$

is a pair $(\mathbf{F}, \kappa)$ of an A_∞ quasi-equivalence $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ and an element $\kappa \in \mathrm{CC}^{-}(\mathcal{D})$, such that, if $\mathbf{F}_\# : \mathrm{CC}^{-}(\mathcal{C}) \to \mathrm{CC}^{-}(\mathcal{D})$ denotes the induced quasi-isomorphism between negative cylic chains,

$$\mathbf{F}_\# \lambda - \eta = d_{eq} \kappa.$$

(here d_{eq} denotes the differential on $\mathrm{CC}^{-}(\mathcal{D})$).

3.2.3 Other definitions in the literature

3.2.4 Relation to topological field theories

@@INSERT A DISCUSSION OF THE WORK OF KONTSEVICH-VLASSOPOULOUS, COSTELLO, KONSTEVICH-SOIBELMAN, LURIE.

3.2.5 Morita Invariance

In this section, we prove that the notion of a sCY structure on a category is suitably Morita invariant, in the following sense.

Theorem 3.1. *Let $\mathcal{X} \subset \mathcal{C}$ be a full subcategory of $\mathcal{C}$. Suppose that $\mathcal{X}$ split-generates $\mathcal{C}$. Suppose that $\mathcal{X}$ has a sCY structure. Then $\mathcal{C}$ has a compatible sCY structure, (meaning that the pull-back map from $\mathcal{C}$ to $\mathcal{X}$ sends the sCY structure of $\mathcal{C}$ into that of $\mathcal{X}$). Conversely, if $\mathcal{C}$ has a sCY structure, then its pull-back to $\mathcal{X}$ defines a sCY structure. Moreover, these associations are mutually inverse up to quasi-isomorphism.*

@@TODO:REVISE/SHORTEN THE PROOF. We'll give a somewhat leisurely discussion of this fact, using standard Morita-type techniques; namely by looking at functors and bimodules associated to the inclusion

$$i : \mathcal{X} \subset \mathcal{C}. \tag{3.22}$$

For a pair of functors between a pair of categories,

$$f : \mathcal{C} \longrightarrow \mathcal{D} \tag{3.23}$$

$$g : \mathcal{C}' \longrightarrow \mathcal{D}' \tag{3.24}$$

recall that there is a naturally induced pull-back operation on bimodules

$$(f \otimes g)^* : \mathcal{D}\text{–mod–}\mathcal{D}' \longrightarrow \mathcal{C}\text{–mod–}\mathcal{C}', \tag{3.25}$$

given, on the level of chain complexes, by

$$(f \otimes g)^*(\mathcal{B})(X, Y) := \mathcal{B}(f(X), g(Y)) \tag{3.26}$$

for a pair of objects $X \in \text{ob } \mathcal{C}$, $Y \in \text{ob } \mathcal{C}'$. There are natural formula for the bimodule multiplications involving the bimodule multiplication of $\mathcal{B}$ composed with sequences of higher order terms for the functors f and g (omitted for now).

Definition 3.5. *Let $f : \mathcal{C} \to \mathcal{D}$ be an A_∞ functor. Define the* **graph** *of f to be the $\mathcal{C}-\mathcal{D}$ bimodule*

$$\Gamma_f := (f \otimes id)^* \mathcal{D}_\Delta, \tag{3.27}$$

and the **transposed graph** *of f to be the $\mathcal{D}-\mathcal{C}$ bimodule*

$$\Gamma_f^T := (id \otimes f)^* \mathcal{D}_\Delta. \tag{3.28}$$

Via convolution (i.e., one-sided tensor product), a $\mathcal{C}-\mathcal{D}$ bimodule $\mathcal{B}$ can be thought of as giving a functor

$$\Phi_{\mathcal{B}} := \cdot \otimes_{\mathcal{C}} \mathcal{B} \tag{3.29}$$

from right $\mathcal{C}$ modules to right $\mathcal{D}$ modules. In particular, we can *compose* a $\mathcal{C}-\mathcal{D}$ bimodule $\mathcal{B}_1$ with at $\mathcal{D}-\mathcal{E}$ bimodule $\mathcal{B}_2$, via taking the (one-sided) tensor product over $\mathcal{D}$

$$\mathcal{B}_1 \otimes_{\mathcal{D}} \mathcal{B}_2. \tag{3.30}$$

Now, let us return to our situation: that of a fully faithful (naive, meaning no higher A_∞ terms) inclusion

$$i : \mathcal{X} \subset \mathcal{C} \tag{3.31}$$

which split-generates $\mathcal{C}$. The graph Γ_i and transposed graph Γ_i^T give us a $\mathcal{X}-\mathcal{C}$ and $\mathcal{C}-\mathcal{X}$ bimodule respectively.

Proposition 3.2. *In the given situation, Γ_i and Γ_i^T are quasi-inverses. That is, there are natural quasi-isomorphisms of bimodules*

$$\Gamma_i \otimes_{\mathcal{C}} \Gamma_i^T \xrightarrow{\sim} \mathcal{X}_\Delta \tag{3.32}$$

$$\Gamma_i^T \otimes_{\mathcal{X}} \Gamma_i \xrightarrow{\sim} \mathcal{C}_\Delta \tag{3.33}$$

Proof. The quasi-isomorphism (3.32) always holds for a fully faithful inclusion, whereas (3.33) relies upon (and in fact, is equivalent to) split-generation. [`Insert proof here`] □

Proposition 3.3. *Let $\mathcal{B}$ be a $\mathcal{C}-\mathcal{C}$ bimodule. Then, the following bimodules are quasi-isomorphic:*

$$\Gamma_i^T \otimes_{\mathcal{X}} (i \otimes 1)^* \mathcal{B} \simeq \mathcal{B} \tag{3.34}$$

$$(1 \otimes i)^* \mathcal{B} \otimes_{\mathcal{X}} \Gamma_i \simeq \mathcal{B} \tag{3.35}$$

Proof. It suffices to prove the first of these quasi-isomorphisms, (3.34). We note that, since tensoring with the diagonal is quasi-isomorphic to the identity,

$$\begin{aligned} (i \otimes 1)^* \mathcal{C} &\simeq (i \otimes 1)^* (\mathcal{C}_\Delta \otimes_{\mathcal{C}} \mathcal{B}) \\ &\simeq \Gamma_i \otimes_{\mathcal{C}} \mathcal{B}, \end{aligned} \tag{3.36}$$

from which (3.34) is an immediate consequence, using Proposition 3.2. □

Pullback by i induces a map

$$i^* := (i \otimes i)^* \mathcal{C}\text{–mod–}\mathcal{C} \to \mathcal{X}\text{–mod–}\mathcal{X}, \tag{3.37}$$

which, by the above proposition, is quasi-isomorphic to

$$\mathcal{C} \to \Gamma_i \otimes_{\mathcal{C}} \mathcal{B} \otimes_{\mathcal{C}} \Gamma_i^T. \tag{3.38}$$

Using graph bimodules, we can also define a left-adjoint to pull-back i^*, denoted

$$\begin{aligned} i_* : \mathcal{X}\text{–mod–}\mathcal{X} &\to \mathcal{C}\text{–mod–}\mathcal{C}, \\ \mathcal{B}' &\mapsto \Gamma_i^T \otimes_{\mathcal{X}} \mathcal{B}' \otimes_{\mathcal{X}} \Gamma_i. \end{aligned} \tag{3.39}$$

Proposition 3.4. *i_* is left adjoint to i^* on the level of homology (perhaps more carefully, on the level of the homotopy category of bimodules).*

Proof. We note that

$$\begin{aligned}
\hom_{\mathcal{C}\text{–mod–}\mathcal{C}}(i_*\mathcal{B},\mathcal{N}) &= \hom_{\mathcal{C}\text{–}\mathcal{C}}(\Gamma_i^T \otimes_{\mathcal{X}} \mathcal{B} \otimes_{\mathcal{X}} \Gamma_i, \mathcal{N}) && (3.40)\\
&= \hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{B} \otimes_{\mathcal{X}\text{–}\mathcal{X}} \left((\Gamma_i^t)^{op} \otimes_{\mathbb{K}} \Gamma_i\right), \mathcal{N}) && (3.41)\\
&\simeq \hom_{\mathcal{X}\text{–}\mathcal{X}}(\mathcal{B}, \hom_{\mathcal{C}\text{–}\mathcal{C}}((\Gamma_i^t)^{op} \otimes_{\mathbb{K}} \Gamma_i, \mathcal{N})). && (3.42)
\end{aligned}$$

where in (3.41) we have reinterpreted the one-sided tensor product of the $\mathcal{X}$–$\mathcal{X}$ bimodule $\mathcal{B}$ with the bimodules Γ_i^T and Γ_i on each side over $\mathcal{X}$ as the one-sided tensor product of the $\mathcal{X}\otimes\mathcal{X}^{op}$ module $\mathcal{B}$ over $\mathcal{X}\otimes\mathcal{X}^{op}$ with the $(\mathcal{X}\otimes\mathcal{X}^{op})$–$(\mathcal{C}\otimes\mathcal{C}^{op})$-*bimodule* $(\Gamma_i^t)^{op}\otimes_{\mathbb{K}}\Gamma_i$. In (3.42) we have simply applied hom-tensor adjunction for such tensor products, noting that there is a way to take hom from a $(\mathcal{X}\otimes\mathcal{X}^{op})$–$(\mathcal{C}\otimes\mathcal{C}^{op})$ bimodule to a $\mathcal{C}\otimes\mathcal{C}^{op}$ bimodule such that the result is an $\mathcal{X}\otimes\mathcal{X}^{op}$ bimodule.

The adjunction finally follows from the following fact:

$$\text{There is a natural quasi-isomorphism } i^*\mathcal{N} \xrightarrow{\sim} \hom_{\mathcal{C}\text{–}\mathcal{C}}((\Gamma_i^t)^{op} \otimes_{\mathbb{K}} \Gamma_i, \mathcal{N}). \tag{3.43}$$

In fact, we simply note that

$$\hom_{\mathcal{C}\text{–}\mathcal{C}}((\Gamma_i^t)^{op} \otimes_{\mathbb{K}} \Gamma_i, \mathcal{N}) = i^* \hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta, \mathcal{N}) \tag{3.44}$$

and then that this quasi-isomorphism is a pull-back of the natural Yoneda-type quasi-isomorphism

$$\mathcal{N} \xrightarrow{\sim} \hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta, \mathcal{N}). \tag{3.45}$$

□

The adjunction above holds for any functor i between a pair of categories, but for our specific situation $\mathcal{X} \subset \mathcal{C}$ we note that up to weak equivalence, i_* and i^* are inverses on the level of objects:

Corollary 3.4. *For any $\mathcal{C}$-bimodule $\mathcal{B}$ and any $\mathcal{X}$-bimodule $\mathcal{B}'$, we have quasi-isomorphisms*

$$\begin{aligned}
i_* i^* \mathcal{B} &\xrightarrow{\sim} \mathcal{B} && (3.46)\\
i^* i_* \mathcal{B}' &\xrightarrow{\sim} \mathcal{B}' && (3.47)
\end{aligned}$$

Proof. This follows from Proposition 3.2. □

Corollary 3.5. *i_* and i^* induce (quasi)-inverse quasi-equivalences of categories of bimodules*

$$i_* : \mathcal{X}\text{–mod–}\mathcal{X} \rightleftarrows \mathcal{C}\text{–mod–}\mathcal{C} : i^* \tag{3.48}$$

and restrict to quasi-equivalences of subcategories of perfect bimodules.

Proof. It suffices to verify that the functor $i_* i^*$ induces a quasi-isomorphism on morphism spaces; the case of $i^* i_*$ is analogous. This follows from the calculation that for $\mathcal{B}_0$ and $\mathcal{B}_1$ bimodules over $\mathcal{C}$, the induced map on morphism spaces

$$\hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{B}_0, \mathcal{B}_1) \longrightarrow \hom_{\mathcal{C}\text{–}\mathcal{C}}(i^* i_* \mathcal{B}_0, i^* i_* \mathcal{B}_1), \tag{3.49}$$

when composed with the quasi-isomorphism induced by Corollary 3.4

$$\hom_{\mathcal{C}-\mathcal{C}}(i^* i_* \mathcal{B}_0, i^* i_* \mathcal{B}_1) \xrightarrow{\sim} \hom_{\mathcal{C}-\mathcal{C}}(\mathcal{B}_0, \mathcal{B}_1) \tag{3.50}$$

is the identity on homology.

For the statement about perfectness, we note the following: if (K, L) is a pair of objects in $\mathcal{X}$, then the (K, L) Yoneda bimodules over $\mathcal{X}$ is sent via i_* to the (K, L) Yoneda bimodules over $\mathcal{C}$ (as every object over $\mathcal{X}$ is an object over $\mathcal{C}$), and vice versa. We conclude that i_* preserves perfectness. Also, any Yoneda bimodule over $\mathcal{C}$ is equivalent to a summand of a complex of Yoneda bimodules in $\mathcal{C}$ using objects from $\mathcal{X}$, by taking the tensor products of (summands of) resolutions of each factor. Thus, if (K', L') is a pair of objects in $\mathcal{C}$, the Yoneda bimodule over (K', L') is sent via i^* to a perfect bimodule over $\mathcal{X}$; hence i^* also preserves perfectness. $\square$

The above calculation will allow us to give relatively short proofs that $\mathcal{X}$ and $\mathcal{C}$ have equivalent Hochschild theories.

Proposition 3.5. *Let $i : \mathcal{X} \to \mathcal{C}$ be as above. Let $\mathcal{B}$ be a bimodule over $\mathcal{C}$ and denote by $i^*\mathcal{B} := (i \otimes i)^*\mathcal{B}$ its pullback to $\mathcal{X}$. Then, we have a quasi-isomorphism of Hochschild chain and co-chain complexes.*

$$\mathrm{HH}_*(\mathcal{X}, i^*\mathcal{B}) \simeq \mathrm{HH}_*(\mathcal{C}, \mathcal{B}) \tag{3.51}$$

$$\mathrm{HH}^*(\mathcal{X}, i^*\mathcal{B}) \simeq \mathrm{HH}^*(\mathcal{C}, \mathcal{B}). \tag{3.52}$$

Proof. To establish (3.51), consider the chain complex computing the following bimodule tensor product

$$\Gamma_i^T \otimes_{\mathcal{C}\text{–mod–}\mathcal{X}} (i \otimes 1)^*\mathcal{B}. \tag{3.53}$$

We note that this chain complex can be interpreted as a Hochschild homology group of $\mathcal{X}$

$$\mathrm{HH}_*(\mathcal{X}, (i \otimes 1)^*\mathcal{B} \otimes_{\mathcal{C}} \Gamma_i^T) \tag{3.54}$$

and as one of $\mathcal{C}$

$$\mathrm{HH}_*(\mathcal{C}, \Gamma_i^T \otimes_{\mathcal{X}} (i \otimes 1)^*\mathcal{B}), \tag{3.55}$$

which respectively compute the left and right hand side of (3.51), by the invariance of Hochschild homology under quasi-isomorphisms of bimodules and Proposition 3.3 (note: need to also state the easier proposition involving $(i \otimes 1)^*\mathcal{B} \otimes_{\mathcal{C}} \Gamma_i^T$).

The statement about Hochschild cohomology is an immediate consequence of Corollary 3.5, and the fact that i_* (and i^*) send the diagonal bimodule of $\mathcal{X}$ to that of $\mathcal{C}$, up to quasi-isomorphism (and vice versa). $\square$

Corollary 3.6. *If $\mathcal{X}$ and $\mathcal{C}$ are smooth, there is a quasi-isomorphism of $\mathcal{C}-\mathcal{C}$ bimodules*

$$i_*\mathcal{X}^! \xrightarrow{\sim} \mathcal{C}^! \tag{3.56}$$

and a quasi-isomorphism of $\mathcal{X}-\mathcal{X}$ bimodules

$$\mathcal{X}^! \simeq i^*\mathcal{C}^! \tag{3.57}$$

Proof. One quasi-isomorphism follows from the other, so we will verify the first. We compute that

$$\begin{aligned} i_*\mathcal{X}^! &= \Gamma_i^T \otimes_{\mathcal{X}} \hom_{\mathcal{X}-\mathcal{X}}(\mathcal{X}_\Delta, \mathcal{X}_\Delta \otimes_{\mathbb{K}} \mathcal{X}_\Delta) \otimes_{\mathcal{X}} \Gamma_i \\ &\simeq \hom_{\mathcal{X}-\mathcal{X}}(\mathcal{X}_\Delta, \Gamma_i \otimes_{\mathbb{K}} \Gamma_i^T) \\ &\simeq \hom_{\mathcal{C}-\mathcal{C}}(i_*\mathcal{X}_\Delta, i_*(\Gamma_i \otimes_{\mathbb{K}} \Gamma_i^T)) \\ &\simeq \hom_{\mathcal{C}-\mathcal{C}}(\mathcal{C}_\Delta, \mathcal{C}_\Delta \otimes_{\mathbb{K}} \mathcal{C}_\Delta) \\ &= \mathcal{C}^!, \end{aligned} \tag{3.58}$$

where the first quasi-isomorphism used the perfectness of Γ_i and Γ_i^T, the second used the fullness of i_*, and the third uses Proposition 3.2. $\square$

Now, let $i : \mathcal{X} \hookrightarrow \mathcal{C}$ be the particularly simple Morita equivalence given as in (3.22). Corollary 3.5 shows that $\mathcal{X}$ is smooth if and only if $\mathcal{C}$ is.

Now, let $\sigma \in CC_*^-(\mathcal{X}, \mathcal{X})$ be a sCY structure, i.e. a cycle in negative cyclic chains such that the induced element in Hochschild chains is *non-degenerate*, inducing a quasi-isomorphism $\mathcal{X}^! \simeq \mathcal{X}_\Delta$ via the identification

$$CH_*(\mathcal{X}, \mathcal{X}) \simeq \hom_{\mathcal{X}-\mathcal{X}}(\mathcal{X}^!, \mathcal{X}_\Delta) \tag{3.59}$$

It is known that i induces a quasi-isomorphism of Hochschild complexes between $\mathcal{X}$ and $\mathcal{C}$, in a manner compatible up to homotopy with chain-level circle actions; hence one has a homotopy-commutative diagram of quasi-equivalences

$$\begin{tikzcd} CC_*^-(\mathcal{X}, \mathcal{X}) \arrow[r, "i_*", "\sim"'] \arrow[d, "\pi_{\mathcal{X}}"] & CC_*^-(\mathcal{C}, \mathcal{C}) \arrow[d, "\pi_{\mathcal{C}}"] \\ CH_*(\mathcal{X}, \mathcal{X}) \arrow[r, "i_*", "\sim"'] & CH_*(\mathcal{C}, \mathcal{C}) \end{tikzcd} \tag{3.60}$$

In particular, the cycle $i_*\sigma$ represents the unique homology class in the negative cyclic homology of $\mathcal{C}$ corresponding to $[\sigma]$. The following Lemma immediately verifies that $i_*\sigma$ defines a sCY structure on $\mathcal{C}$.

Lemma 3.2. *There exists a map $i_\sharp$ inducing a homotopy commutative diagram of identifications*

$$\begin{tikzcd} CH_*(\mathcal{X}, \mathcal{X}) \arrow[r, "i_*", "\sim"'] \arrow[d, "\mathrm{ev}_{\mathcal{X}}", "\sim"'] & CH_*(\mathcal{C}, \mathcal{C}) \arrow[d, "\mathrm{ev}_{\mathcal{C}}", "\sim"'] \\ \hom_{\mathcal{X}-\mathcal{X}}(\mathcal{X}^!, \mathcal{X}_\Delta) \arrow[r, "i_\sharp", "\sim"'] & \hom_{\mathcal{C}-\mathcal{C}}(\mathcal{C}^!, \mathcal{C}_\Delta) \end{tikzcd} \tag{3.61}$$

Moreover, $i_\sharp$ preserves non-degenerate elements.

Proof. The map $i_\sharp$ is constructed as the composition of quasi-isomorphisms

$$\begin{aligned} \hom_{\mathcal{X}\text{–}\mathcal{X}}(\mathcal{X}^!, \mathcal{X}_\Delta) &\simeq \hom_{\mathcal{C}\text{–}\mathcal{C}}(i_*\mathcal{X}^!, i_*\mathcal{X}_\Delta) \\ &\simeq \hom_{\mathcal{C}\text{–}\mathcal{C}}(\mathcal{C}^!, \mathcal{C}_\Delta \end{aligned} \tag{3.62}$$

where the first quasi-isomorphism uses the fact that i_* is full, and the second uses Corollary 3.6 and Proposition 3.2. The verification of homotopy-commutativity is a diagram chase, and the fact that it preserves non-degenerate elements follow essentially from fullness of i_*, and the fact that the diagram (3.61) homotopy commutes is a diagram chase... `INSERT DETAILS HERE`. $\square$

3.3 Geometric sCY structures on the wrapped Fukaya category

In this section, we describe the geometric sCY structure on the wrapped Fukaya category that has been constructed under suitable hypotheses by the second-named author in [21, 20]. The main point is that this sCY structure can be characterized by a *negative cyclic open-closed map* from the negative cyclic chains of the wrapped Fukaya category to the homotopy fixed points of the (twisted) symplectic co-chain complex (this is the complex computing the *twisted symplectic cohomology*, whose definition we will recall below). Namely, there is a map:

$$\widetilde{\mathcal{OC}}^- : \mathrm{CC}^-_{*-n}(\mathcal{W}_b) \to (SC^*_b(X))^{hS^1}. \tag{3.63}$$

where the homotopy fixed points are with respect to a canonical S^1 structure possessed by $SC^*_b(X)$ which we recall below. In Proposition 3.7, we show that a chain level choice of unit $1 \in C^*(X)$ determines a canonical element $\tilde{1} \in (SC^*_b(X))^{hS^1}$, coming from a natural map $C^*(X)[u] \to (SC^*_b(X))^{hS^1}$. The main Theorem we will use from [21, 20] is:

Theorem 3.2 ([21, 20])**.** *If there exists a cycle $\tilde{\sigma} \in \mathrm{CC}^-_{*-n}(\mathcal{W}_b)$ such that $[\tilde{\sigma}]$ maps to $[\tilde{1}]$, then it is cohomologically unique and determines a geometric sCY structure on $\mathcal{W}_b$. (Also,* (3.63) *is an isomorphism).*

Below we recall some of the geometric ingredients of this map in more detail, and then recall why such an element exists for $X = T^*M$, using [5, 21].

@@REmainder of this section needs revision.

3.3.1 Overview

In order to describe smooth Calabi-Yau structures, it is necessary to define *symplectic cohomology*, which is the closed string analogue of the wrapped Fukaya category. Symplectic cohomology, a type of *Hamiltonian Floer cohomology*, is formally the Morse cohomology of a certain *action functional* on the free loop space of X, depending on a choice of Hamiltonian H:

$$\begin{aligned} \mathcal{A}_H :& \mathcal{L}X \to \mathbb{R} \\ \mathcal{A}_H(x) :=& - \int x^* \lambda + \int_0^1 H(x(t))dt \end{aligned} \tag{3.64}$$

Time-dependent Hamiltonians $H_t : S^1 \times X \to \mathbb{R}$ fit straightforwardly into this viewpoint by considering the functional $\mathcal{A}_{H_t} = - \int x^*\lambda + \int_0^1 H_t(x(t))dt$. Critical points of $\mathcal{A}_{H_t}$ are time-1 orbits of the (in general time dependent) Hamiltonian vector field X_{H_t}. As in Morse theory, for a Liouville (hence non-compact) target X, the resulting homology theory is not completely invariant of choices, but rather depends on the behavior of H near ∞. We call the result *symplectic cohomology* when one uses H of *large growth*, as in (2.7) (equivalently, one can take the direct limit over all H with linear growth). We recall the construction of symplectic cohomology in 3.3.2.

The inclusion of constant loops $X \to \mathcal{L}X$ induces a canonical map

$$PSS : H^*(X) = H_{2n-*}(\bar{X}, \partial \bar{X}) \to SH^*(X) \tag{3.65}$$

and hence a distinguished element, the image of 1 or $[\bar{X}]$, which by abuse of notation we also call 1 or $[\bar{X}]$ respectively. Among other structures, $SH^*(X)$ has a pair-of-pants product, making PSS a ring map (in particular, $1 \in SH^*(X)$ is the unit for this structure).

Where do we recall the construction?

As formally the Morse homology for a function on an (infinite dimensional) space $\mathcal{L}X$ with S^1 action, one expects the symplectic co-chain complex $SC^*(X)$ to inherit an S^1 structure (certainly the singular homology of a space with S^1 action does!). Unlike the case of singular homology, this S^1 action is not *strict*, in the sense of CITE; in order to have a well-defined theory, one breaks S^1-symmetry in both the Hamiltonian and almost complex structure, which essentially means working with a non-invariant Morse function and metric on $\mathcal{L}X$. It was noted by Seidel [**?**]… in §3.3.4 we recall the proof of the following fact, due to Bourgeois-Oancea [**?**]:

Next, the inclusion of constant loops $X \hookrightarrow \mathcal{L}X$ is naturally S^1 equivariant, with respect to the trival action on X and loop rotation on $\mathcal{L}X$. This leads one to expect the following, which we prove in §REF:

3.3.2 (twisted) Symplectic cohomology

There is an added subtlety in the definition, coming from the fact that non-trivial time 1 orbits of an autonomous Hamiltonian such as H occur in S^1 families, and thus are degenerate. In order to define Floer cohomology, we break this S^1 symmetry by choosing $F : S^1 \times X \to \mathbb{R}$ a smooth non-negative function, with

$$F \text{ and } \lambda(X_F) \text{ uniformly bounded in absolute value; and} \tag{3.66}$$

$$\begin{aligned} &\text{all time-1 periodic orbits of } Y_{S^1}, \text{ the (time-dependent) Hamiltonian vector field} \\ &\text{corresponding to } H_{S^1}(t,m) = H(m) + F(t,m), \text{ are non-degenerate.} \end{aligned} \tag{3.67}$$

The second condition can be ensured by, for example, choosing F to be supported near orbits of X_H as a small multiple of a standard Morse function on S^1. Fixing such a choice, define

$$\mathcal{O}$$

to be the set of (time-1) periodic orbits of H_{S^1}.

Then, given an element $y \in \mathcal{O}$, we can define the *degree* of y to be

$$\deg(y) := n - CZ(y) \tag{3.68}$$

where CZ is the Conley-Zehnder index of y. Now, define the **symplectic co-chain complex twisted by background class** $b \in H^2(X, \mathbb{Z}_2)$ over $\mathbb{K}$ to be

$$CH^i_b(X; H, F, J_t) = \bigoplus_{y \in \mathcal{O}, deg(y)=i} |o_y|_{\mathbb{K}} \otimes \kappa^b_y, \tag{3.69}$$

where

- the **orientation line** o_y is a real vector space associated to every orbit in $\mathcal{O}$ via a process described in [4]*§C.6 and $|o_y|$ denotes the free abelian group generated by the two possible orientations on o_y, with the relationship that their sum vanishes.
- the **background line** κ^b_y is the free abelian group generated by the two possible spin structures on the restriction $y^*(E_b)$ of the vector bundle E_b to y, with the relation that their sum vanishes (note that Spin structures on S^1 are acted upon freely and transitively by $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$), hence there are two).

We remark that the κ^b_y can be replaced with the trivial line if $b = 0$.

Given an S^1 dependent family $J_t \in \mathcal{J}_1(M)$, consider maps

$$u : (-\infty, \infty) \times S^1 \to M \tag{3.70}$$

converging exponentially at each end to a time-1 periodic orbit of H_{S^1} and satisfying Floer's equation

$$(du - X_{S^1} \otimes dt)^{0,1} = 0. \tag{3.71}$$

Here, the cylinder $A = (-\infty, \infty) \times [0, 1]$ is equipped with coordinates s, t and a complex structure j with $j(\partial s) = \partial_t$; in these coordinates (3.71) becomes

$$\partial_s u = -J_t(\partial_t u - X). \tag{3.72}$$

Given time 1 orbits $y_0, y_1 \in \mathcal{O}$, denote by $\tilde{\mathcal{M}}(y_0; y_1)$ the set of maps u converging to y_0 when $s \to -\infty$ and y_1 when $s \to +\infty$. Standard analysis equips this set with a topology and a natural $\mathbb{R}$ action coming from translation in the s direction. For generic J_t, the moduli space is smooth of dimension $\deg(y_0) - \deg(y_1)$ with free $\mathbb{R}$ action unless it is of dimension 0.

Definition 3.6. *Define*

$$\mathcal{M}(y_0; y_1) \tag{3.73}$$

to be the quotient of $\tilde{\mathcal{M}}(y_0; y_1)$ *by the* $\mathbb{R}$ *action whenever it is free, and the empty set when the* $\mathbb{R}$ *action is not free.*

Construct the analogous bordification $\overline{\mathcal{M}}(y_0; y_1)$ by adding **broken cylinders**

$$\overline{\mathcal{M}}(y_0; y_1) = \coprod \mathcal{M}(y_0; x_1) \times \mathcal{M}(x_1; x_2) \times \cdots \times \mathcal{M}(x_k; y_1) \tag{3.74}$$

A strong form of a maximum principle for solutions to (3.72), along with standard analysis, ensures that

Lemma 3.3 (see e.g., [4], Lemma 5.2)**.** *For generic* J_t*, the moduli space* $\overline{\mathcal{M}}(y_0, y_1)$ *is a compact manifold with boundary of dimension* $\deg(y_0) - \deg(y_1) - 1$*. The boundary is covered by the closure of the images of natural inclusions*

$$\mathcal{M}(y_0; y) \times \mathcal{M}(y; y_1) \to \overline{\mathcal{M}}(y_0; y_1). \tag{3.75}$$

Moreover, for each y_1*,* $\overline{\mathcal{M}}(y_0; y_1)$ *is empty for all but finitely many choices of* y_0*.*

For a regular $u \in \mathcal{M}(y_0; y_1)$ with $\deg(y_0) = \deg(y_1) + 1$, by the discussion in [**?**] (but using the language of [37] as in [4]*§C) we get an isomorphism of orientation lines

$$\mu_u : o_{y_1} \longrightarrow o_{y_0}. \tag{3.76}$$

As described in [3], because $u(s,t)$ (exponentially) converges to y_0 and y_1 at $\pm\infty$, a Spin structure on either $y_i^*(E_b)$ induces a Spin structure on $u^*(E_b)$. In particular, u induces a canonical isomorphism

$$\kappa^b_{y_1} \to \kappa^b_{y_0} \tag{3.77}$$

Thus we can define a differential

$$d : CH^*(M; H, F_t, J_t) \longrightarrow CH^*(M; H, F_t, J_t) \tag{3.78}$$

$$d([y_1]) = \sum_{y_0; \deg(y_0) = \deg(y_1)+1} \sum_{u \in \mathcal{M}(y_0; y_1)} (-1)^{\deg(y_1)} \partial_u([y_1]), \tag{3.79}$$

where ∂_u is the tensor product of (3.76) and (3.77).

Lemma 3.4.

$$d^2 = 0.$$

Call the resulting group $SH^*_b(X)$. Among other TFT structures, (twisted) symplectic cohomology is a ring via the *pair of pants product.*

Remark 3.2. *The grading conventions for* $SH^*_b(X)$ *are determined by saying the identity element* e *lives in degree zero, and that the product map is degree zero.*

3.3.3 The open-closed map

There is a geometric open-closed map

$$\mathcal{OC}^{nu} = \check{\mathcal{OC}} \oplus \hat{\mathcal{OC}} : \mathrm{CH}^{nu}_{*-n}(\mathcal{F}, \mathcal{F}) \longrightarrow SC^*(M), \tag{3.80}$$

with source the *non-unital Hochschild complex*, which is described in [21] (the map with source the usual cyclic bar complex was originally described in [4]).

The first component $\check{\mathcal{OC}}$ of (3.80) is controlled by a space denoted

$$\overline{\check{\mathcal{R}}}^1_d, \tag{3.81}$$

the Deligne-Mumford compactification of the abstract moduli space of discs with d boundary positive punctures $z_1, \ldots, z_d$ labeled in counterclockwise order and 1 interior negative puncture p_{out}, with an asymptotic marker pointing at p_{out} pointing towards z_d. One can equip (3.81) with the structure of a manifold with corners, with codimension 1 boundary strata described in [4]*§C.3. We trivialize the main component of $\check{\mathcal{R}}^1_d$ by choosing the (unique) unit disc representative of any element fixing the position of z_d and p_{out} at 1 and 0. The ordered (angular) coordinates of the remaining marked points induce the (anti-)canonical orientation

$$-\, dz_1 \wedge \cdots \wedge dz_{d-1}. \tag{3.82}$$

The second component $\hat{\mathcal{OC}}$ is controlled by the space

$$\mathcal{R}^{1,free}_d \tag{3.83}$$

parametrising discs with d boundary punctures $z_1, \ldots, z_d$ and an interior puncture z_{out}, with asymptotic marker pointing *anywhere between* z_1 *and* z_d (so the dimension is one higher than (3.81)). For concreteness, we work with an alternate, but diffeomorphic model

$$\hat{\mathcal{R}}^1_d, \tag{3.84}$$

the abstract moduli space of discs with $d+1$ boundary punctures z_f, z_1, …, z_d and an interior puncture z_{out} with asymptotic marker pointing towards the boundary point z_f, modulo automorphism. The point z_f is marked as "auxiliary," meaning that we forget it before assigning asympotic and boundary conditions (but before compactification, the position of z_f is remembered by the direction of the asymptotic marker). Trivializing the main component by fixing z_f and p_{out}, we obtain the orientation

$$-\, dz_1 \wedge \cdots \wedge dz_d. \tag{3.85}$$

The Deligne-Mumford compactification

$$\overline{\hat{\mathcal{R}}}^1_d, \tag{3.86}$$

as a manifold with corners, is equal to the compactification $\overline{\check{\mathcal{R}}}^1_{d+1}$, except from the point of view of assigning Floer data, as we will be forgetting the point z_f instead of fixing asymptotics for it. Thus, we track any stratum containing z_f:

- we treat the main component (containing z_{out} and k boundary marked points) as belonging to $\overline{\hat{\mathcal{R}}}^1_{k-1}$ if it contains z_f and $\overline{\check{\mathcal{R}}}^1_k$ otherwise; and

- If the ith boundary marked point of any non-main component was z_f, we view it as an element of $\mathcal{R}^{k,f_i}$, the space of discs with 1 output and k input marked points removed from the boundary, with the ith point marked as "forgotten" (see [21, §A]. This is abstractly diffeomorphic to $\mathcal{R}^k$, but by definition, a Floer datum for any element S of $\mathcal{R}^{k,f_i}$ is a choice of data for the corresponding element $\pi_i(S)$ in which the ith point has been forgotten. Moreover, any univeral and consistent Floer data for the family is chosen to be independent of the position of z_i.

  We treat any other non-main component as belonging to $\mathcal{R}^k$ as usual.

Thus, the codimension-1 boundary of the Deligne-Mumford compactification is covered by the natural inclusions of the following strata

$$\overline{\mathcal{R}}^m \times_i \overline{\hat{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq i < d-m+1 \tag{3.87}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} \overline{\check{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq j \leq m,\ 1 \leq k \leq m \tag{3.88}$$

where the notation $\times_j$ means that the output of the first component is identified with the jth boundary input of the second.

The map which deletes z_f $\pi_f : \hat{\mathcal{R}}^1_d \to \mathcal{R}^{1,free}_d$ extends to a map $\overline{\pi}_f$ from the compactification $\overline{\hat{\mathcal{R}}}^1_d$ as follows: we call a component T of a representative S of $\overline{\mathcal{R}}^1_d$ the *main component* if it contains the interior marked point, and the *secondary component* if its output is attached to the main component. Then, $\overline{\pi}_f$ puts the auxiliary point z_f back in, eliminates any component which is not main or secondary which has only one non-auxiliary marked point p, and labels the positive marked point below this component by p. Given a representative S of $\overline{\hat{\mathcal{R}}}^1_d$, we call $\overline{\pi}_f(S)$ the *associated reduced surface*.

A *universal and consistent Floer datum for the non-unital open-closed map* is, as before, an inductive choice of Floer data for every d and the reduced surface corresponding to every component of every representative of (3.81) and (3.86), agreeing (up to conformal equivalence) to infinite order with previously chosen Floer data on boundary strata (including those for the A_∞ structure). This datum exists by an inductive argument. Fixing such a datum $(\mathbf{D}_{\check{\mathcal{OC}}}, \mathbf{D}_{\hat{\mathcal{OC}}})$, for a d tuple of objects $L_1, \ldots, L_d$, d chords $x_i \in \hom_{\mathcal{W}}(L_i, L_{i+1 \bmod d})$, and an orbit $y_{out} \in \mathcal{O}$, we obtain Gromov-compactified moduli spaces

$$\begin{gathered} \overline{\check{\mathcal{R}}}^1_d(y; x_d, \ldots, x_1) \\ \overline{\hat{\mathcal{R}}}^1_d(y; x_d, \ldots, x_1) \end{gathered} \tag{3.89}$$

whose interiors consists of parametrized families of Floer curves

$$\begin{aligned}
&\{(S,u)|S \in \check{\mathcal{R}}^1_d : u : S \to M, (du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer data given by } \mathbf{D}_{\check{\mathcal{OC}}}(S)\} \\
&\{(S,u)|S \in \hat{\mathcal{R}}^1_d : u : \overline{\pi}_f(S) \to M, (du - X \otimes \alpha)^{0,1} = 0 \text{ using the Floer data given by } \mathbf{D}_{\hat{\mathcal{OC}}}(S)\}
\end{aligned} \tag{3.90}$$

satisfying the usual moving boundary conditions and asymptotics

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \bmod d} \\ \lim_{s \to \pm\infty} u \circ \epsilon^k(s, \cdot) = x_k & \\ \lim_{s \to -\infty} u \circ \delta(s, \cdot) = y & \end{cases}, \tag{3.91}$$

The boundary consists of maps from strata corresponding to the boundary strata of (3.81) as well as semi-stable breakings.

Lemma 3.5 (Transversality, index calculations, and compactness)**.** *For generic choices of universal and consistent Floer data, $\overline{\check{\mathcal{R}}}^1_d(y; x_d, \ldots, x_1)$ is a compact manifold (with boundary) of dimension*

$$\deg(y) - n + d - 1 - \sum_{i=1}^d \deg(x_i). \tag{3.92}$$

and $\overline{\check{\mathcal{R}}}^1_d(y; x_d, \ldots, x_1)$ is a compact manifold (with boundary) of dimension

$$\deg(y) - n + d - \sum_{i=1}^d \deg(x_i). \tag{3.93}$$

For inputs and outputs such that (3.92) = 0, elements $u \in \check{\mathcal{R}}^1_d(y; x_d, \ldots, x_1)$ are rigid and induce (using (3.82) and [4]*Lemma C.4, a minor generalization of [37]*(12b-d)) isomorphisms of orientation lines

$$(\check{\mathcal{R}}^d_1)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y. \tag{3.94}$$

The length d part of $\check{\mathcal{OC}} := \oplus_d \check{\mathcal{OC}}_d$ is a sum over all such maps for varying y:

$$\check{\mathcal{OC}}_d : \bigoplus_{L_1, \ldots, L_d} \hom_{\mathcal{W}}(L_d, L_1) \otimes \hom_{\mathcal{W}}(L_{d-1}, L_d) \otimes \cdots \otimes \hom_{\mathcal{W}}(L_1, L_2) \to SC^*(M) \tag{3.95}$$

$$\check{\mathcal{OC}}_d([x_d] \otimes \cdot \otimes [x_1]) := \tag{3.96}$$

$$\sum_{\deg(y) = n - d + 1 + \sum \deg(x_i)} \sum_{u \in \overline{\check{\mathcal{R}}}^d_1(y; x_d, \ldots, x_1)} (-1)^{\deg(x_d) + \sum_{k=1}^d k \deg(x_k)} (\check{\mathcal{R}}^d_1)_u([x_d], \ldots, [x_1]). \tag{3.97}$$

Similarly, for inputs and outputs such that (3.93) = 0, the rigid elements $u \in \hat{\mathcal{R}}^1_d(y; x_d, \ldots, x_1)$ induce isomorphisms

$$(\hat{\mathcal{R}}^d_1)_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y. \tag{3.98}$$

which give the relevant component of the map $\hat{\mathcal{OC}}$:

$$\hat{\mathcal{OC}}_d : \bigoplus_{L_1,\ldots,L_d} \hom_{\mathcal{W}}(L_d, L_1) \otimes \hom_{\mathcal{W}}(L_{d-1}, L_d) \otimes \cdots \otimes \hom_{\mathcal{W}}(L_1, L_2) \to SC^*(M) \tag{3.99}$$

$$\hat{\mathcal{OC}}_d([x_d] \otimes \cdot \otimes [x_1]) := \tag{3.100}$$

$$\sum_{\deg(y)=n-d+1+\sum \deg(x_i)} \sum_{u \in \check{\bar{\mathcal{R}}}_1^d(y; x_d,\ldots,x_1)} (-1)^{\sum_{k=1}^d k \deg(x_k)} (\hat{\mathcal{R}}_1^d)_u([x_d], \ldots, [x_1]). \tag{3.101}$$

A codimension-1 boundary analysis of (3.89) implies that

Proposition 3.6 ([21], @@CITE PROPOSITION)**.**

$$\check{\mathcal{OC}} \circ b = d_{SC} \circ \check{\mathcal{OC}}. \tag{3.102}$$

$$\check{\mathcal{OC}} \circ (1 - t) + \hat{\mathcal{OC}} \circ b' = d_{SC} \circ \hat{\mathcal{OC}}. \tag{3.103}$$

Remark 3.3. *Equation (3.102) of the Proposition, which is due to [4], implies that $\check{\mathcal{OC}}$ considered as a map from the sub-complex $\check{\mathrm{CH}}_*(\mathcal{F},\mathcal{F})$ is also a chain map. Since the inclusion $\check{\mathrm{CH}}_*(\mathcal{F},\mathcal{F}) \subset \mathrm{CH}_*^{nu}(\mathcal{F},\mathcal{F})$ is a quasi-isomorphism, we conclude that $[\check{\mathcal{OC}}] = [\mathcal{OC}^{nu}]$. However, the complex $\check{\mathrm{CH}}_*(\mathcal{F},\mathcal{F})$ does not have a visible circle action, unless $\mathcal{F}$ is equipped with strict units).*

3.3.4 Circle action on the closed sector

The actions $\Delta_k := \mu^{k|1}(\beta, \ldots, \beta, \cdot) : SC^*(X) \to SC^{*-2k+1}(X)$ constituting the weak S^1 structure come from counts of parametrized moduli spaces of solutions. It is convenient to build the parameters (which are products of S^1's and simplices), which determine amounts of rotation present in the gradient flow equation, as extra data on the cylinder itself, as done in [21].

Definition 3.7. *A* r**-point angle-decorated cylinder** *consists of a semi-infinite or infinite cylinder $C \subseteq (-\infty, \infty) \times S^1$, along with a collection of auxiliary points $p_1, \ldots, p_r \in C$, satisfying*

$$(p_1)_s \leq \cdots \leq (p_r)_s, \tag{3.104}$$

where $(a)_s$ denotes the $s \in (-\infty, \infty)$ coordinate. (In particular, points are allowed to be equal) The **heights** *associated to this data are the s coordinates*

$$h_i = (p_i)_s,\ i = 1, \ldots, r \tag{3.105}$$

and the **angles** *associated to C are the S^1 coordinates*

$$\theta_i := (p_1)_t,\ i \in 1, \ldots, r. \tag{3.106}$$

The **cumulative rotation** of an r-point angle-decorated cylinder is the first angle:

$$\eta := \eta(C, p_1, \ldots, p_r) = \theta_1. \tag{3.107}$$

The i**th incremental rotation** of an r-point angle-decorated cylinder is the difference between the ith and $i-1$st angles:

$$\kappa_i^{inc} := \theta_i - \theta_{i+1} \text{ (where } \theta_{r+1} = 0). \tag{3.108}$$

Definition 3.8. *The* **moduli space of r-point angle-decorated cylinders**

$$\mathcal{M}_r \tag{3.109}$$

is the space of r-point angle-decorated infinite cylinders, modulo translation.

For an element of this moduli space, the angles and relative heights of the auxiliary points give a non-canonical isomorphism

$$\mathcal{M}_r \simeq (S^1)^r \times [0,\infty)^{r-1}. \tag{3.110}$$

In particular (3.109) is an open manifold with corners, with corner strata given by loci where heights of the auxiliary points p_i are coincident. Given an arbitrary representative C of $\mathcal{M}_r$ with associated heights $h_1, \ldots, h_r$, we can always find a translation $\tilde{C}$ satisfying $\tilde{h}_r = -\tilde{h}_1$; we call this the *standard representative* associated to C.

Given a representative C of this moduli space, and a fixed constant δ, we fix a positive cylindrical end around $+\infty$

$$\begin{aligned} \epsilon^+ : [0,\infty) \times S^1 &\to C \\ (s,t) &\mapsto (s + h_r + \delta, t) \end{aligned} \tag{3.111}$$

and a negative cylindrical end around $-\infty$ (note the angular rotation in t!):

$$\begin{aligned} \epsilon^- : (-\infty, 0] \times S^1 &\to C \\ (s,t) &\mapsto (s - (h_1 - \delta)), t + \theta_1). \end{aligned}$$

These ends are disjoint from the p_i and vary smoothly with C.

There is a compactification $\overline{\mathcal{M}}_r$ of $\mathcal{M}_r$ consisting of **broken r-point angle-decorated cylinders**. The stratum consisting of s-fold broken configurations lies in the codimension s boundary, with the manifolds-with-corners structure explicitly defined by local gluing maps using the ends (3.111) and (3.112). Note that the gluing maps rotate the bottom cylinder in order to match an end (3.111) with (3.112), and identify the remaining free ends with the ones already chosen on the glued space.

In particular, the compactification $\overline{\mathcal{M}}_r$ has codimension-1 boundary covered by the images of the natural inclusion maps

$$\overline{\mathcal{M}}_{r-k} \times \overline{\mathcal{M}}_k \longrightarrow \partial\mathcal{M}_r, \; 0 < k < r \tag{3.112}$$

$$\overline{\mathcal{M}}_r^{i,i+1} \longrightarrow \partial\mathcal{M}_r, \; 1 \leq i < r, \tag{3.113}$$

where $\overline{\mathcal{M}}_r^{i,i+1}$ denotes the compactification of the locus where ith and $i+1$st heights are coincident. With regards to the this stratum, for $r > 1$ there is a projection map, a forgetful map which remembers first of the coordinates with coincident heights:

$$\begin{aligned} \pi_i : \overline{\mathcal{M}}_r^{i,i+1} &\longrightarrow \mathcal{M}_{r-1} \\ (h_1, \ldots, h_i, h_{i+1} = h_i, h_{i+2}, \ldots, h_r) &\longmapsto (h_1, \ldots, h_i, h_{i+2}, \ldots, h_r) \\ (\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_r) &\longmapsto (\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+2}, \ldots, \theta_r). \end{aligned} \tag{3.114}$$

Definition 3.9. *A* **Floer datum for an r-point angle-rotated cylinder** $\tilde{C} := (C, p_1, \ldots, p_r)$ *consists of the following choices:*

- *The positive and negative strip-like ends on $\epsilon^\pm : C^\pm \to C$ chosen in (3.111)-(3.112).*
- *The one-form on C given by $\alpha = dt$.*
- *A Hamiltonian $H_{\tilde{C}} : C \to \mathcal{H}(M)$ compatible with the positive and negative strip-like ends, meaning that*

$$(\epsilon^\pm)^* H_C = H, \tag{3.115}$$

  for a previously fixed choice H.
- *A time-dependent perturbation term $F_{\tilde{C}} : C \to \mathcal{H}(M)$, supported on a neighborhood of the orbits of H, and compatible with positive and negative ends*

$$(\epsilon^\pm)^* F_C = F_t \tag{3.116}$$

- *A surface dependent complex structure $J_{\tilde{C}} : C \to \mathcal{J}_1(M)$ also compatible with $\epsilon^\pm$.*

$$(\epsilon^\pm)^* J_{\tilde{C}} = J_t \tag{3.117}$$

  for our previously fixed choice J_t.

Definition 3.10. *A universal and consistent choice of Floer data for the weak S^1 action is an inductive choice of Floer data, for each k and each representative $S = (C, p_1, \ldots, p_k)$ of $\overline{\mathcal{M}}_k$, satisfying the following conditions at boundary strata:*

- *Near a boundary stratum of the form (3.112) the data agrees with the product of Floer data coming from lower-dimensional spaces, up to conformal equivalence. Moreover, with respect to the boundary chart structure induced by the gluing map, the Floer data agrees to infinite order with the data obtained by gluing.*
- *At a boundary stratum of the form (3.113), the Floer datum for S is conformally equivalent to one pulled back from $\overline{\mathcal{M}}_{k-1}$ via the forgetful map π_i.*

Lemma 3.6 (CITE [21])**.** *Universal and consistent choices of Floer data exist.*

Picking a universal and consistent choice of Floer data for the weak S^1 action, for $(x^+, x^-) \in \mathcal{O}$, we define for each $k \geq 1$,

$$\mathcal{M}_k(x^+; x^-) \tag{3.118}$$

to be the parametrized space of maps

$$\{S = (C, p_1, \ldots, p_r) \in \mathcal{M}_k,\ u : C \to M| \begin{cases} \lim_{s\to\pm\infty}(\epsilon^{\pm})^* u(s, \cdot) & = x^{\pm} \\ (du - X_{H_S} \otimes dt)^{(0,1)_S} & = 0. \end{cases} \tag{3.119}$$

meaning that u solves Floer's equation with respect to the Hamiltonian H_S and complex structure J_S chosen for the given element S. Standard methods imply that boundary of the Gromov bordification $\overline{\mathcal{M}}_k(x^+; x^-)$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_r(y; x^-) \times \overline{\mathcal{M}}_{k-r}(x^+; y) \to \partial\overline{\mathcal{M}}_k(x^+; x^-) \tag{3.120}$$

$$\overline{\mathcal{M}}_k^{i,i+1}(x^+; x^-) \to \partial\overline{\mathcal{M}}_k(x^+; x^-), \tag{3.121}$$

along with the usual semi-stable strip breaking boundaries

$$\begin{aligned} \overline{\mathcal{M}}_k(y; x^-) \times \overline{\mathcal{M}}(x^+; y) \to \partial\overline{\mathcal{M}}_k(x^+; x^-) \\ \overline{\mathcal{M}}(y; x^-) \times \overline{\mathcal{M}}_k(x^+; y) \to \partial\overline{\mathcal{M}}_k(x^+; x^-) \end{aligned} \tag{3.122}$$

Lemma 3.7. *For generic choices of Floer data for the weak S^1 action, the moduli spaces $\overline{\mathcal{M}}_k(x^+; x^-)$ are smooth compact manifolds of dimension*

$$\deg(x^+) - \deg(x^-) + (2k - 1). \tag{3.123}$$

As usual, signed counts of rigid elements of this moduli space for varying x^+ and x^- (using induced maps on orientation lines) give the matrix coefficients for the overall map

$$\delta_k : CF^*(M) \to CF^{*-2k+1}(M). \tag{3.124}$$

Lemma 3.8 (CITE [21])**.** *For each k,*

$$\sum_{i=0}^{k} \delta_i \delta_{k-i} = 0. \tag{3.125}$$

Proof. This follows from a codimension 1 boundary analysis; the key point being that the portions of the codimension-1 boundary corresponding to strata $\mathcal{M}_r^{i,i+1}$ contribute the zero operation (as the choice of Floer data is invariant under forgetting a marked point, and hence solutions to Floer's equation on this locus are never rigid). □

3.3.5 Inclusion of constant loops

We recall now the definition of a PSS version of inclusion of constant loops:

$$C^*(T^*M) \overset{PD}{\cong} C_{2n-*}(T*M, S^*M) \to SC_b^*(T^*M). \tag{3.126}$$

First, choose

- a sub-closed one-form α (*subclosed* means $d\alpha \leq 0$) on the cylinder C with $\alpha = 0$ near $+\infty$ and dt near $-\infty$.
- A domain dependent almost complex structure $J_C : C \to \mathcal{J}(T^*M)$ equal to J_t near $-\infty$ and independent of s, t near $+\infty$.

Now, given N a manifold with boundary equipped with a map to (T^*M, S^*M), and an orbit $y \in \mathcal{O}$, we define

$$\mathcal{M}(y, N) \tag{3.127}$$

to be the space of maps $u : C \to T^*M$ satisfying Floer's equation

$$(du - X \otimes \alpha)^{0,1} = 0 \tag{3.128}$$

with respect to the one-form α and J_C, with asyptotics

$$\lim_{s\to+\infty} u(s,t) \in N \text{ is independent of } t, \text{ and } \lim_{s\to-\infty} u(s,t) = y. \tag{3.129}$$

INSERT DISCUSSION ABOUT INDUCED CHOICE OF ORIENTATION LINE HERE

For generic choices of J_C, (3.127) is a smooth manifold of dimension $\operatorname{codim}(N) - |y|$. Moreover, the usual compactness result applies to show that the Gromov bordification $\overline{\mathcal{M}}(y, N)$, whose codimension 1 strata consists of $\mathcal{M}(y, \partial N)$ and $\coprod_{|y_0|=|y|-1} \mathcal{M}(y, y_0) \times \mathcal{M}(y, N)$, is compact if N is properly embedded.

In particular, defining

$$PSS(N) := \sum_{y; |y| = \operatorname{codim}(N)} \sum_{u \in \mathcal{M}(y, \partial N)} |\mathcal{M}_u^N| \tag{3.130}$$

we immediately see that PSS is a chain map...

We can think of $C_*(\bar{X}, \partial\bar{X})$ as an S^1-complex with trivial S^1-action, meaning that $\Delta_i = 0$ for $i > 0$. Then,

Proposition 3.7. *There is a natural S^1-equivariant enhancement*

$$\widetilde{PSS} : C_*(\bar{X}, \partial\bar{X}) \to SC^*(X) \tag{3.131}$$

with respect

Proof. Fix a choice of Hamiltonian in the definition of symplectic cohomology to be C^2 small and time-independent in the interior of $\bar{X}$. In this case, all of the surface dependent Hamiltonians appearing on the moduli spaces $\mathcal{M}_r$ can also be chosen to be C^2-small and time-independent near the zero section. The result of such a choice is that on the (action-filtered) subcomplex consisting of critical points of H, the operations $\Delta_1, \ldots, \Delta_r, \ldots,$ will be zero on the chain level. (this is because solutions to the parametrized Floer equation on trajectories with asymptotics lying in this subcomplex persist for all parameters in $\mathcal{M}_r$ and thus will never be rigid). $\square$ Incomplete proof...

3.3.6 The cyclic open-closed map

In this section, we summarize the construction [21] of an enhancement of $\mathcal{OC}$ to a degree n morphism of S^1-complexes. This enhancement amounts to the data of degree zero chain maps

$$\mathcal{OC}^{k|1} : \overline{C_{-*}(S^1)}^{\otimes k} \otimes \mathrm{CH}_*(\mathcal{W}) \to SC^*(X)[n], \tag{3.132}$$

with $\mathcal{OC}^{0|1} = \mathcal{OC}$, satisfying the A_∞ module equations. Equivalently, this is a family of maps of degree $n - 2k$

$$\mathcal{OC}^k = \mathcal{OC}^{k|1}(\underbrace{\beta, \ldots, \beta}_{k \text{ times}}, \cdot) : \mathrm{CH}_*(\mathcal{W}) \to SC^*(X), \tag{3.133}$$

with $\mathcal{OC}^0 = \mathcal{OC}$, satisfying, for each s:

$$\sum_{i=0}^{s} \delta_i \mathcal{OC}^{d-i} = \mathcal{OC}^{s-1} \circ B^{nu} + \mathcal{OC}^s \circ b_{\mathrm{CH}^{nu}}. \tag{3.134}$$

(where $\mathcal{OC}^{-1} = 0$). With respect to the decomposition of chain complexes $\mathrm{CH}^{nu}_*(\mathcal{W}) = \check{\mathrm{CH}}_*(\mathcal{W}) \oplus \hat{\mathrm{CH}}_*(\mathcal{W})$, we can write $\mathcal{OC}^k = \check{\mathcal{OC}}^k \oplus \hat{\mathcal{OC}}^k$, translating (3.134) into the pair of equations, for each s:

$$\sum_{i=0}^{s} \delta_i \check{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^{k-1} B^{nu} + \check{\mathcal{OC}}^k b \tag{3.135}$$

$$\sum_{i=0}^{s} \delta_i \hat{\mathcal{OC}}^{k-i} = \hat{\mathcal{OC}}^k b' + \check{\mathcal{OC}}^k (1-t). \tag{3.136}$$

To define $\check{\mathcal{OC}}^k$, $\hat{\mathcal{OC}}^k$, we will define, for each d, two moduli spaces, in the following order:

$${}_k\check{\mathcal{R}}^1_{d+1} \tag{3.137}$$

$${}_k\hat{\mathcal{R}}^1_d, \tag{3.138}$$

and understand their compactifications. For $k = 0$, these moduli spaces are simply $\check{\mathcal{R}}^1_{d+1}$ and $\hat{\mathcal{R}}^1_d$ respectively.

The first space (3.137) is the moduli space of discs with with $d+1$ positive boundary marked points $z_0, \ldots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \ldots, p_k$ (without an asymptotic marker), marked as *auxiliary.* Also, choosing a representative of an element this moduli space which fixes z_0 at 1 and z_{out} at 0 on the unit disc, the p_i should be *strictly radially ordered*; that is,

$$0 < |p_1| < \cdots < |p_k| < \frac{1}{2}. \tag{3.139}$$

Using the above representative, one can talk about the *angle*, or *argument* of each auxiliary interior marked point,

$$\theta_i := \arg(p_i). \tag{3.140}$$

We require that with respect to the above representative,

$$\text{the asymptotic marker on } z_{out} \text{ points in the direction } \theta_1 \text{ (or towards } z_0 \text{ if } k = 0). \tag{3.141}$$

(equivalently one could define $\theta_{k+1} = 0$, so that θ_1 is always defined). For every representative $S \in {}_k\check{\mathcal{R}}^1_{d+1}$,

$$\begin{array}{l}\text{fix a negative cylindrical end around } z_{out} \text{ not containing any } p_i\text{, compatible with the}\\ \text{direction of the asymptotic marker, or } \textit{equivalently compatible with the angle } \theta_1.\end{array} \tag{3.142}$$

The compactification of (3.137) is a real blow-up of the ordinary Deligne-Mumford compactification, in the sense of [?] (see [?] for a first discussion in the context of Floer theory). As discussed in CITE, the codimension 1 boundary of the compactified check moduli space ${}_k\overline{\check{\mathcal{R}}}^1_{d+1}$ is covered by the images of the natural inclusions of the following strata:

$$\overline{\mathcal{R}^s} \times {}_k\overline{\check{\mathcal{R}}}^1_{d-s+2} \tag{3.143}$$

$${}_s\overline{\check{\mathcal{R}}}^1_{d} \times \overline{\mathcal{M}}_{k-s} \tag{3.144}$$

$${}_{k-1}\overline{\check{\mathcal{R}}}^{S^1}_{d+1} \tag{3.145}$$

$${}_k^{i,i+1}\overline{\check{\mathcal{R}}}^1_{d+1} \tag{3.146}$$

The strata (3.145)-(3.146), which refer to the loci where $|p_k| = \frac{1}{2}$ and $|p_i| = |p_{i+1}|$ respectively, describe the boundary loci of the ordering condition (3.139) and hence come equipped with a natural manifold with corners structure. The strata (3.143)-(3.144) have manifold with corners structure given by standard local gluing maps using fixed choices of strip-like ends near the boundary. For (3.143) this is standard, and for (3.144), the local gluing map uses the cylindrical ends (3.142) and (3.111) (in other words, one rotates the r-pointed angle cylinder by an amount commensurate to the angle of the first marked point p_{k-s+1} on the disk before gluing).

Associated to the stratum (3.146) where p_i and p_{i+1} have coincident magnitudes, there is a forgetful map, with one dimensional fibers

$$\check{\pi}_i : {}_k^{i,i+1}\overline{\check{\mathcal{R}}}^1_{d+1} \to {}_{k-1}\overline{\check{\mathcal{R}}}^1_{d+1} \tag{3.147}$$

which forgets the point p_{i+1}.

The space (3.138) is defined to be the moduli space of discs with with $d+1$ positive boundary marked points $z_f, z_1, \dots, z_d$ labeled in counterclockwise order, 1 interior negative puncture z_{out} equipped with an asymptotic marker, and k additional interior marked points $p_1, \dots, p_k$ (without an asymptotic marker), marked as *auxiliary*, staisfying a *strict radial ordering* condition as before: for any representative element with z_f fixed at 1 and z_{out} at 0, we require (3.139) to hold, as well as condition (3.141). The boundary marked point z_f is also marked as auxiliary, but abstractly, we see that ${}_k\hat{\mathcal{R}}^1_d \cong {}_k\check{\mathcal{R}}^1_{d+1}$.

In codimension 1, the compactification ${}_k\overline{\hat{\mathcal{R}}}^1_d$ has boundary covered by inclusions of the following strata:

$$\overline{\mathcal{R}}^s \times {}_k\overline{\hat{\mathcal{R}}}^1_{d-s+2} \tag{3.148}$$

$$\overline{\mathcal{R}}^{m,f_k} \times_{d-m+1} {}_k\overline{\check{\mathcal{R}}}^1_{d-m+1} \quad 1 \leq k \leq m \tag{3.149}$$

$${}_s\overline{\hat{\mathcal{R}}}^1_d \times \overline{\mathcal{M}}_{k-s} \tag{3.150}$$

$${}_{k-1}\overline{\hat{\mathcal{R}}}^{S^1}_d \tag{3.151}$$

$${}^{i,i+1}_{k}\overline{\hat{\mathcal{R}}}^1_d \tag{3.152}$$

Once more, on strata (3.152) where p_i and p_{i+1} have coincident magnitudes, define the map

$$\hat{\pi}_i : {}^{i,i+1}_{k}\overline{\hat{\mathcal{R}}}^1_d \to {}_{k-1}\overline{\hat{\mathcal{R}}}^1_d. \tag{3.153}$$

to be the one forgetting the point p_{i+1} (so again, this map has one-dimensional fibers). On the stratum (3.151), which is the locus where $|p_k| = \frac{1}{2}$, there is also a map of interest

$$\hat{\pi}_{boundary} : \overline{\hat{\mathcal{R}}}^{S^1}_d \to \overline{\mathcal{R}}^{S^1}_d \tag{3.154}$$

which simply forgets the position of z_f.

We orient the moduli spaces (3.137)-(3.138) as follows: picking, on a slice of the automorphism action which fixes the position of z_d at 1 and z_{out} at 0, the volume forms

$$-r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k \tag{3.155}$$

$$r_1 \cdots r_k dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{d-1} \wedge dz_f \wedge dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_k \wedge d\theta_k. \tag{3.156}$$

Above, (r_i, θ_i) denote the polar coordinate positions of the points p_i.

Definition 3.11. *A* **Floer datum** *on a stable disc S in ${}_k\overline{\check{\mathcal{R}}}^1_{d+1}$ or a stable disc S in ${}_k\overline{\mathcal{R}}^{S^1}_d$ is simply a Floer datum for S. A* **Floer datum** *on a stable disc $S \in {}_k\overline{\hat{\mathcal{R}}}^1_d$ is a Floer datum for $\overline{\pi}_f(S)$.*

Definition 3.12. *A* **universal and consistent Floer datum for the cyclic open-closed map**, *is an inductive choice, for every $k \geq 0$ and $d \geq 1$, of Floer data for every representative $S_0 \in {}_k\overline{\check{\mathcal{R}}}^1_{d+1}$,*

$S_1 \in {}_k\overline{\hat{\mathcal{R}}}^1_d$, *varying smoothly over these three moduli spaces, whose restriction to the boundary stratum is conformally equivalent to a product of Floer data previously chosen on lower dimensional moduli spaces. Near the nodal boundary strata, with regards to gluing coordinates, this choice agrees to infinite order with the Floer data obtained by gluing. Moreover, this choice should satisfy the following additional requirements: For* $S_0 \in {}_k\overline{\check{\mathcal{R}}}^1_{d+1}$,

$$\text{At a boundary stratum of the form (3.146), the Floer datum for } S_0 \text{ is conformally equivalent to the one pulled back from } {}_{k-1}\overline{\check{\mathcal{R}}}^1_{d+1} \text{ via the forgetful map } \check{\pi}_i. \tag{3.157}$$

Finally, for $S_1 \in {}_k\overline{\hat{\mathcal{R}}}^1_d$,

$$\text{The choice of Floer datum on strata containing } \mathcal{R}^{d,f_i} \text{ components should be constant along fibers of the forgetful map } \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}. \tag{3.158}$$

$$\text{The Floer datum on the main component } (S_1)_0 \text{ of } \pi_f(S_1) \text{ should coincide with the Floer datum chosen on } (S_1)_0 \in {}_k\mathcal{R}^{1,free}_d \subset {}_k\mathcal{R}^{S^1}_d. \tag{3.159}$$

$$\text{At a boundary stratum of the form (3.151), the Floer datum on the main component of } S_1 \text{ is conformally equivalent to the one pulled back from } {}_k\overline{\mathcal{R}}^{S^1}_d \text{ via the forgetful map } \hat{\pi}_{boundary}\,. \tag{3.160}$$

$$\text{At a boundary stratum of the form (3.152), the Floer datum for } S_1 \text{ is conformally equivalent to the one pulled back from } {}_{k-1}\overline{\hat{\mathcal{R}}}^1_{d+1} \text{ via the forgetful map } \hat{\pi}_i\,. \tag{3.161}$$

Proposition 3.8. *Universal and conformally consistent choices of Floer data for the S^1-equivariant open-closed map exist.*

Fixing a universal and consistent choice of Floer data for the cyclic open-closed map, we obtain, for any d-tuple of Lagrangians $L_0, \ldots, L_{d-1}$, and asympotic conditions

$$\begin{aligned} \vec{x} &= (x_d, \ldots, x_1),\ x_i \in \chi(L_i, L_{i+1-\mathrm{mod} d}) \\ y_{out} &\in \mathcal{O} \end{aligned} \tag{3.162}$$

compactified moduli spaces

$${}_k\overline{\check{\mathcal{R}}}^1_{d+1}(y_{out}, \vec{x}) \tag{3.163}$$

$${}_k\overline{\hat{\mathcal{R}}}^1_d(y_{out}, \vec{x}) \tag{3.164}$$

of maps into M with source an arbitrary element S of the moduli spaces (3.137) and (3.138) respectively (strictly speaking, for (3.138) the source is $\pi_f(S)$), satisfying Floer's equation using the Floer datum chosen for the given S, and asymptotic and moving boundary conditions

$$\begin{cases} u(z) \in \psi^{\rho_S(z)} L_i & \text{if } z \in \partial S \text{ lies between } z_i \text{ and } z_{i+1 \bmod d} \\ \lim_{s\to\pm\infty} u \circ \epsilon^k(s,\cdot) = x_k & \\ \lim_{s\to-\infty} u \circ \delta(s,\cdot) = y & \end{cases}. \tag{3.165}$$

Proposition 3.9. *For generic choices of Floer data, the Gromov-type compactifications (3.163) - (3.164) are smooth compact manifolds of dimension*

$$\dim({}_k\overline{\check{\mathcal{R}}}^1_{d+1}(y_{out},\vec{x})) = \deg(y_{out}) - n + d - 1 - \sum_{i=0}^{d} \deg(x_i) + 2k; \tag{3.166}$$

$$\dim({}_k\overline{\hat{\mathcal{R}}}^1_{d}(y_{out},\vec{x})) = \deg(y_{out}) - n + d - \sum_{i=0}^{d} \deg(x_i) + 2k. \tag{3.167}$$

For rigid elements u in the moduli spaces (3.163) - (3.164), (which occurs for asymptotics $(y,\vec{x})$ satisfying (3.166) $= 0$ or (3.167) $= 0$ respectively), the orientations (3.155) and (3.156), along with [4]*Lemma C.4 induce isomorphisms of orientation lines

$$({}_k\check{\mathcal{R}}^1_{d+1})_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y \tag{3.168}$$

$$({}_k\hat{\mathcal{R}}^1_{d})_u : o_{x_d} \otimes \cdots \otimes o_{x_1} \to o_y. \tag{3.169}$$

Summing the application of these isomorphisms over all u defines the $|o_{y_{out}}|_{\mathbb{K}}$ component of three families of operations $\check{\mathcal{OC}}^k$, $\mathcal{OC}^{S^1,k}$, $\hat{\mathcal{OC}}^k$ up to a sign twist:

$$\check{\mathcal{OC}}^k([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out}) = d-n-2k+1+\sum \deg(x_i)} \; \sum_{u \in {}_k\overline{\check{\mathcal{R}}}^1_d(y_{out};\vec{x})} (-1)^{\check{\star}_d} ({}_k\check{\mathcal{R}}^1_d)_u([x_d],\ldots,[x_1]); \tag{3.170}$$

$$\hat{\mathcal{OC}}^k([x_d],\ldots,[x_1]) := \sum_{\deg(y_{out}) = d-n-2k+\sum \deg(x_i)} \; \sum_{u \in {}_k\overline{\hat{\mathcal{R}}}^1_d(y_{out};\vec{x})} (-1)^{\hat{\star}_d} ({}_k\hat{\mathcal{R}}^1_d)_u([x_d],\ldots,[x_1]). \tag{3.171}$$

where the signs are given by

$$\check{\star}_d = \deg(x_d) + \sum_i i \cdot \deg(x_i). \tag{3.172}$$

$$\hat{\star}_d = \sum_i i \cdot \deg(x_i). \tag{3.173}$$

At this point, one can analyze the operation associated to the codimension-1 boundary strata. However, there is a term corresponding to the stratum ${}_k\overline{\mathcal{R}}^{S^1}_d$ that is not obviously relatable to the existing operations or zero because of extra degeneracy:

Proposition 3.10 (Prop. REF from [21]). *There exist choices of universal and consistent Floer data on ${}_k\check{\overline{\mathcal{R}}}^1_d$ and ${}_k\hat{\overline{\mathcal{R}}}^1_d$ such that the operation associated to the boundary stratum ${}_k\overline{\mathcal{R}}^{S^1}_d$ by*

$$\mathcal{OC}^{S^1,k}([x_d],\dots,[x_1]) := \sum_{\deg(y_{out})=d-n-2k+\sum\deg(x_i)} \sum_{u\in {}_k\overline{\mathcal{R}}^{S^1}_d(y_{out};\vec{x})} (-1)^{\hat{\star_d}}({}_k\mathcal{R}^{S^1}_d)_u([x_d],\dots,[x_1]) \tag{3.174}$$

(using the boundary orientation) is equal to $\hat{\mathcal{OC}}_d \circ B^{nu}$ on the chain level.

INSERT STRATEGY OF ABOVE PROPOSITION HERE.

The main idea above is to make a choice of Floer datum for the strata ${}_k\overline{\mathcal{R}}^{S^1}_d$ and ${}_k\hat{\overline{\mathcal{R}}}^1_d$ such that…

Proposition 3.11 (Thm. REF from [21]). *The resulting operations $\hat{\mathcal{OC}}$ and $\check{\mathcal{OC}}$ satisfy the equations (3.135)-(3.136).*

Passing to negative cyclic chains, we see that $\widetilde{\mathcal{OC}} = \sum_{k=0}^{\infty} u^k \mathcal{OC}^k$ gives a map $CC_*^-(\mathcal{W}) \to (SC^*(X))^{hS^1}$ which is a quasi-isomorphism if $\mathcal{OC}$ is, fitting into the following strictly commutative diagram

$$\begin{tikzcd} \mathrm{CC}_*^-(\mathcal{W}) \arrow[r] \arrow[d, "\widetilde{\mathcal{OC}}"] & \mathrm{CH}_*(\mathcal{W}) \arrow[d, "\mathcal{OC}"] \\ (SC^*(X))^{hS^1} \arrow[r] & SC^*(X) \end{tikzcd} , \tag{3.175}$$

where the horizontal maps are the chain maps arising from projection to $u = 0$.

3.3.7 Geometric sCY structures

Let $\mathcal{B}$ be a finite full sub-category of the wrapped Fukaya category of X, $\mathcal{W}$. In this section we define a natural geometric class of Hochschild and negative cyclic chains for $\mathcal{B}$ coming from the open-closed map. We summarize results of GANATRA proving that these classes in fact define (weak and strong respectively) sCY structures for the wrapped Fukaya category, and of ABOUZAID that any such $\mathcal{B}$ admitting such a structure split-generates $\mathcal{W}$.

Definition 3.13. *A* **geometric volume form***, or* **geometric fundamental class** *for $\mathcal{B}$ is an element σ in the Hochschild chains of $\mathcal{B}$ which maps via $\mathcal{OC}$ to e, the unit in $SH^*(X)$. If $\mathcal{B}$ admits a geometric fundamental class, we call $\mathcal{B}$ an* **essential collection***, or equivalently say that $\mathcal{B}$ satisfies the* **Abouzaid criterion** *[4].*

Definition 3.14. *An S^1***-invariant geometric fundamental class** *for $\mathcal{B}$ is an element*

$$\hat{\sigma} \in \mathrm{CC}^-(\mathcal{B}) \tag{3.176}$$

which maps via $\widetilde{\mathcal{OC}}$ to $\tilde{e}$, the S^1 invariant fundamental class of constant loops, via the map $\widetilde{\mathcal{OC}}$.

It is clear from the commutative diagram (3.175) that any S^1-invariant fundamental class for $\mathcal{B}$, when projected to Hocshchild chains, is a geometric fundamental class. Conversely, we have the following:

Theorem 3.3 ([21] CITE combined with [20])**.** *If $\sigma \in \mathrm{CH}_*(\mathcal{B}, \mathcal{B})$ is a geometric fundamental class for $\mathcal{B}$, then there exists an S^1 invariant geometric fundamental class for $\mathcal{B}$, $\hat{\sigma}$ projecting to σ.*

Proof. In [20] it is shown that if $\mathcal{B}$ admits a fundamental class, then $\mathcal{OC}$ is a quasi-isomorphism. By a spectral sequence argument or alternatively by homotopy invariance for S^1-equivariant homology theories, one can deduce that $\widetilde{\mathcal{OC}}$ is a quasi-isomorphism as well. Hence, the class $[\hat{e}]$ has a pre-image. $\square$

The proof also establishes that such classes, if they exist, are homologically unique. One of the main results of [20] is that any geometric fundamental class is *non-degenerate* and hence constitutes a weak sCY structure for $\mathcal{B}$:

Theorem 3.4. *Suppose $\mathcal{B}$ admits a geometric fundamental class σ. Then $\mathcal{B}$ is smooth and $\mathrm{Ev}_\sigma : \mathcal{B}^! \to \mathcal{B}$ is a quasi-isomorphism.*

Proof. `Insert a brief description of the proof.` $\square$

Corollary 3.7. *If $\mathcal{B}$ admits a geometric fundamental class σ, then it admits a uniquely determined sCY structure, characterized as the (homologically unique) S^1-invariant fundamental class $\hat{\sigma}$ of $\mathcal{B}$.*

Finally, it is worth mentioning (and this was in fact the original motivation for geometric fundamental classes!) Abouzaid's generation result:

Theorem 3.5 ([4])**.** *If $\mathcal{B}$ admits a geometric fundamental class, then $\mathcal{B}$ split-generates $\mathcal{W}$.*

The corollary of all of these and the Morita theory developed in REF is the following:

Corollary 3.8. *Suppose $\mathcal{B}$ admits a geometric fundamental class, for some finite sub-category $\mathcal{B} \subset \mathcal{W}$. Then, $\mathcal{B}$ split-generates $\mathcal{W}$ and there exists a unique geometrically specified sCY structure $\hat{\sigma}$ on $\mathcal{W}$, characterized as the S^1-invariant fundamental class on $\mathcal{B}$.*

3.4 Geometric sCY structures on the string topology category

Let M^n be a closed, connected, oriented manifold, and ΩM its based loop space. The main result of this section is the following.

Theorem 5. *The chain algebra $C_*(\Omega M)$ is a smooth Calabi-Yau (sCY) algebra.*

3.4.1 Background Homological Algebra

Let R be a differential graded algebra, P and Q right R-modules and S a left R-module, we write

$$Rhom_R(P,Q) \quad \text{and} \quad Q \otimes_R^L S$$

to mean the right-derived homomorphisms and left-derived tensor product respectively. Explicit models of these spaces can be given using two-sided bar constructions in the usual way.

Let $\alpha \in P \otimes_R^L S$. Then one has a well-defined (up to homotopy) "evaluation homomorphism", which we view as a cap product,

$$\cap\alpha : Rhom_R(P,Q) \to Q \otimes_R^L S.$$

If α has dimension d, then $\cap\alpha$ is a map of chain complexes of degree $-d$. A specific morphism modeling $\cap\alpha$ can be viewed using bar complexes, as follows. We suppose $\alpha \in B(P,R,S)$. Then the cap product is a homomorphism

$$\cap\alpha : Hom_R(B(P,R,R),Q) \to B(Q,R,S).$$

See [18] [33] for a precise definition. This cap product operation makes sense in the setting of A_∞-algebras as well, by again using two sided bar complexes to construct explicit models. We consider the special case when A is an A_∞-algebra, $R = A \otimes A^{op}$, $\alpha \in B(A, A\otimes A^{op}, A) \simeq A \otimes^L_{A\otimes A^{op}} A \simeq CH_*(A,A)$. Then we have an operation

$$\lambda_\alpha = \cap\alpha : Hom_{A\otimes A^{op}}(B(A, A\otimes A^{op}, A\otimes A^{op}), A\otimes A^{op}) \to B(A\otimes A^{op}, A\otimes A^{op}, A). \tag{3.177}$$

The source of our map is, by definition, $A^!$. It is a model for $Rhom_{A\otimes A^{op}}(A, A\otimes A^{op})$, the "bimodule dual" of A. The right hand side admits a natural bimodule equivalence to A, so we can think of the cap product as a map

$$\lambda_\alpha : A^! \to A.$$

This construction can be viewed as a "double duality" construction

$$\begin{aligned}\lambda : B(A, A\otimes A^{op}, A) &\to Hom_{A\otimes A^{op}}(Hom_{A\otimes A^{op}}(B(A, A\otimes A^{op}, A\otimes A^{op}), A\otimes A^{op}), A)\\ &= Hom_{A\otimes A^{op}}(A^!, A)\end{aligned}$$

which is an equivalence when A is smooth. In particular this perspective makes it clear that the cap product $\lambda_\alpha : A^! \to A$ is a map of $A\otimes A^{op}$-modules (i.e A-bimodules).

3.4.2 The free loop space, the Hochschild complex, and homotopy fixed points

Let $LM = Map(S^1, M)$ be the free loop space of the closed manifold M. The natural S^1-action defined by rotating the loops has as its fixed points the constant loops, $c : M \hookrightarrow LM$, where $c(x) : S^1 \to M$ is given by $c(x)(t) = x \ \forall t \in S^1$. The homotopy fixed points of this action is defined to be the equivariant mapping $LM^{hS^1} = Map_{S^1}(ES^1, LM)$, where ES^1 is a contractible, free $S^1 - CW$ complex. The following observation states that in the case of LM, the homotopy fixed points are also equivalent to M.

Lemma 6. *There is a natural homotopy equivalence*

$$LM^{hS^1} \simeq M.$$

Proof. $LM^{hS^1} = Map_{S^1}(ES^1, LM)$. Now the full (not necessarily equivariant) mapping space $Map(ES^1, LM)$ is by the standard adjunction, homeomorphic to $Map(ES^1 \times S^1, M)$. Since the S^1 action on M is trivial, this adjunction restricts to the equivariant mapping space to give a homeomorphism

$$LM^{hS^1} = Map_{S^1}(ES^1, LM) \cong Map(ES^1 \times_{S^1} S^1, M).$$

But the orbit space $ES^1 \times_{S^1} S^1$ is contractible. So the mapping space $Map(ES^1 \times_{S^1} S^1, M)$ is homotopy equivalent to M. The lemma now follows. □

This lemma defines a chain homotopy equivalence (well-defined up to chain homotopy)

$$e : C_*(M) \simeq C_*(LM^{hS^1}) = C_*(Map_{S^1}(ES^1, LM)), \tag{3.178}$$

where $C_*(-)$ refers to singular chains.

Consider the natural map from the chains of a mapping space to the space of homomorphisms between chain complexes,

$$\sigma : C_*(Map_{S^1}(ES^1, LM)) \to Hom_{C_*(S^1)}(C_*(ES^1), C_*(LM)). \tag{3.179}$$

Since ES^1 is a free, contractible S^1-CW complex, the augmentation of its chains, $C_*(ES^1) \to k$ gives a free, acyclic resolution of the ground field k as a $C_*(S^1)$-module. Moreover, chains $C_*(S^1)$ is equivalent to the exterior algebra on one generator in dimension 1, $C_*(S^1) \simeq \Lambda(\Delta)$. Thus in the derived category σ may be viewed as a map

$$\sigma : C_*(LM^{hS^1}) \to Rhom_{\Lambda(\Delta)}(k, C_*(LM)).$$

Now the Goodwillie equivalence $CH_*(C_*(\Omega M), C_*(\Omega M)) \xrightarrow{\simeq} C_*(LM)$ is an equivalence of $\Lambda(\Delta)$-modules. So σ may be viewed as a map

$$\sigma : C_*(LM^{S^1}) \to Rhom_{\Lambda(\Delta)}(k, CH_*(C_*(\Omega M), C_*(\Omega M)) = CC^-(C_*(\Omega M)),$$

where the target complex is the negative cyclic chain complex of the algebra $C_*(\Omega M)$. Composing σ with e defines a morphism in the homotopy category,

$$\iota_* : C_*(M) \xrightarrow[\simeq]{e} C_*(LM^{hS^1}) \xrightarrow{\sigma} CC^-(C_*(\Omega M)). \tag{3.180}$$

Definition 2. *Let M be an oriented, closed n-manifold. Let $[M] \in H_n(M;\mathbb{Z})$ be its fundamental class. We define a cycle in $\gamma_M \in CC^-(C_*(\Omega M))$ to be of* **fundamental type** *if the homology class it represents $\{\gamma_M\} \in HC^-(C_*(\Omega M))$ is equal to $\iota_*([M])$.*

A more precise version of Theorem 5 is the following.

Theorem 7. *Let M be a closed, oriented n-manifold. Then any cycle $\gamma_M \in CC^-(C_*(\Omega M))$ of fundamental type defines an sCY-structure on $C_*(\Omega M)$.*

Proof. Let $\gamma_M \in CC^-(C_*(\Omega M))$ be a cycle of fundamental type. Let $\pi : CC^-(C_*(\Omega M)) \to CH_*(C_*(\Omega M), C_*(\Omega M))$ be the projection from the negative cyclic chains to the Hochschild chains. Let $\alpha = \pi(\gamma_M)$. To prove this theorem we need to show that the induced bimodule homomorphism given by the cap product (3.177)

$$\lambda_\alpha : A^! \to A$$

is an equivalence, where $A = C_*(\Omega M)$. To do this it is enough to compute in homology, which, by the definition of λ_α is the cap product

$$\cap\alpha : Ext_{A\otimes A^{op}}(A, A\otimes A^{op}) \to Tor_{A\otimes A^{op}}(A, A\otimes A^{op}) \cong H_*(\Omega M).$$

Now recall that since $A = C_*(\Omega M)$ is a connective Hopf algebra, then for any $A\otimes A^{op}$-bimodule P, there are isomorphisms

$$\Phi^* : Ext_A(k, P^{Ad}) \xrightarrow{\cong} Ext_{A\otimes A^{op}}(A, P) = HH^*(A, P) \quad \text{and}$$
$$\Phi_* : Tor_A(k, P^{Ad}) \xrightarrow{\cong} Tor_{A\otimes A^{op}}(A, P) = HH_*(A, P)$$

where P^{Ad} is the induced adjoint A-module given by conjugation, where one uses the canonical antiautomorphism of the Hopf algebra A. We remark that since $A = C_*(\Omega M)$ which is equivalent to $C_*(G)$ for an appropriate topological group G, then this antiautomorphism on A is induced by the antiautomorphism on G given by $g \to g^{-1}$. See [33] for details. Now recall that $Tor_{C_*(\Omega M)}(k,k) \cong H_*(M)$. Furthermore, since γ_M is of fundamental type, then in homology, $[\alpha] \in HH_*(C_*(\Omega M), C_*(\Omega M)) \cong H_*(LM)$ is the image of the fundamental class $[M] \in H_n(M)$ under the inclusion of the constant loops, $c : M \to LM$. Then as observed by Malm in [33], the following diagram commutes:

$$\begin{tikzcd}
Ext_A(k, P^{Ad}) \arrow[r, "\cong"] \arrow[d, "\cap[M]"'] & HH^*(A,P) \arrow[d, "\lambda_\alpha"] \\
Tor_A(k, P^{Ad}) \arrow[r, "\cong"'] & HH_*(A,P)
\end{tikzcd}$$

where the left hand vertical map is the cap product operation $Ext_A(k, P^{Ad}) \times Tor_A(k,k) \to Tor_A(k, P^{Ad})$. Now as described by Dwyer-Greenlees -Iyengar in [17], $A = C_*(\Omega M)$, $Ext_A(k, P^{Ad}) \cong$

$H^*(M, \{P^{Ad}\})$ and $Tor_A(k, P^{Ad}) = H_*(M, \{P^{Ad}\})$ where the coefficient systems are given by the $C_*(\Omega M)$-module P^{Ad}, and capping with the fundamental class $[M]$ is the twisted Poincaré duality isomorphism. This implies that on homology, the operation λ_α is an isomorphism, which in turn implies that on the chain level, $\lambda_\alpha : A^! \to A$ is an equivalence of $A \otimes A^{op}$-modules. This proves the theorem.

□

4 Comparison of Calabi-Yau structures

The goal of this section is to prove the following theorem.

Theorem 4.1. *The A_∞ equivalence $\mathbf{F} : \mathcal{W}_b^{conor}(T^*M) \to \mathcal{S}_M$ given in Theorem 2.1 intertwines the geometric sCY structures on $\mathcal{W}_b^{conor}(T^*M)$ and $\mathcal{S}_M$, and hence is an equivalence of sCY categories.*

4.1 Overview of the proof

In the two sCY categories under study in this paper, the co-trace elements $\sigma_{\mathcal{W}} \in CC^-(\mathcal{W}_b^{conor}(T^*M))$ and $\sigma_{\mathcal{S}} \in C^-(\mathcal{S}_M)$ are both constructed using similar, but not obviously equivalent constructions. We compare those constructions in detail in this section.

Recall that in a previous section, we constructed a functor $F : \mathcal{W}_b^{conor}(T^*M) \to \mathcal{S}_M$ that is an equivalence of A_∞-categories. Our goal is to prove that F is an equivalence of sCY-categories, and for this we need the following theorem.

Theorem 8. *The induced map on negative cyclic chains, $F_* : CC^-(\mathcal{W}_b^{conor}(T^*M)) \to CC^-(\mathcal{S}_M)$ preserves the co-trace elements. That is, $F_*(\sigma_{\mathcal{W}}) = \sigma_{\mathcal{S}}$.*

Now recall that Abouzaid [?] proved that the wrapped Fukaya category $\mathcal{W}_b^{conor}(T^*M)$ is generated by the cotangent fiber of a fixed basepoint $x_0 \in M$. In particular on the level of Hochschild and cyclic chains, we have chain equivalences,

$$CH_*(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{\simeq} CH_*(\mathcal{W}_b^{conor}(T^*M))$$
$$CC^-_*(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{\simeq} CC^-_*(\mathcal{W}_b^{conor}(T^*M))$$

Similarly, in the string topology category $\mathcal{S}_M$, it was shown in [?] that the endomorphisms of the basepoint, which is the chains of the based loop space $End_{x_0}(\mathcal{S}_M) = C_*(\Omega M)$ generates $\mathcal{S}_M$. In particular there are chain equivalences

$$CH_*(C_*(\Omega M)) \xrightarrow{\simeq} CH_*(\mathcal{S}_M)$$
$$CC^-(C_*(\Omega M) \xrightarrow{\simeq} CC^-(\mathcal{S}_M)$$

Furthermore we know that the functor $F : \mathcal{W}_b^{conor}(T^*M) \to \mathcal{S}_M$ takes the cotangent fiber $T^*_{x_0}M$ to the basepoint viewed as a submanifold, $x_0 \in M$, and by [**?**] we know it induces a quasi-isomorphism on the endomorphism rings,

$$F : End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M) \xrightarrow{\simeq} C_*(\Omega M). \tag{4.1}$$

In particular we have induced equivalences of negative cyclic chain groups,

$$F_* : CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{\simeq} CC^-(C_*(\Omega M)). \tag{4.2}$$

So to prove Theorem 8 it suffices to prove the following.

Theorem 9. *The induced homomorphism*

$$F_* : CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \to CC^-(C_*(\Omega M))$$

preserves the co-trace cycles up to homotopy. That is, $F_*(\sigma_{\mathcal{W}})$ is homologous to $\sigma_{\mathcal{S}}$ in $CC^-(C_*(\Omega M))$.

Proof. We now give a proof of this theorem, modulo two propositions that will be proved in the following subsections.

We begin by recalling how these co-trace elements were constructed. For an A_∞ $C_*(S^1)$ module C_* , let $C_*^{hS^1}$ denote the homotopy fixed point chain complex. That is,

$$C_*^{hS^1} = Rhom_{C_*(S^1)}(k, C_*). \tag{4.3}$$

Let $[M] \in C_*(M)$ be a fundamental cycle giving the orientation. As was seen in a previous section this defines a class in the homotopy fixed points of the symplectic cohomology, $[M] \in SC^*(T^*M)^{hS^1}$. A co-trace element $\sigma_{\mathcal{W}} \in CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M))$ was defined to be any cycle that maps to a cycle homologous to $[M]$ under the open-closed map $CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{\tilde{\mathcal{OC}}} SC^*(T^*M)^{hS^1}$. Since this map is a chain equivalence, $\sigma_{\mathcal{W}}$ is well defined up to homotopy. That is, any two choices for such a cycle are homologous.

Now it was also shown that under the map $SC^*(T^*M)^{hS^1} \xrightarrow{\mathcal{CCL}} C_*(LM)^{hS^1}$, $[M]$ maps to a fundamental cycle in $C_*(M)$, thought of as a subcomplex of $C_*(LM)^{hS^1}$ by viewing $M \hookrightarrow LM$ as the constant loops, which constitute the S^1-fixed points of the rotation circle action on LM. Since $SC^*(T^*M)^{hS^1} \xrightarrow{\mathcal{CCL}} C_*(LM)^{hS^1}$ is also an equivalence, we have the following characterization of the co-trace cycle.

Lemma 10. *A co-trace cycle $\sigma_{\mathcal{W}} \in CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M))$ is characterized by the fact that it maps to a class homologous to the fundamental cycle $[M] \in C_*(LM)^{hS^1}$ under the composition*

$$CC_*^-(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{\tilde{\mathcal{OC}}} SC^*(T^*M)^{hS^1} \xrightarrow{\mathcal{CCL}} C_*(LM)^{hS^1}.$$

Now recall that it was shown above that the following is a homotopy commutative diagram:

$$\begin{tikzcd} CC^-_*(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \arrow[r, "F_*"] \arrow[d, "\tilde{\mathfrak{O}}\mathfrak{C}"'] & CC^-(C_*(\Omega M)) \arrow[d, "\mathcal{A}"] \\ SC^*(T^*M)^{hS^1} \arrow[r, "\mathfrak{CCL}"'] & C_*(LM)^{hS^1} \end{tikzcd}$$

where $\mathcal{A}$ is the extension defined earlier of the map defined by Abouzaid [?]. It was shown above that all maps in this diagram, including $\mathcal{A}$, are equivalences. Therefore we have the following alternative characterization of the co-trace cycle.

Lemma 11. *A co-trace cycle $\sigma_{\mathcal{W}} \in CC^-_*(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M))$ is characterized by the fact that it maps to a class homologous to the fundamental cycle $[M] \in C_*(LM)^{hS^1}$ under the composition*

$$CC^-_*(End_{\mathcal{W}_b^{conor}(T^*M)}(T^*_{x_0}M)) \xrightarrow{F_*} CC^-(C_*(\Omega M)) \xrightarrow{\mathcal{A}} C_*(LM)^{hS^1}.$$

Now recall that by the construction of the string topology co-trace element $\sigma_{\mathcal{S}} \in CC^-(C_*(\Omega M))$ we have the following:

Lemma 12. *A co-trace cycle in the string topology category $\mathcal{S}_M$, is any cycle $\sigma_{\mathcal{S}} \in CC^-(C_*(\Omega M))$ that maps to a class homologous to the fundamental cycle $[M] \in C_*(LM)^{hS^1}$ under the map*

$$\mathcal{G} : CC^-(C_*(\Omega M)) \xrightarrow{\simeq} B_*(C_*(G))$$

which is induced by the equivariance of Goodwillie's equivalence, $\mathcal{G} : CH_*(C_*(\Omega M)) \xrightarrow{\simeq} C_*(LM)$.

Comparing Lemmas 11 and 12, to show that $F_*(\sigma_{\mathcal{W}}) \simeq \sigma_{\mathcal{S}}$, and thereby prove Theorem 9, it suffices to show that the preimage of the fundamental cycle $[M] \in C_*(LM)^{hS^1}$ under the two equivalences $\mathcal{A}$ and $\mathcal{G}$ are homologous in $CC^-(C_*(\Omega M))$. We accomplish this in two steps, carried out in the following two subsections of this section. The first step is an equivariant analysis of a well known map from the simplicial bar-construction of a group, to the cyclic bar construction,

$$\iota : B(G) \to N^{cy}(G).$$

The S^1-equivariant properties may well be known, but it does not seem to be in the literature. Therefore this analysis may be of independent interest. When one applies singular chains to this analysis, it implies results about chain level bar construction and the Hochschild chains In particular in the first subsection of this section we prove the following.

Lemma 13. *There is an explicit map of chain complexes, $\iota_* : B_*(C_*(G)) \to CC^-(C_*(G))$ that makes the following diagram homotopy commute*

$$\begin{tikzcd} B_*(C_*(G)) \arrow[r, "="] \arrow[d, "\iota_*"'] & B_*(C_*(G)) \arrow[d] \\ CC^-(C_*(G)) \arrow[r, "\mathcal{G}"'] & C_*(L(BG))^{hS^1}. \end{tikzcd}$$

Given this lemma, we can consider the case when G is a topological group homotopy equivalent to ΩM. In that case there is an explicit equivalence $p : B_*(C_*(\Omega M)) \to C_*(M)$. And so there is a cycle, well defined up to homotopy, $\sigma(M) \in B_*(C_*(\Omega M))$ that maps to a cycle homologous to $[M] \in C_*(L(M))^{hS^1}$ under the composition

$$B_*(C_*(\Omega M)) \xrightarrow{\iota_*} CC^-(C_*(\Omega M)) \xrightarrow{\mathcal{G}} C_*(LM)^{hS^1}.$$

In the second subsection of this section, we prove the following.

Lemma 14. *The compositions $\mathcal{G} \circ \iota_*$ and $\mathcal{A} \circ \iota_* : B_*(C_*(\Omega M)) \xrightarrow{\iota_*} CC^-(C_*(\Omega M)) \to C_*(LM)^{hS^1}$ are homotopic.*

Assuming Lemmas 13 and 14 for now, we can complete the proof of Theorem 9 as follows.

Since there is an explicit equivalence $C_*(M) \simeq B_*(C_*(\Omega M))$, Lemma 13 implies there is a unique (up to homotopy) fundamental cycle $[\tilde{M}] \in B_*(C_*(\Omega M))$ with the property that $\mathcal{G} \circ \iota_*([\tilde{M}] \simeq [M] \in C_*(LM)^{hS^1}$. By Lemma 12 this implies that $\iota_*([\tilde{M}]) \simeq \sigma_{\mathcal{S}}$. But by Lemma 14, this implies $\mathcal{A} \circ \iota_*([\tilde{M}]) \simeq [M] \in C_*(LM)^{hS^1}$, and therefore $\mathcal{A}(\sigma_{\mathcal{S}}) \simeq [M]$. But by Lemma 11 we also know that $\mathcal{A}(F_*(\sigma_{\mathcal{W}}) \simeq [M]$. Since $\mathcal{A}$ is an equivalence, this means that

$$F_*(\sigma_{\mathcal{W}}) \simeq \sigma_{\mathcal{S}}$$

which was the statement of Theorem 9. $\square$

4.2 Comparison of closed loops

We recall a variant definition of chain map first introduced by [1]:

$$H^*(\mathcal{CL}) : SH^*_b(T^*M) \to H_{n-*}(\mathcal{L}M). \tag{4.4}$$

Recall that we have denoted by S^+ the positive half of the cylinder $[0,\infty)_s \times S^1_t$. Pick a domain dependent primary Hamiltonian term $H_{S^+} : S^+ \to C^\infty(X, \mathbb{R})$, satisyfing the following conditions:

$$\text{near } s = \infty,\ \mathcal{J}_{S^+} \text{ and } H_{S^+} \text{ are independent of } s \text{ and equal to } J_t,\ H_t = H + F_t \tag{4.5}$$

$$\text{near } s = 0,\ H_{S^+} = 0 \text{ in a neighborhood of the zero section.} \tag{4.6}$$

(If one desires as in [?]*§3.2, H_{S^+} can be constructed as $\tilde{H}_{C^+} + F_t$, where $\tilde{H}_{C^+}$ restricts to H for large s and $-F_t$ near zero for small s). For a Hamiltonian orbit $y \in \mathcal{O}$, consider the space

$$\mathcal{R}^1(y) \tag{4.7}$$

of finite energy maps $u : S^+ \to T^*M$ solving Floer's equation

$$(du - (X_{H_{S^+}} \otimes dt))^{0,1} = 0 \tag{4.8}$$

(using the almost complex structure J_{S+}) with boundary conditions on the zero section and asymptotic conditions on y:

$$\begin{cases} u(0,t) \in M \\ \lim_{s\to+\infty} u(s,\cdot) = y. \end{cases} \tag{4.9}$$

The space (4.7) admits a natural Gromov compatification $\overline{\mathcal{R}}^1(y)$ whose codimension 1 boundary strata are covered by natural inclusions

$$\mathcal{R}^1(y_0) \times \mathcal{M}(y_0; y) \to \partial\overline{\mathcal{R}}^1(y). \tag{4.10}$$

Standard transversality and compactness arguments ensure that $\overline{\mathcal{R}}^1(y)$ is a compact manifold with boundary of dimension $n - |y|$. An examination of orientations shows that (strictly speaking, this is a singular chains variant of the cited Lemma):

Lemma 4.1 ([?], (3.15))**.** *There exists a choice of fundamental chains for $\overline{\mathcal{R}}^1(y)$ and $\overline{\mathcal{M}}(y_0; y)$ in singular homology such that*

$$\partial[\overline{\mathcal{R}}^1(y)] = \sum_{y_0} (-1)^{n+|y_0|}[\overline{\mathcal{R}}^1(y_0)] \times [\overline{\mathcal{M}}(y_0; y)]. \tag{4.11}$$

Define $ev : \overline{\mathcal{R}}^1(y) \to \mathcal{L}M$ to be the map which takes an element, and restricts to its boundary, using basepoint $-1 \in S^1 = \partial D^2$ and the length parametrisation induced by the metric, and denote by ev_* the induced map on normalized singular chains. Then we can define

$$\mathcal{CL} : SC^*_b(T^*M) \to C^{\mathcal{L}}_{n-*}(\mathcal{L}M) \tag{4.12}$$

$$\mathcal{CL}([y_1]) = (-1)^{|y_1|}\mathrm{ev}_*([\overline{\mathcal{R}}^1(y_1)]). \tag{4.13}$$

Note that by (4.11) $\partial\mathcal{CL}(y_1) = (-1)^{|y_1|}\mathrm{ev}_*(\partial[\overline{\mathcal{R}}^1(y_1)]) = (-1)^{|y_1|}\sum_{y_0}(-1)^{n+|y_0|}ev_*([\overline{\mathcal{R}}^1(y_0)]\times[\overline{\mathcal{M}}(y_0; y_1)]))$. Now, on a boundary component $\mathcal{R}^1(y_0) \times \mathcal{M}(y_0; y)$, note that ev factors as

$$\mathcal{R}^1(y_0) \times \mathcal{M}(y_0; y) \to \mathcal{R}^1(y_0) \xrightarrow{\bar{ev}} \mathcal{L}M. \tag{4.14}$$

where the first map is projection to the first component. Hence, by Lemma A.1, we see that $ev_*([\overline{\mathcal{R}}^1(y_0)] \times [\overline{\mathcal{M}}(y_0; y_1)])$ is zero unless the component $\mathcal{M}(y_0; y_1)$ are rigid, hence

$$\begin{aligned} \mathrm{ev}_*(\partial[\overline{\mathcal{R}}^1(y_1)]) &= (-1)^{|y_1|} \sum_{y_0; |y_0| = |y_1|+1} (-1)^{n+|y_0|}\#([\overline{\mathcal{M}}(y_0; y_1)])ev_*([\overline{\mathcal{R}}^1(y_0)]) \\ &= \mathcal{CL}(dy_1). \end{aligned} \tag{4.15}$$

This verifies

Proposition 4.1 ([1], see [?] for these particular conventions.)**.** $\mathcal{CL}$ *is a degree n chain map.*

Theorem 4.2 ([?])**.** *The homology level map $[\mathcal{CL}]$ is an isomorphism.*

The following proposition is new:

Proposition 4.2. *$\mathcal{CL}$ extends to an equivariant morphism of S^1 structures of degree n; namely, there are maps*

$$\mathcal{CL}_r : SC_b^*(T^*M) \to C^{\mathcal{L}}_{n-*-2r}(\mathcal{L}M), \tag{4.16}$$

with $\mathcal{CL}_0 = \mathcal{CL}$, satisfying, for each k

$$\mathcal{CL}_k \circ d + \sum_{i=0}^{k-1} \mathcal{CL}_i \circ \Delta^{SC}_{k-i} = \Delta^{\mathcal{L}M} \circ \mathcal{CL}_{k-1} + \partial \circ \mathcal{CL}_k \tag{4.17}$$

We write $\widetilde{\mathcal{CL}}$ for the morphism in the category of complexes with S^1-structure, that is,

$$\begin{aligned} \widetilde{\mathcal{CL}} = \oplus_k \mathcal{CL}^{k|1} : \bigoplus_{k\geq 0} \overline{C}_{-*}(S^1)^{\otimes k} \otimes SC_b^*(T^*M) \to C_{n-*}(M), \\ \mathcal{CL}^{r|1}(\underbrace{\beta, \ldots, \beta}_{k \text{ times}}, y) := \mathcal{CL}_r(y). \end{aligned} \tag{4.18}$$

In order to prove this proposition, we need to introduce new moduli spaces:

Definition 4.1. *An ordered r-marking for S^+ consists of an r-tuple of points $p_1, \ldots, p_r$ on S^+, satisfying*

$$0 \leq (p_1)_s \leq (p_2)_s \leq \cdots \leq (p_r)_s. \tag{4.19}$$

The **moduli space of half-cylinders with ordered r-markings***, denoted*

$$\mathcal{H}_r \tag{4.20}$$

is simply the space of all such pairs $(S^+, p_1, \ldots, p_r)$, with its canonical (complex) orientation. The case $r = 0$ fits into the above description, as the space of (S^+) (which is a point).

For a fixed δ, we equip any element $(S^+, p_1, \ldots, p_r) \in \mathcal{H}_r$ with the positive strip-like end

$$\begin{aligned} \epsilon^+ : [0, \infty) \times S^1 \to C \\ (s, t) \mapsto (s + (p_r)_s + \delta, t). \end{aligned} \tag{4.21}$$

Construct a compactification by allowing the points p_i to escape to positive ∞ and break off a *broken r-point angle-decorated cylinder*, in the sense of Definition 3.8.

$$\overline{\mathcal{H}}_r := \coprod_s \coprod_{j_1, \ldots, j_s; j_1 \geq 0, j_i > 0, \sum j_i = r} \mathcal{H}_{j_1} \times \mathcal{M}_{j_2} \times \cdots \times \mathcal{M}_{j_s}. \tag{4.22}$$

At a codimension s stratum of the form (4.22), the manifolds with corner structure is determined by connect summing with respect to the cylindrical ends (4.21), (3.112), and (3.111) (note that the ends (3.112) are rotated, so the gluing map also involves rotating the bottom component). To be a little more concrete, the boundary chart

$$\mathcal{H}_j \times \mathcal{M}_{r-j} \times [0, 1) \to \overline{\mathcal{H}}_r \tag{4.23}$$

associates to a tuple $(S^+, p_1, \ldots, p_r), (C, q_1, \ldots, q_{r-j}), \lambda = \frac{\log \rho}{1+\log \rho}$ (so $\rho \in [1, \infty]$) the following half-cylinder with ordered r marking: denoting by (s_i, t_i) the s and t coordinates of p_i and (h_i, θ_i) the s and t coordinates of q_i, the result of gluing is

$$\begin{aligned}(S^+,&(s_1, t_1 + \theta_1), (s_2, t_2 + \theta_1), \ldots, (s_j, t_j + \theta_1),\\ &(s_j + \rho, \theta_1), (s_j + \rho + h_2 - h_1, \theta_2), \ldots, (s_j + 2\delta + \rho + h_{r-j} - h_1, \theta_{r-j}))\end{aligned} \tag{4.24}$$

All together, the codimension 1 boundary of $\overline{\mathcal{H}}_r$ is covered by strata of the form (4.22) as well as loci

$$\overline{\mathcal{H}}_k \times \overline{\mathcal{M}}_{r-k} \longrightarrow \partial\mathcal{H}_r,\ 0 < k < r \tag{4.25}$$

$$\overline{\mathcal{H}}_r^{i,i+1} \longrightarrow \partial\mathcal{H}_r,\ 1 \leq i < r, \tag{4.26}$$

$$\overline{\mathcal{H}}_r^0 \longrightarrow \partial\mathcal{H}_r. \tag{4.27}$$

where consecutive points p_i, p_{i+1} have coincident heights (4.25), and the locus (4.27) where p_1 has height 0.

There are natural maps

$$\mathfrak{o}_i : \mathcal{H}_r \to S^1,\ i = 1, \ldots, r \tag{4.28}$$

recording the S^1 coordinates of p_i: $\mathfrak{o}_i(S^+, p_1, \ldots, p_r) := (p_i)_t$. With the proviso that on a stratum $\mathcal{H}_j$, we declare $\mathfrak{o}_{j+1} = 1 \in S^1$, we see that, for instance, $\mathfrak{o}_1$ extends continuously to a function on $\overline{\mathcal{H}}_r$, which we also call the *basepoint map*

$$\mathfrak{bp}_r = exp^{\pi i} \cdot \mathfrak{o}_1 : \overline{\mathcal{H}}_r \to S^1 \tag{4.29}$$

(Clearly this is continuous across the local gluing charts of the form $\mathcal{H}_j \times \mathcal{M}_{r-j} \times [0, 1) \to \mathcal{H}_r$ for $j \geq 1$, and one can check it for $j = 0$ using the explicit chart maps (4.24). More concretely, on the stratum where $j = 0$, the basepoint map returns $\mathfrak{o}_1 = 1$, which the gluing map (4.24) identifies with higher strata). Next, the difference between $\mathfrak{o}_1$ and $\mathfrak{o}_2$, with the proviso that on a stratum $\mathcal{H}_j$, $\mathfrak{o}_{j+1} = \mathfrak{o}_{j+2} = 1 \in S^1$, also extends continuously to $\overline{\mathcal{H}}_r$, and is called the *basepoint difference map*

$$\mathfrak{d} := \mathfrak{o}_1 \mathfrak{o}_2^{-1} : \overline{\mathcal{H}}_r \to S^1 \tag{4.30}$$

The strata (4.26) and (4.27) each possess maps of interest. First, there is a projection map which only remembers the point p_i (in essence we have only forgotten the angle of p_{i+1}, as its height and p_i's are coincident)

$$\begin{aligned}\pi_i : \mathcal{H}_r^{i,i+1} &\longrightarrow \mathcal{H}_{r-1}\\ (S^+, p_1, \ldots, p_r) &\mapsto (p_1, \ldots, p_i, p_{i+2}, \ldots, p_r).\end{aligned} \tag{4.31}$$

π_i is compatible with the choice of positive ends (4.21) and hence π_i extends to compactifications

$$\pi_i : \overline{\mathcal{H}}_r^{i,i+1} \to \overline{\mathcal{H}}_{r-1}. \tag{4.32}$$

In particular the map $\mathfrak{o}_{i+1} : \mathcal{H}_r \to S^1$ induces an isomorphism

$$(\pi_i, \mathfrak{o}_{i+1}) : \overline{\mathcal{H}}_r^{i,i+1} \xrightarrow{\sim} \overline{\mathcal{H}}_{r-1}. \tag{4.33}$$

Next, for (4.27) there is a map which forgets the point p_1 (once more, we are only really forgetting the angle, or t parameter, of p_1):

$$\begin{aligned} \pi_0 : \mathcal{H}_r^0 &\longrightarrow \mathcal{H}_{r-1} \\ (S^+, p_1, \ldots, p_r) &\mapsto (p_2, \ldots, p_r). \end{aligned} \tag{4.34}$$

It is worth noting that π_0 has fiber S^1, and in fact we choose an identification

$$(\pi_0, \mathfrak{d}) : \mathcal{H}_r^0 \xrightarrow{\sim} \mathcal{H}_{r-1} \times S^1. \tag{4.35}$$

where $\mathfrak{d}$ is the *basepoint difference map* in (4.30).

Definition 4.2. *A* **Floer datum for a half cylinder with ordered** r **markings** $\tilde{S} := (S^+, p_1, \ldots, p_r)$ *consists of the following choices:*

- *The positive end on $\epsilon^+ : C^\pm \to S^+$ chosen in (4.21)*
- *The one-form on S^+ given by $\alpha = dt$.*
- *A Hamiltonian $H_{\tilde{S}} : S^+ \to \mathcal{H}(M)$ compatible with the positive strip-like end, meaning*

$$(\epsilon^+)^* H_{\tilde{S}} = H, \tag{4.36}$$

  for a previously fixed choice H.
- *A time-dependent perturbation term $F_{\tilde{S}} : S^+ \to \mathcal{H}(M)$, supported on a neighborhood of the orbits of H, and compatible with the positive end*

$$(\epsilon^+)^* F_{\tilde{S}} = F_t \tag{4.37}$$

- *A surface dependent complex structure $J_{\tilde{S}} : S \to \mathcal{J}_1(M)$ also compatible with ϵ^+.*

$$(\epsilon^\pm)^* J_{\tilde{C}} = J_t \tag{4.38}$$

  for our previously fixed choice J_t.

Definition 4.3. *A universal and consistent choice of Floer data for the weak S^1 action is an inductive choice of Floer data, for each k and each representative $S = (C, p_1, \ldots, p_k)$ of $\overline{\mathcal{H}}_k$, satisfying the following conditions at boundary strata:*

- *Near $s = 0$, $H_{\tilde{S}} + F_{\tilde{S}} = 0$ in a neighborhood of the zero section.*

- *Near a boundary stratum of the form (4.25) the data agrees with the product of Floer data coming from lower-dimensional spaces, up to conformal equivalence. Moreover, with respect to the boundary chart structure induced by the gluing map, the Floer data agrees to infinite order with the data obtained by gluing.*
- *At a boundary stratum of the form (4.26), the Floer datum for S is conformally equivalent to one pulled back from $\mathcal{H}_{r-1}$ via the forgetful map π_i.*
- *At a boundary stratum of the form (4.27), the Floer datum for S is conformally equivalent to one pulled back from $\mathcal{H}_{r-1}$ via the forgetful map π_0.*

For a Hamiltonian orbit $y \in \mathcal{O}$, consider the space

$$\mathcal{H}_r(y) \tag{4.39}$$

of the parametrized moduli space of pairs $\{C = (S^+, p_1, \ldots, p_r) \in \mathcal{H}_r, u\}$ where u is a finite energy map $u : S^+ \to T^*M$ solving Floer's equation

$$(du - (X_{H_{S^+}} \otimes dt))^{0,1} = 0 \tag{4.40}$$

for the universal and consistent choice of Floer data for the weak S^1 action on C, with boundary conditions on the zero section and asymptotic conditions on y:

$$\begin{cases} u(0,t) \in M \\ \lim_{s \to +\infty} u(s, \cdot) = y. \end{cases} \tag{4.41}$$

The space (4.39) admits a natural Gromov compactification $\overline{\mathcal{H}}_r(y)$ whose codimension 1 boundary strata are covered by natural inclusions

$$\mathcal{H}_{r-k}(y_0) \times \mathcal{M}_k(y_0; y) \to \partial\overline{\mathcal{H}}_r(y), \tag{4.42}$$

$$\mathcal{H}_r(y_0) \times \mathcal{M}(y_0; y) \to \partial\overline{\mathcal{H}}_r(y), \tag{4.43}$$

$$\mathcal{H}_r^{i,i+1}(y) \tag{4.44}$$

$$\mathcal{H}_r^{0}(y). \tag{4.45}$$

Standard transversality and compactness arguments ensure that $\overline{\mathcal{H}}_r(y)$ is a compact manifold with boundary of dimension $n - |y| + 2r$. Moreoever, the consistency condition on Floer data implies that (4.31) and (4.34) induce maps between moduli spaces of maps, and in particular that there are identifications

$$(\pi_i, \mathfrak{o}_{i+1}) : \mathcal{H}_r^{i,i+1}(y) \cong \mathcal{H}_{r-1}(y) \times S^1 \tag{4.46}$$

$$(\pi_0, \mathfrak{o}) : \mathcal{H}_r^{0}(y) \cong \mathcal{H}_{r-1}(y) \times S^1. \tag{4.47}$$

Inductively, given that we have already chosen fundamental chains on $\mathcal{H}_{r-1}(y)$, choose fundamental chains on $\mathcal{H}_r^{i,i+1}(y)$ and $\mathcal{H}_r^0(y)$ which are *product-like*, meaning

There is a mistake: we need to use the length parametrisation in the target to choose such a splitting, and fundamental chain. In the end, these chains will differ by a degree 1 self-map of S^1, so certainly represent the same homology class

$$[\mathcal{H}_r^{i,i+1}(y)] = [\overline{\mathcal{H}}_{r-1}(y)] \times [S^1] \tag{4.48}$$

$$[\mathcal{H}_r^0(y)] = [\overline{\mathcal{H}}_{r-1}(y)] \times [S^1] \tag{4.49}$$

with respect to the identifications (4.33), (4.35).

An examination of orientations shows that

Lemma 4.2. *There exists a choice of fundamental chains for $\overline{\mathcal{H}}_r(y)$ and $\overline{\mathcal{M}}(y_0; y)$ in singular homology such that*

$$\begin{aligned}\partial[\overline{\mathcal{H}}_r(y)] = &\sum_{k\geq 1}^{r} \sum_{y_0} (-1)^{n+|y_0|} [\overline{\mathcal{H}}_{r-k}(y_0)] \times [\overline{\mathcal{M}}_k(y_0; y)] \\ &+ \sum_{y_0} (-1)^{n+|y_0|} [\overline{\mathcal{H}}_r(y_0)] \times [\overline{\mathcal{M}}(y_0; y)] \\ &+ [\mathcal{H}_r^{i,i+1}(y)] + [\mathcal{H}_r^0(y)].\end{aligned} \tag{4.50}$$

Now, there is a natural map from $\overline{\mathcal{H}}_r(y)$ to the loop space of M once we have chosen a basepoint (using the length parametrisation induced by a metric); on the open stratum $\mathcal{H}_r(y)$ one can capture the dependence on choice of basepoint as a map

$$\tilde{ev} : \mathcal{H}_r(y) \times S^1 \to \mathcal{L}M \tag{4.51}$$

$$\tilde{ev}((u, \theta_0))(t) := u(0, t + \theta_0). \tag{4.52}$$

We use the function $\mathfrak{bp} = \exp(i\pi)\mathfrak{o}_1 : \overline{\mathcal{H}}_r(y) \to S^1$ to provide such a basepoint, giving us an evaluation map

$$ev = \tilde{ev} \circ (id, \mathfrak{bp}) : \mathcal{H}_r(y) \to \mathcal{L}M. \tag{4.53}$$

Because $\mathfrak{bp}$ extends continuously to the compactification $\overline{\mathcal{H}}_r(y)$, ev extends to $ev : \overline{\mathcal{H}}_r(y) \to S^1$, so we can define

$$\mathcal{CL}_r(y) := ev_*([\overline{\mathcal{H}}_r(y)]). \tag{4.54}$$

Proof of Proposition 4.2. Let's observe first that

$$\begin{aligned}\partial ev_*([\overline{\mathcal{H}}_r(y)]) = &\sum_{k\geq 1}^{r} \sum_{y_0} (-1)^{n+|y_0|} ev_*([\overline{\mathcal{H}}_{r-k}(y_0)] \times [\overline{\mathcal{M}}_k(y_0; y)]) \\ &+ \sum_{y_0} (-1)^{n+|y_0|} ev_*([\overline{\mathcal{H}}_r(y_0)] \times [\overline{\mathcal{M}}(y_0; y)]) \\ &+ ev_*([\mathcal{H}_r^{i,i+1}(y)]) + ev_*([\mathcal{H}_r^0(y)]).\end{aligned} \tag{4.55}$$

We note now that $ev : \mathcal{H}_{r-k}(y_0) \times \mathcal{M}_k(y_0; y) \to \mathcal{L}M$ factors through projection to the first factor, in particular ev_* of the relevant fundamental chain is zero unless $\dim \mathcal{M}_k(y_0; y) = 0$, similarly for the boundary locus $\mathcal{H}_r(y_0) \times \mathcal{M}(y_0; y)$. So the first sum becomes

$$\begin{aligned}\sum_{k\geq 1}^{r} \sum_{y_0; |y_0|=|y|+1} &(-1)^{n+|y_0|} ev_*([\overline{\mathcal{H}}_{r-k}(y_0)]) n([\overline{\mathcal{M}}_k(y_0; y)]) \\ &= \sum_{k\geq 1} \mathcal{CL}_{r-k}(\Delta_k^{SC}(y)).\end{aligned} \tag{4.56}$$

Similarly, the second sum becomes

$$\begin{aligned}\sum_{y_0; |y_0|=|y|+1-2k} &(-1)^{n+|y_0|} ev_*([\overline{\mathcal{H}}_r(y_0)]) \#([\overline{\mathcal{M}}(y_0; y)]) \\ &= \mathcal{CL}_r(dy).\end{aligned} \tag{4.57}$$

Now, with respect to the isomorphism $\mathcal{H}_r^{i,i+1}(y) \cong \mathcal{H}_{r-1}(y) \times S^1$, We note that ev factors as projection onto the first factor followed by evaluation; hence, since the fundamental chain $[\mathcal{H}_r^{i,i+1}(y)])$ is product-like, with non-trivial degree in the S^1 factor, we conclude that $ev_*([\mathcal{H}_r^{i,i+1}(y)]) = 0$ in the normalized chain complex.

However, the same is not true for the space $\mathcal{H}_r^0(y)$: indeed, our ev map depends on the choice of basepoint p_1, which is forgotten by projecting to $\mathcal{H}_{r-1}(y)$. Instead, we see that there is a commutative diagram of maps

$$\begin{tikzcd} \overline{\mathcal{H}}_{r-1}(y) \times S^1 \arrow[r, "ev_{r-1}\times id"] & \mathcal{L}M \times S^1 \arrow[d, "mult"] \\ \overline{\mathcal{H}}_r^0(y) \arrow[u, "(\pi_0, \mathfrak{d})", "\sim"'] \arrow[r, "ev_r"] & \mathcal{L}M \end{tikzcd} \tag{4.58}$$

(here $mult : \mathcal{L}M \times S^1 \to \mathcal{L}M$ denotes the natural S^1 action). More precisely, (4.58) commutes because ev_r is by definition $\tilde{ev} \circ (id, \exp(i\pi)\mathfrak{o}_1)$, whereas $ev_{r-1} \circ \pi_0$ (thought of as a function on $\mathcal{H}_r^0$) is by definition $\tilde{ev} \circ (id, \exp(i\pi)\mathfrak{o}_2)$; these maps differ precisely by multiplication by $\mathfrak{d} := \mathfrak{o}_1 \mathfrak{o}_2^{-1}$.

By naturality of the Eilenberg-Zilber map, and our choice of fundamental chain compatible with (4.49), we have that

$$\begin{aligned} ev_*[\overline{\mathcal{H}}_r^0(y)] &= mult_*(ev_{r-1} \times id)_*[\overline{\mathcal{H}}_{r-1}(y)] \times [S^1] \\ &= mult_*((ev_{r-1})_*([\overline{\mathcal{H}}_{r-1}(y)]) \times [S^1]) \\ &= \Delta^{\mathcal{L}M}(\mathcal{CL}_{r-1}(y)). \end{aligned} \tag{4.59}$$

$\square$

4.3 The inclusions of constant loops

We first recall the proof of the following fact:

Proposition 4.3 ([?]*Lemma 3.6)**.** *There is a homotopy-commutative diagram of chain complexes:*

$$\begin{tikzcd} C_{2n-*}(T^*M, S^*M) \arrow[r, "PSS"] \arrow[d, "\cong"] & SC^*_b(T^*M) \arrow[d, "\mathcal{CL}"] \\ C_{n-*}(M) \arrow[r] & C_{n-*}(\mathcal{L}M). \end{tikzcd} \tag{4.60}$$

Proof. Choose a family $(J^r_{C^+}, \beta_r)$ of domain dependent almost complex structures and 1-forms on C^+, for $r \in [0, \infty)$ with β_r all sub-closed (meaning $d\beta_r \leq 0$), which are $(I_{C^+}, 0)$ for $r = 0$, and that result from gluing (I_{C^+}, β) and (I_{PSS}, β_{PSS}) for r near ∞. The meaning being that for r very large, β^r agrees with dt in a larger and larger neighborhood of the boundary component, with support of $d\beta$ pushed to ∞. Define

$$\mathcal{R}_{[1,\infty)}(N) \tag{4.61}$$

to be the paramatetrized space of solutions to Floer's equation

$$\{(r, u) | r \in [0, \infty), u : C_+ \to T^*M, (du - X_H \otimes \beta_r)^{0,1} = 0\} \tag{4.62}$$

where $(0, 1)$ is taken with respect to $J^r_{C^+}$, with boundary conditions on the zero section, $u(0, \cdot) \in M$ and such that $\lim_{s\to+\infty} u(s, t)$ converges to a point of N. One can in the usual fashion construct the Gromov bordification, $\overline{\mathcal{R}}_{[1,\infty)}(N)$, whose codimension 1 boundary is covered by the inclusions of the natural subspaces

$$\overline{\mathcal{R}}_0(N) \tag{4.63}$$

$$\overline{\mathcal{R}}^1(y) \times \mathcal{M}(y, N) \tag{4.64}$$

$$\mathcal{R}_{[0,\infty)}(\partial N) \tag{4.65}$$

where the first moduli space is the space of solutions for $r = 0$, and the remaining moduli spaces are self-explanatory. For generic choices of $(J^r_{C^+}, \beta_r)$ $\overline{\mathcal{R}}_{[1,\infty)}(N)$ is a smooth manifold of dimension $\mathrm{codim}(N) + 1$, and so we can choose a fundamental chain $[\overline{\mathcal{R}}_{[1,\infty)}(N)]$ which is compatible with the (4.63) - (4.65) in the usual sense. As usual, length parametrization induced by a metric gives a map $\mathrm{ev} : \mathcal{R}_{[1,\infty)}(N) \to \mathcal{L}M$, so one can define $H(N) := \mathrm{ev}_*([\overline{\mathcal{R}}_{[1,\infty)}(N)])$. The compatibility with boundary strata means $\partial H(N)$, the total boundary of H, is a sum of evaluations of chains coming from (4.63) - (4.65). The last term (4.65) is evidently $H \circ \partial$, and the second last term (4.64) corresponds to $\mathcal{CL} \circ PSS$ (by the usual argument; strictly speaking the boundary chain is the Eilenberg-Zilber of the tensor product of fundamental chains, but since evaluation factors through projection to the first component, the image under ev_* is degenerate and hence zero unelss the elements of $\mathcal{M}(y, N)$ are rigid). For the first term, we note that all solutions to $(du - X \otimes \beta^0)^{0,1} = (du)^{0,1} = 0$ with the given boundary conditions must in fact be constant, and hence ev_* corresponds to intersecting N with the zero section M, followed by the inclusion of constant loops. $\square$

Proposition 4.4. *The diagram (4.60) enhances to a homotopy-commutative diagram in the category of chain complexes with S^1 action. In particular, passing to homotopy fixed points (or negative cyclic chains), we obtain a commutative diagram*

$$\begin{tikzcd} C_{2n-*}(T^*M, S^*M) \otimes \mathbb{K}[u] \arrow[r, "\widetilde{PSS}"] \arrow[d, "\cong"] & (SC^*_b(T^*M))^{hS^1} \arrow[d, "\widetilde{\mathcal{CL}}"] \\ C_{n-*}(M) \otimes \mathbb{K}[u] \arrow[r] & (C_{n-*}(\mathcal{L}M))^{hS^1}. \end{tikzcd} \tag{4.66}$$

Proof. As in Proposition 3.7, There is a short argument in this case which bypasses the somewhat tedious cyclic homotopies in favor of making specific choices of Hamiltonian and circle action. First, note we choose the $C_{-*}(S^1)$ action to be trivial on $C_{n-*}(M)$ and $C_{2n-*}(T^*M, S^*M)$. Next, note that on the subcomplex of $C_{n-*}(\mathcal{L}M)$ consisting of chains of constant loops, the S^1 action is again trivial on the chain level, at least on the normalized chain complex (as one can show rather easily it factors through degenerate chains).

Now, further fix a choice of Hamiltonian for $SC^*(T^*M)$ as in Proposition 3.7 to be C^2 small, Morse (with critical points just along M) and time-independent in a neighborhood of the zero section. As previously argued, on the (action filtered) subcomplex consisting of critical points of H, the operations $\Delta_1, \ldots, \Delta_r, \ldots,$ are zero on the chain level. This subcomplex maps to the subcomplex of constant loops, for since the Hamiltonian term on the entire half cylinder involved in $\mathcal{CL}$ is C^2 small, solutions must necessarily be t-independent. By the same argument as before, the higher order maps $\mathcal{CL}_r$ can also be chosen to be zero on the chain level. □

Insert citation of Floer

Incomplete argument.

4.4 Comparison of cyclic open-closed maps

4.4.1 Abouzaid's model of Goodwillie's map

We recall a modification of Goodwillie's map introduced in [3]:

$$\mathcal{A} : CH_*(C_{-*}(\Omega_p M)) \to \tilde{C}_*(\mathcal{L}M), \tag{4.67}$$

and adapted here to singular chains (the reference [3] uses cubical chains).

As in the Goodwillie map, the inclusion from the based loop space to the free loop space

$$\iota : \Omega_p M \to \mathcal{L}M, \tag{4.68}$$

gives the length 0 part of the chain map:

$$\mathcal{A}^{(0)} : C_{-*}\Omega_p M \to \tilde{C}_{-*}(\mathcal{L}M) \tag{4.69}$$

as

$$\mathcal{A}^{(0)}(x) := (-1)^{|x|} pr_* \iota_*(x), \tag{4.70}$$

where $|x|$ denotes the degree of a singular chain x.

Citation needed

Going further, Abouzaid defines, as a special case of Goodwillie's CITE construction, a family of products

$$\sharp_t : \Omega_p M \times \Omega_p M \to \mathcal{L}M, \ t \in [0,1] \tag{4.71}$$

which are essentially the concatenation of the first and second loop, with new start point varying along the first loop (so as t varies from 0 to 1, this induces a homotopy between the composition $\alpha_1 \cdot \alpha_2$ and $\alpha_2 \cdot \alpha_1$ as elements of $\mathcal{L}M$). If γ_1 has length l_1 and γ_2 has length l_2, then $\gamma_1 \sharp_t \gamma_2$ has length $l_1 + l_2$, and is parametrized as:

$$\gamma_1 \sharp_t \gamma_t(s) = \begin{cases} \gamma_1(s + tl_1) & s \in [0, (1-t)l_1] \\ \gamma_2(s - (1-t)l_1) & s \in [(1-t)l_1, (1-t)l_1 + l_2] \\ \gamma_1(s - l_2 - (1-t)l_1) & \text{otherwise} \end{cases} \tag{4.72}$$

Given two singular chains $\alpha \in C_i(\Omega_p M)$, $\beta \in C_j(\Omega_p M)$, we obtain a map

$$F_{\alpha,\beta} : \Delta^i \times \Delta^j \times I \to \mathcal{L}M \tag{4.73}$$

$$F_{\alpha,\beta}(\vec{s}_1, \vec{s}_2, t) = \alpha(\vec{s}_1) \sharp_t \beta(\vec{s}_2); \tag{4.74}$$

i.e., a *product singular chain* in the language of Definition A.3. As in Section A.1.1 we can *singularize* (i.e., divide this chain into a sum of singular simplices) to obtain an element of singular chains $sing(F_{\alpha,\beta})$; now define

$$\mathcal{A}^{(1)}(\beta, \alpha) := -pr_*(sing(F_{\alpha,\beta})). \tag{4.75}$$

All together, we define

$$\begin{aligned} &\mathcal{A} : CH_*(C_{-*}(\Omega_p M)) \to \tilde{C}_{-*}(\mathcal{L}M) \\ &\mathcal{A}(\alpha_k \otimes \cdots \otimes \alpha_1) := \begin{cases} \mathcal{A}^{(0)}(\alpha_1) & k = 1 \\ \mathcal{A}^{(1)}(\alpha_2, \alpha_1) & k = 2 \\ 0 & k > 2 \end{cases} \end{aligned} \tag{4.76}$$

Lemma 4.3. *$\mathcal{A}$ is a chain map.*

Proof. For words not of length 2 or 3, this is straightforward. For a word $\alpha_3 \otimes \alpha_2 \otimes \alpha_1$, we need to show that

$$\begin{aligned} &\mathcal{A}(\alpha_3 \otimes \mu_2(\alpha_2, \alpha_1)) + (-1)^{|\alpha_1|} \mathcal{A}(\mu_2(\alpha_3, \alpha_2) \otimes \alpha_1) \\ &\quad + (-1)^{|\alpha_2| + (|\alpha_1|+1)(|\alpha_3|+|\alpha_2|+1)} \mathcal{A}(\mu_2(\alpha_1, \alpha_3) \otimes \alpha_2) = 0. \end{aligned} \tag{4.77}$$

Given a Hochschild chain $x_3 \otimes x_2 \otimes x_1$, where $x_i : \Delta^{n_i} \to \Omega_q M$, note that by associativity and naturality of the Eilenberg-Zilber map discussed in Section A.1.1, $\mathcal{A}(\mu^2(x_3, x_2), x_1)$ is the singularization of the product-singular chain

$$\begin{aligned} &\Delta^{n_1} \times \Delta^{n_3} \times \Delta^{n_2} \times I \to \mathcal{L}M \\ &\qquad (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) \mapsto x_1(\mathbf{v}_1)\sharp_t(x_3(\mathbf{v}_3) \cdot x_2(\mathbf{v}_2)). \end{aligned} \tag{4.78}$$

Suppose $|\alpha_3| = i$, $|\alpha_2| = j$, $|\alpha_1| = k$, and consider the following three maps

$$\gamma_s : \Delta^i \times \Delta^j \times \Delta^k \times I \to \mathcal{L}M, \ s = 1, 2, 3 \tag{4.79}$$

given by

$$\gamma_1(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1, t) = (\alpha_2(\mathbf{v}_2) \cdot \alpha_1(\mathbf{v}_1))\sharp_t \alpha_3(\mathbf{v}_3) \tag{4.80}$$

$$\gamma_2(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1, t) = \alpha_1(\mathbf{v}_1)\sharp_t(\alpha_3(\mathbf{v}_3) \cdot \alpha_2(\mathbf{v}_2)) \tag{4.81}$$

$$\gamma_3(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1, t) = \alpha_2(\mathbf{v}_2)\sharp_t(\alpha_1(\mathbf{v}_1) \cdot \alpha_3(\mathbf{v}_3)) \tag{4.82}$$

This proof is incomplete; missing e.g., sign analysis and chains of length 2.

It is easy to see that γ_2 and γ_3 fit together in the sense of Definition A.5. $\square$

Lemma 4.4. *$\mathcal{A}$ is strictly S^1-equivariant, meaning that $\mathcal{A} \circ B = \Delta \circ \mathcal{A}$.*

Proof. It is easy to see that $\mathcal{A}^{(1)} \circ B(\sigma) = \Delta \circ \mathcal{A}^{(0)}$ agree as singular chains in $\mathcal{L}M$, and hence agree in the relevant quotient complex $\tilde{C}_{-*}\mathcal{L}M$. Hence, it follows that $\Delta \circ \mathcal{A}^{(1)} = 0$ as an element of $\tilde{C}_{-*}\mathcal{L}M$.

Let τ and σ be singular chains of dimension i and j respectively. Denoting $\bar{\kappa} := \Delta\mathcal{A}^{(1)}(\sigma \otimes \tau)$, we see that $\bar{\kappa}$ is the singularization of the prod-singular chain $\kappa : \Delta^i \times \Delta^j \times I \times I \to \mathcal{L}M$ given by

$$\kappa(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2)(s) := \tau(\mathbf{s}_1)\sharp_{t_1}\sigma(\mathbf{s}_2)(s + (l_1 + l_2)t_2) \tag{4.83}$$

(here $l_1 + l_2$ denotes the length of the paths τ and σ, implicitly dependent on $\mathbf{s}_1 \in \Delta^i, \mathbf{s}_2 \in \Delta^j$).

The argument in words is to show, up to relations in $\tilde{C}^{\mathcal{L}}_{-*}\mathcal{L}M$, that $sing(\kappa)$ is degenerate. More specifically, we can split κ up into three singular generalized prisms, which then fit together in a different order to a singular generalized prism which is visibly independent of one of its coordinates. In words (4.83) is the $i + j + 2$ prod-singular chain whose last coordinate parametrizes different starting points around the circle of the ordinary concatenation $\tau(\mathbf{s}_1) \cdot \sigma(\mathbf{s}_2)$, but this loop of start points begins at a point on σ determined by the variable t_1 (which was the original start point).

We can divide the generalized prism $\Delta^i \times \Delta^j \times I \times I$ into three sub-generalized prisms as follows:

$$\begin{aligned} &\square_1 := \{(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2) | t_2 \in [0, (1 - t_1) \cdot \frac{l_1}{l_1 + l_2}]\} \\ &\square_2 := \{(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2) | t_2 \in [(1 - t_1) \cdot \frac{l_1}{l_1 + l_2}, \frac{l_2}{l_1 + l_2} + \frac{l_1}{l_1 + l_2}(1 - t_1)]\} \\ &\square_3 := \{(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2) | t_2 \in [\frac{l_2}{l_1 + l_2} + \frac{l_1}{l_1 + l_2}(1 - t_1), 1]\} \end{aligned} \tag{4.84}$$

This argument requires revision.

In an abuse of notation, denote by $\kappa|_{\square_r}$ the singular generalized prism $i+j+2$ chain obtained by composition $\kappa|_{\square_r}$ with the canonical map which shrinks the last factor of $\Delta^i \times \Delta^j \times I \times I$ to $\square_r$ (by just shrinking and shifting the last coordinate by the right amount). In $\tilde{C}_{-*}$ we see that

$$sing(\kappa) \sim sing(\kappa|_{\square_1}) + sing(\kappa|_{\square_2}) + sing(\kappa|_{\square_3}) \tag{4.85}$$

Now, we see that the sequence $\kappa|_{\square_2}, \kappa|_{\square_3}, \kappa|_{\square_1}$ also fits together, in this order, so we have

$$sing(\kappa) \sim sing(\tilde{\kappa}), \tag{4.86}$$

where we denote by $\tilde{\kappa}$ the result of gluing in this new order. But $\tilde{\kappa}$ is independent of t_1, and is hence a degenerate generalized singular prism chain. More precisely, observe that

$$\tilde{\kappa}(\mathbf{s}_1, \mathbf{s}_2, t_1, t_2)(s) := \tau(\mathbf{s}_1)\sharp_0\sigma(\mathbf{s}_2)(s + (l_1 + l_2)t_2). \tag{4.87}$$

We conclude that $sing(\tilde{\kappa})$ is zero in the normalized singular chain complex. $\square$

4.4.2 Adding a marked point to the outgoing segment

As in [3]*§5.1, denote by $\mathcal{P}_{d,1}$ the moduli space of half discs with d incoming boundary punctures and one marked point on the outgoing boundary segment, and $\overline{\mathcal{P}}_{d,1}$ its Gromov compactification. Associate to it the moduli space of maps

$$\overline{\mathcal{P}}_{d,1}(q, \vec{x}, q) := \overline{\mathcal{P}}_d(q, \vec{x}, q) \times_{\overline{\mathcal{P}}_d} \overline{\mathcal{P}}_{d,1} \tag{4.88}$$

where the fiber product is taken with respect to the natural map

$$\overline{\mathcal{P}}_{d,1} \to \overline{\mathcal{P}}_d \tag{4.89}$$

which on the top stratum forgets the position of the outgoing marked point. (Alternatively, one can define the moduli space (4.88) by using a Floer datum which is independent of the position of the outgoing marked point, and the fibre-product relation will be a consequence).

Using the length parametrization of the outgoing segment, fix an identification

$$f_d : \overline{\mathcal{P}}_d(q, \vec{x}, q) \times [0,1] \to \overline{\mathcal{P}}_{d,1}(q, \vec{x}, q) \tag{4.90}$$

In particular, we can pick a *product-like* fundamental chain for $\overline{\mathcal{P}}_{d,1}(q, \vec{x}, q)$:

$$[\overline{\mathcal{P}}_{d,1}(q, \vec{x}, q)] := (f_d)_*[\overline{\mathcal{P}}_d(q, \vec{x}, q)] \times [[0,1]]. \tag{4.91}$$

Lemma 4.5 (Abouzaid [3], (5.3))**.** *Let ι_0, ι_1 denote the maps $\overline{\mathcal{P}}_d(q, \vec{x}, q) \to \overline{\mathcal{P}}_{d,1}(q, \vec{x}, q)$ obtained by restricting f_d defined in* (4.90) *to 0 and 1 respectively. Then, the fundmental chains of $\overline{\mathcal{P}}_{d,1}(q, \vec{x}, q)$ satisfy the following inductive relation in the quotient complex of singular chains $\tilde{C}$ defined in Section*

??

$$
\begin{aligned}
\partial[\overline{\mathcal{P}}_{d,1}(q,\vec{x},q)] = (-1)^{1+||\vec{x}||}(\iota_1([\overline{\mathcal{P}}_d(q,\vec{x},q)]) - \iota_0([\overline{\mathcal{P}}_d(q,\vec{x},q)]))+ \\
\sum_{\vec{x}^1\cup\vec{x}^2=\vec{x}} (-1)^{\flat}\left((-1)^{1+||\vec{x}^2||}[\overline{\mathcal{P}}_{d_1,1}(q,\vec{x}^1,q)]\times[\overline{\mathcal{P}}_{d_2}(q,\vec{x}^2,q)] + [\overline{\mathcal{P}}_{d_1}(q,\vec{x}^1,q)]\times[\overline{\mathcal{P}}_{d_2,1}(q,\vec{x}^2,q))\right) \\
+\sum_x (-1)^{\sharp+\dim(\overline{\mathcal{R}}(y,\vec{x}^2))}[\overline{\mathcal{P}}_{d_1,1}(q,\vec{x}^1,q)]\times[\overline{\mathcal{R}}(x,\vec{x}^2)]
\end{aligned}
\tag{4.92}
$$

Remark 4.1. *If one did not want to work with the quotient of singular or cubical chains by "chains that fit together," as in Definition A.5, we can still ensure the consistency conditions of Lemma 4.5 at the expense of being unable to use a product fundamental chain for $\overline{\mathcal{P}}_{d,1}(q,\vec{x},q)$. For the purpose of identifying the resulting map with a version of Goodwillie's map, this fundamental chain can be homotoped to a product-like chain, though the resulting homotopy has an additional boundary in which a half-disc breaks off of the homotopy. The relevant homotopy can be iterated, leading to the higher order terms of the Goodwillie map, which so conveniently do not appear in the simplified model of [3].*

4.4.3 Adding a forgotten point

For $1 \le i \le d$, let

$$\mathcal{P}_d^{f_i} \tag{4.93}$$

denote the moduli space of half discs $\mathcal{P}_d$, with the ith incoming boundary marked point z_i put back in (so it is not a puncture) and marked as auxiliary, or *forgotten*. For $d > 1$, the *i-forgetful map*

$$\mathcal{F}_{d,i} : \mathcal{P}_d^{f_i} \to \mathcal{P}_{d-1} \tag{4.94}$$

associates to a half disc S the half disc S obtained by forgetting the point z_i; the fiber of (4.94) is topologically an (open) interval. (4.94) extends to Deligne-Mumford compactifications in the usual fashion, by forgetting z_f and collapsing any resulting unstable components.

For $d > 1$ and any collection $\{x_1, \dots, x_{d-1}\}$ of chords in $\chi(T_q^*M, T_q^*M)$, we get a parametrized moduli space

$$\overline{\mathcal{P}}^{f_i}(q,\vec{x},q) := \overline{\mathcal{P}}_d^{f_i} \times_{\mathcal{P}_d} \mathcal{P}(q,\vec{x},q). \tag{4.95}$$

(Put another way, this is equivalently the moduli space of maps satisfying Floer's equation and specified asymptotics, with source equal to $\mathcal{F}_{d,i}(S)$ for S an arbitrary element of $\overline{\mathcal{P}}^{f_i}(q,\vec{x},q)$, using the universal and consistent choice of Floer datum for $\overline{\mathcal{P}}^{f_i}$ which is pulled back via $\mathcal{F}^{d,i}$).

Using the length parametrization of the segment between z_{i-1} and z_{i+1} in the target, we obtain an identification

$$\overline{\mathcal{P}}(q,\vec{x},q)\times[0,1] \xrightarrow{i} \overline{\mathcal{P}}^{f_i}(q,\vec{x},q) \tag{4.96}$$

(see [3]*(5.2) for a similar parametrization). A product of fundamental chains for (4.96) gives rise to a preferred choice of fundamental chain for $\overline{\mathcal{P}}^{f_i}(q,\vec{x},q)$.

As before, restriction to the outgoing boundary segment induces an evaluation map

$$\mathrm{ev}^{f_i} : \overline{\mathcal{P}}^{f_i}(q, \vec{x}, q) \to \Omega_q M \tag{4.97}$$

We notice that $\mathrm{ev}^{f_i}(i(u,t)) = \mathrm{ev}(u)$, where i is the identification (4.96). In particular, $ev^{f_i} \circ i$ factors through projection $\overline{\mathcal{P}}(q, \vec{x}, q) \times [0,1] \to \overline{\mathcal{P}}(q, \vec{x}, q)$. We conclude that, by Lemma CITE , insert reference to eilenberg-zilber section

Proposition 4.5. *For $d > 1$,*

$$ev_*^{f_i}([\overline{\mathcal{P}}^{f_i}(q, \vec{x}, q)]) = ev_*^{f_i} i_*([\overline{\mathcal{P}}(q, \vec{x}, q)] \times [[0,1]]) = 0. \tag{4.98}$$

as an element of chains.

We also get, for $d = 1$ a moduli space

$$\overline{\mathcal{P}}^{f_i}(q, \{\}, q) \tag{4.99}$$

The *no escape Lemma* or maximum principle implies that the value of $|p|$ restricted to any element u of (4.99) obtains its maximum at the asymptotics of u, where (since z_f does not contribute asymptotic conditions) $|p| = 0$. In particular,

Lemma 4.6. *The only elements of* (4.99) *are constant maps.*

Corollary 4.1. *For $d = 1$, $ev_*([\overline{\mathcal{P}}^{f_i}(q, \{\}, q)]) = e_q$, where $e_q \in C_{-*}(\Omega_q M)$ is the strict identity element (represented by the zero simplex on a constant path of parametric length zero).*

4.4.4 Annuli

Denote by $\check{\mathcal{C}}_1^-$ the moduli space of annuli, with one marked point p_{out} on one boundary component and a boundary (incoming) puncture z_{in} on the other component, that are biholomorphic to

$$(\{z \in \mathbb{C} | 1 \leq z \leq r\}, p_{out} = -1, z_{in} = r), \tag{4.100}$$

for some r.

Denote by $\check{\mathcal{C}}_d^-$ the moduli space of annuli formed from $\mathcal{C}_1^-$ by adding $d-1$ additional marked points added to the component containing z_{in}; we now label the marked points on this component by $z_1, \ldots, z_d = z_{in}$. Call the component containing p_{out} the *outgoing circle*, and choosing a representative of $\check{\mathcal{C}}_d^-$ with fixed position (4.100) as in (4.100), fix the orientation

$$dr \wedge dz_1 \wedge \cdots \wedge dz_{d-1}. \tag{4.101}$$

The Deligne-Mumford compactification has codimension 1 strata covered by the inclusions of the products

$$\overline{\mathcal{R}}^1 \times \overline{\check{\mathcal{R}}}_d^1 \tag{4.102}$$

$$\overline{\mathcal{P}}_{d_1,1} \times \overline{\mathcal{P}}_{d_2}, \quad d_1 + d_2 = d, d_1 \geq 1 \tag{4.103}$$

$$\overline{\check{\mathcal{C}}}_{d_1}^- \times \overline{\mathcal{R}}_{d_2}, \quad d_1 + d_2 - 1 = d, d_1 \geq 1, d_2 \geq 2 \tag{4.104}$$

Insert definition of Floer datum

Definition 4.4. *A* **Floer datum** *for a stable annulus* $S \in \overline{\check{\mathcal{C}}}_d^-$ *is a Floer datum for* S *in the sense of Definition 2.2 with the following additional conditions/requirements:*

- *The* **closed 1-form** α_S *need not be zero when restricted to the outgoing boundary component;*
- *The* **time-shifting map** $\rho : \partial S \to [1, \infty)$ *is equal to 1 on the outgoing boundary component;*
- *The* **total Hamiltonian perturbation** $H_S + F_S$ *is equal to zero in a neighborhood of the zero section* $M \subset T^*M$ *near the outgoing boundary component.*

A *universal and consistent Floer datum for the annulus* consists of an inductive choice for each $d \geq 1$ and each $T \in \overline{\check{\mathcal{C}}}_d^-$ of a Floer datum, smoothly varying in T, whose restriction to each boundary stratum is conformally equivalent to the product of Floer data coming from lower-dimensional spaces. As usual, on a stratum, the Floer datum should agree to infinite order with the Floer data obtained by gluing, using the boundary/corner charts of $\overline{\check{\mathcal{C}}}_d^-$. As usual, one can prove that universal and consistent choices of Floer data for the annulus exist.

Thus, given a sequence of inputs $\vec{x} = \{x_1, \ldots, x_d\}$, define the moduli space

$$\overline{\check{\mathcal{C}}}_d^-(\vec{x}) \tag{4.105}$$

to be the parametrized space of maps

$$\{(S, u) : S \in \overline{\check{\mathcal{C}}}_d^-, u : S \to T^*M\} \tag{4.106}$$

Maybe it makes sense to split the time-dependent perturbation from the autonomous piece in this Floer equation and Floer data.

which map the outgoing boundary component to M and the other boundary components to T_q^*M, with asymptotic condition $\psi^{w_k, S} x_k$ at the kth incoming end, which solve Floer's equation using the chosen Floer datum for S:

$$(du - X_S \otimes \alpha_S)^{0,1} = 0. \tag{4.107}$$

For generic choices of Floer data, the Gromov compactification (4.105) is a compact manifold of dimension

$$d - \sum_{k=1}^{d} |x_k|. \tag{4.108}$$

The boundary of (4.105) is covered by the images of natural inclusions

$$\overline{\mathcal{R}}^1(y) \times \overline{\mathcal{R}}^1_d(y, \vec{x}),\ y \in \mathcal{O} \tag{4.109}$$

$$\overline{\mathcal{P}}_{d_1,1}(q, \vec{x}^1, q) \times \overline{\mathcal{P}}_{d_2}(q, \vec{x}^2, q),\ 0 \leq r < d_2 \leq d = d_1 + d_2 \tag{4.110}$$

$$\overline{\check{\mathcal{C}}}_d^-(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2}(x; \vec{x}^2),\ 1 \leq d_1 < d = d_1 + d_2 - 1,\ x \in \mathcal{X} \tag{4.111}$$

where in the second type of stratum $\vec{x}^1 = (x_{r+1}, \ldots, x_{r+d_1})$ and $\vec{x}^2 = (x_{r+d_1+1}, \ldots, x_d, x_1, \ldots, x_r)$ and in the last type of stratum x is an element of $\vec{x}^1$, which, when removed from $\vec{x}^1$ and replaced with $\vec{x}^2$, produces a cyclic reordering of the list $\vec{x}$. The cases of (4.111) differ according to whether x_d ends up in $\vec{x}^1$ or $\vec{x}^2$.

Lemma 4.7 ([3], §5.4)**.** *For each d, there exists an inductive choice of (relative) fundamental chains for $\overline{\check{\mathcal{C}}}_d^-(\vec{x})$ for all sequences $\vec{x}$ satisfying the following relation (in the singular chain complex $\tilde{C}_*(\overline{\check{\mathcal{C}}}_d^-(\vec{x})$:*

$$\begin{aligned}\partial[\overline{\check{\mathcal{C}}}_d^-(\vec{x})] = &\sum(-1)^{(d-1)(n-|y|)}[\overline{\mathcal{R}}^1(y)] \times [\overline{\mathcal{R}}_d^1(y;\vec{x})] + \\ &\sum(-1)^{\frac{n(n+1)}{2}+\diamond_r^{d_1}}[\overline{\mathcal{P}}_{d_1,1}(q,\vec{x}^1,q)] \times [\overline{\mathcal{P}}_{d_2}(q,\vec{x}^2,q)] + \sum_{x_d\in\vec{x}^1}(-1)^{1+\sharp}[\overline{\check{\mathcal{C}}}_d^-(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2}(x;\vec{x}^2)] \\ &+ \sum_{x_d\in\vec{x}^2}(-1)^{d_1+1+\diamond_r^{d_1}+d_2|\vec{x}^2|}[\overline{\check{\mathcal{C}}}_d^-(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2}(x;\vec{x}^2)]\end{aligned} \tag{4.112}$$

where the sign $\sharp$ is as in (2.101) *and the other sign is*

$$\diamond_r^{d_1} = r(d+1) + \left(\sum_{k=1}^{r}|x_k|\right)\cdot\left(\sum_{k=r+1}^{d}|x_k|\right) + d_2\left(\sum_{k=r+1}^{r+d_1}|x_k|\right). \tag{4.113}$$

As before, using the point p as a basepoint and length parametrization in the target, by restricting an element u to its outgoing boundary component, one obtains a map

$$ev : \overline{\check{\mathcal{C}}}_d^-(\vec{x}) \to \mathcal{L}M \tag{4.114}$$

and we can define

$$\mathcal{H} : \mathrm{CH}_*(CW_b^*(T_q^*M)) \to \tilde{C}_{-*}(\mathcal{L}M) \tag{4.115}$$

by linearly extending

$$x_d \otimes \cdots \otimes x_1 \mapsto (-1)^{|x_d|+\star+d|\vec{x}|} ev_*([\overline{\check{\mathcal{C}}}_d^-(\vec{x})]) \tag{4.116}$$

Fix sign in map— star isn't right, cross isn't working in LaTeX

The compatible choices of Floer data made, and subsequently fundamental chains, along with a careful verification of signs, proves that

Lemma 4.8 ([3], Lemma 5.4)**.** *$\mathcal{H}$ is a homotopy between $(-1)^{\frac{n(n+1)}{2}}\mathcal{G}\circ\mathcal{F}_\#$ and $\mathcal{CL}\circ\mathcal{OC}$; that is,*

$$(-1)^{\frac{n(n+1)}{2}}\mathcal{A}\circ\mathcal{F}_\# - \mathcal{CL}\circ\mathcal{OC} = d\mathcal{H} + \mathcal{H}d. \tag{4.117}$$

All together, this establishes:

Proposition 4.6 (Abouzaid [3], Prop. 1.6)**.** *The following diagram homotopy-commutes up to an overall sign of $(-1)^{n(n+1)/2}$:*

$$\begin{tikzcd} \mathrm{CH}_*(CW_b^*(T_q^*M)) \arrow[r, "\mathbf{P}_\#"] \arrow[d, "\mathcal{OC}"] & \mathrm{CH}_*(C_{-*}\Omega_q M) \arrow[d, "\mathcal{A}"] \\ SC_b^{*+n}(T^*M) \arrow[r, "\mathcal{CL}"] & \tilde{C}_{-*}(\mathcal{L}M) \end{tikzcd} \tag{4.118}$$

To begin the process of studying cyclic homotopies, we need to first extend this homotopy-commutative diagram to one of the form

$$\begin{tikzcd} \mathrm{CH}^{nu}_*(CW^*_b(T^*_q M)) \arrow[r, "\mathbf{P}_\#"] \arrow[d, "\mathcal{OC}^{nu}"] & \mathrm{CH}^{nu}_*(C_{-*}\Omega_q M) \arrow[d, "\mathcal{A}^{nu}"] \\ SC^{*+n}_b(T^*M) \arrow[r, "\mathcal{CL}"] & \tilde{C}_{-*}(\mathcal{L}M) \end{tikzcd} \tag{4.119}$$

Insert: extension to the non-unital Hochschild complex

where the sources have been replaced by non-unital Hochschild complexes. To this extent, we need to define a chain homotopy $\hat{\mathcal{H}}$ constructing the portion of the chain homotopy with source an element of $\hat{\mathrm{CH}}_*(CW^*_b(T^*_q M)$.

For $d \geq 1$, denote by $\hat{\mathcal{C}}^-_d$ the same moduli space of annuli as $\check{\mathcal{C}}^-_{d+1}$, where the puncture z_{in} is replaced with an auxiliary (filled in) boundary marked point z_f; the orientation on this moduli space is thus

$$dr \wedge dz_1 \wedge \cdots \wedge dz_d. \tag{4.120}$$

The codimension-1 boundary of the natural compactification $\overline{\hat{\mathcal{C}}}^-_d$ is covered by the images of the natural inclusions

$$\overline{\mathcal{R}}^1 \times \overline{\hat{\mathcal{R}}}^1_d \tag{4.121}$$

$$\overline{\mathcal{P}}_{d_1,1} \times \overline{\mathcal{P}}^{f_i}_{d_2}, \quad d_1 + d_2 = d, d_1 \geq 1, 1 \leq i \leq d_2 \tag{4.122}$$

$$\overline{\hat{\mathcal{C}}}^-_{d_1} \times \overline{\mathcal{R}}^{d_2}, \quad d_1 + d_2 - 1 = d, d_1 \geq 1, d_2 \geq 2 \tag{4.123}$$

$$\overline{\check{\mathcal{C}}}^-_{d_1} \times \overline{\mathcal{R}}^{d_2, f_i}, \quad d_1 + d_2 - 1 = d, d_1 \geq 1, d_2 \geq 2 \tag{4.124}$$

where (4.121) is as in (4.109), (4.122) is as in (4.110) except the second factor now contains the auxiliary point z_f in the ith position, for some $i \leq d_2$, (4.123) refers to the usual boundary bubbling at a point excluding z_f, and (4.123) refers to boundary bubbling which includes the point z_f (in which case the bubbled stratum is thought of as belonging to discs with a forgotten point, in the sense of Appendix **??**).

Inductively a *universal and consistent Floer datum* for the $\overline{\hat{\mathcal{C}}}^-_d$ in the sense of the previous construction: that is, a smoothly varying choice of Floer data for each component of each representative of $\overline{\hat{\mathcal{C}}}^-_d$, restricting to a Floer datum on each boundary stratum that is conformally equivalent to previous choices, and converging to infinite order to this previous choice with respect to standard boundary and corner gluing charts for the family. In the same fashion as before, for a sequence of inputs $\vec{x}$ one obtains a (Gromov-compactified) moduli space of parametrized solutions to Floer's equation, with specified asymptotics and Lagrangian boundary conditions on M (for the outgoing boundary component and $T^*_q M$ (for the remaining components)

$$\overline{\hat{\mathcal{C}}}^-_d(\vec{x}); \tag{4.125}$$

for generic choices of Floer data this is a compact manifold of dimension

$$d - \sum_{k=1}^{d} |x_k| + 1, \tag{4.126}$$

whose boundary is covered by the images of the natural inclusions

$$\overline{\mathcal{R}}^1(y) \times \overline{\hat{\mathcal{R}}}{}^1_d(y, \vec{x}),\ y \in \mathcal{O} \tag{4.127}$$

$$\overline{\mathcal{P}}_{d_1,1}(q, \vec{x}^1, q) \times \overline{\mathcal{P}}^{f_i}_{d_2}(q, \vec{x}^2, q),\ 0 \leq r < d_2 \leq d = d_1 + d_2 \tag{4.128}$$

$$\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2}(x; \vec{x}^2),\ 1 \leq d_1 < d = d_1 + d_2 - 1,\ x \in \mathcal{X} \tag{4.129}$$

$$\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2, f_i}(x; \vec{x}^2),\ 1 \leq d_1 < d = d_1 + d_2 - 1,\ x \in \mathcal{X} \tag{4.130}$$

where...

Lemma 4.9. *For each d, there exists an inductive choice of relative fundamental chains for $\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x})$ for all sequences $\vec{x}$ satisfying the following relation in the singular chain complex $\tilde{C}_*(\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x}))$:*

$$\partial[\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x})] = \sum (-1)^{(d-1)(n-|y|)} [\overline{\mathcal{R}}^1(y)] \times [\overline{\mathcal{R}}^1_d(y; \vec{x})] + \tag{4.131}$$

$$\sum (-1)^{\frac{n(n+1)}{2} + \diamond_r^{d_1}} [\overline{\mathcal{P}}_{d_1,1}(q, \vec{x}^1, q)] \times [\overline{\mathcal{P}}^{f_i}_{d_2}(q, \vec{x}^2, q)] + \sum_{x_d \in \vec{x}^1} (-1)^{1+\sharp} [\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2}(x; \vec{x}^2)] \tag{4.132}$$

$$+ \sum_{x_d \in \vec{x}^2} (-1)^{d_1 + 1 + \diamond_r^{d_1} + d_2 |\vec{x}^2|} [\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2, f_i}(x; \vec{x}^2)] \tag{4.133}$$

Once more, by restricting an element u to the outgoing boundary component (and using length parametrization in the target), we have a map $\mathrm{ev} : \overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x}) \to \mathcal{L}M$, so we define

$$\hat{\mathcal{H}} : \hat{\mathrm{CH}}_*(CW_b^*(T_q^* M)) \to \tilde{C}_{-*}(\mathcal{L}M) \tag{4.134}$$

as the linear extension of

$$x_d \otimes \cdots \otimes x_1 \mapsto (-1)^{d|\vec{x}|} \mathrm{ev}_*([\overline{\hat{\mathcal{C}}}{}^{-}_d(\vec{x})]). \tag{4.135}$$

Lemma 4.10.

$$(-1)^{\frac{n(n+1)}{2}} \hat{\mathcal{A}} \circ \hat{\mathbf{P}}_\sharp - \mathcal{CL} \circ \hat{\mathcal{OC}} = d\hat{\mathcal{H}} + \hat{\mathcal{H}} b' + \check{\mathcal{H}}(1 - t). \tag{4.136}$$

insert citations/more detail

Proof. As before, one applies ev_* to the relation (4.133), noting that by since evaluation from the stratum $\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2, f_i}(x; \vec{x}^2)$ to $\mathcal{L}M$ factors through projection to $\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x}^1)$, by Lemma CITE, the result is zero unless the elements of $\overline{\mathcal{R}}^{d_2, f_i}$ are rigid (which in turn only holds if $d_2 = 2$, by Lemma CITE). Hence,

$$\mathrm{ev}_*([\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2, f_i}(x; \vec{x}^2)]) = \begin{cases} 0 & d_2 > 2 \\ \mathrm{ev}_*([\overline{\check{\mathcal{C}}}{}^{-}_d(\vec{x})]) = \check{\mathcal{H}}(\vec{x}) & d_2 = 2, i = 1 \\ \mathrm{ev}_*([\overline{\check{\mathcal{C}}}{}^{-}_d(t\vec{x})]) = \check{\mathcal{H}}(\vec{x}) & d_2 = 2, i = 2. \end{cases} \tag{4.137}$$

Similarly, by . . . insert

$$\mathrm{ev}_*([\overline{\mathcal{P}}_{d_1,1}(q, \vec{x}^1, q)] \times [\overline{\mathcal{P}}^{f_i}_{d_2}(q, \vec{x}^2, q)]) = \begin{cases} 0 & d_2 > 1 \\ \mathcal{A}^{(2)}(e, \mathcal{P}^d(x_d, \dots, x_1)) & d_2 = 1. \end{cases} \tag{4.138}$$

□

4.4.5 Degenerate familes of annuli (with auxiliary points)

We define a family of (mostly degenerate) operations, for $k \geq 0$, which arise as the boundary of the family of annuli described in §4.4.6

Definition 4.5. *Let S denote a disc with boundary marked points or punctures $y_1, \dots, y_r$. A (y_i, y_j)* **strip-comparison map** *is a biholomorphism*

$$\phi : \bar{S} \to \bar{Z} \tag{4.139}$$

from the compactification $\bar{S}$ obtained by filling in any punctures to the two-point compactification of the infinite strip $Z = (-\infty, \infty) \times [0, 1]$ sending y_i to $+\infty$ and y_j to $-\infty$.

Any two (y_i, y_j)-strip comparison maps differ by post-composition with translation, hence the following definition is independent of choices:

Definition 4.6. *Let S denote a disc with boundary marked points or punctures $y_1, \dots, y_r$, and an interior marked point p. For $t \in (0, 1)$, p_1 is said to lie at* **height** *t* **relative** *(y_i, y_j) if, for any (y_i, y_j) strip-comparison map ϕ, if $\phi(p_1) \in \mathbb{R} \times \{t\}$. We write*

$$h_{(y_i, y_j)}(p) = t \tag{4.140}$$

or simply $h(p) = t$, if the pair of marked points is implicit.

Define

$$\check{\mathcal{D}}_{d_1, d_2} \tag{4.141}$$

to be the moduli space of broken curves represented by pairs of half discs (S, S'), with d_1+2 boundary punctures labeled $z_{-1}, z_0, z_1, \dots, z_{d_1}$ on S and $d_2 + 2$ boundary punctures $z'_{-1}, z'_0,' z'_1, \dots, z'_{d_2}$ on S', and an additional marked point p on S between z_{-1} and z_0. The boundary punctures $z_{-1}, z_0, z'_{-1}, z'_0$ are labeled as outgoing, and the remaining are incoming. Also, the pair (S, S'): is not disconnected: z_{-1} attached to z_0 and z_0 attached to z'_0 (strictly speaking, one starts with the nodal surface where z_i, z'_i are put in and identified, then remove the nodal points. Or from another view, one could view these as disjoint pairs of strips, and use the "attachment datum" as a sort of diagonal asymptotic condition).

We refer to the pair of boundary segments between z_{-1} and z_0 and between z'_{-1} and z'_0 as the *outgoing boundary segments*. Note that we have an identification (extending to Deligne-Mumford compactifications):

$$\check{\mathcal{D}}_{d_1,d_2} \cong \mathcal{P}_{d_1,1} \times \mathcal{P}_{d_2} \tag{4.142}$$

Going further, for $I \subset \{2, \ldots, k\}$ we define

$$ {}_k^I\check{\mathcal{D}}_{d_1,d_2} \tag{4.143}$$

to be the same moduli space, equipped with k interior marked (auxiliary) points $p_1, \ldots, p_k$, with $p_i \in S$ if $i \in \{2, \ldots, k\} \backslash I$ and $p_i \in S'$ if $i \in \{1\} \cup I$. For such a point $p \in S \cup S'$, denote by

$$h(p) := \begin{cases} h_{(z_{-1},z_0)}(p) & p \in S \\ h_{(z'_{-1},z'_0)}(p) & p \in S', \end{cases} \tag{4.144}$$

where $h_{(y_1,y_2)}$ is the height function relative a pair of points as in Definition 4.6. With respect to the function h, $p_1, \ldots, p_k$ are required to satisfy an ordering condition:

$$h(p_1) \leq h(p_2) \leq \cdots \leq h(p_k) \leq \frac{1}{2}. \tag{4.145}$$

Remark 4.2. *Because of* (4.145)*, the space* (4.143) *does not split as a product of moduli spaces for* $k > 1$*, as opposed to* (4.141)*.*

There is a *reduced* moduli space of interest

$$ {}_k^I\check{\mathcal{D}}^{red}_{d_1,d_2} \tag{4.146}$$

defined to be exactly the same as $\check{\mathcal{D}}_{d_1,d_2}$ except the first auxiliary point $\bar{p}_1$ is now a boundary marked point on the outgoing component of S' (it is still sensible to require (4.145), though this condition is now vacuous for p_1). The *forgetful*, or *rescaling* map

$$\mathcal{F}_1 : {}_k^I\check{\mathcal{D}}_{d_1,d_2} \to {}_k^I\check{\mathcal{D}}^{red}_{d_1,d_2} \tag{4.147}$$

sends p_1 to the point on the outgoing boundary component with the same value of the function w; the fiber of this map away from the locus $h(p_2) = 0$ is topologically an interval. Moreover, the fibration is trivial: in reverse, there is a *scaling* map

$$\mathcal{F}_1 : {}_k^I\check{\mathcal{D}}^{red}_{d_1,d_2} \times [0,1] \to {}_k^I\check{\mathcal{D}}_{d_1,d_2} \tag{4.148}$$

which sends an element $((S, \bar{S}'), t)$ to the element (S, S') where S' is produced from $\bar{S}'$ by inserting the unique point p_1 with $w(p_1) = w(\bar{p}_1)$ and $h(p_1) = th(p_2)$, then deleting $\bar{p}_1$.

Somewhat more drastically, we will want to consider the forgetful map

$$\mathcal{F} : {}_k^I\check{\mathcal{D}}_{d_1,d_2} \to {}_1^{\emptyset}\check{\mathcal{D}}^{red}_{d_1,d_2} = \mathcal{P}_{d_1,1} \times \mathcal{P}_{d_2,1} \tag{4.149}$$

which forgets all points p_i, for $i \geq 2$ and rescales p_1 to a boundary point $\bar{p}_1$ as above. In the stable range $d_2 \geq 1$, we can forget $\bar{p}_1$ as well, leading to a *maximal forgetful map*

$$\mathcal{F}_{max} : {}^I_k\check{\mathcal{D}}_{d_1,d_2} \to \mathcal{P}_{d_1,1} \times \mathcal{P}_{d_2}; \tag{4.150}$$

in the semi-stable case $d_2 = 0$, there is a *semi-maximal forgetful map* obtained by forgetting $p_2, \dots, p_k$ and rescaling p_1 in the sense of (4.147):

$$\mathcal{F}_{semi} : {}^I_k\check{\mathcal{D}}_{d_1,0} \to \mathcal{P}_{d_1,1} \times \mathcal{P}_{0,1}. \tag{4.151}$$

The maps (4.149), (4.150), and (4.151) extend in the usual fashion (described previously) to Deligne-Mumford compactifications.

Similarly, for $I \subset \{2, \dots, k+1\}$ we have

$$ {}^I_k\mathcal{D}^{S^1}_{d_1,d_2} \tag{4.152}$$

which is identical to ${}^I_{k+1}\check{\mathcal{D}}_{d_1,d_2}$, except the $k+1$st auxiliary point is constrained to lie at height exactly $\frac{1}{2}$, so the ordering condition becomes:

$$h(p_1) \leq h(p_2) \leq \cdots \leq h(p_k) \leq h(p_{k+1}) = \frac{1}{2}. \tag{4.153}$$

Finally, we also have have

$$ {}^I_k\hat{\mathcal{D}}^{f_i}_{d_1,d_2},\ 1 \leq i \leq d_1 \tag{4.154}$$

$$ {}^I_k\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2},\ 1 \leq j' \leq d_2 \tag{4.155}$$

defined exactly to be ${}^I_k\hat{\mathcal{D}}_{d_1+1,d_2}$ (respectively ${}^I_k\hat{\mathcal{D}}_{d_1,d_2+1}$) where the ith point on S (respectively j'th point on S') is filled in and marked as auxiliary.

Convenient fact: we do not care about the orientations of these spaces unless $k = 0$, as the resulting operations are zero anyway.

For $k = 0$, the constraint that $p_1 \in I \dots$ induces a splitting

$$\hat{\mathcal{D}}^{f'_j}_{d_1,d_2} \cong \mathcal{P}_{d_1,1} \times \mathcal{P}^{f_i}_{d_2} \tag{4.156}$$

Fix a *universal and conformally consistent Floer datum for degenerate annuli*, which consists of, for every d_1, d_2, and I, and every S_1

We obtain moduli spaces

$$ {}^I_k\check{\mathcal{D}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.157}$$

$$ {}^I_k\mathcal{D}^{S^1}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.158}$$

$$ {}^I_k\hat{\mathcal{D}}^{f_i}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2),\ 1 \leq i \leq d_1 \tag{4.159}$$

$$ {}^I_k\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2),\ 1 \leq j' \leq d_2 \tag{4.160}$$

For $k > 0$, the scaling map extends to the moduli space of maps, producing maps:

$$ {}^{I}_{k}\check{\mathcal{D}}^{red}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \times [0,1] \to {}^{I}_{k}\check{\mathcal{D}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.161} $$

$$ {}^{I}_{k}\check{\mathcal{D}}^{S^1,red}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \times [0,1] \to {}^{I}_{k}\check{\mathcal{D}}^{S^1}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.162} $$

$$ {}^{I}_{k}\hat{\mathcal{D}}^{red,f_i}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \times [0,1] \to {}^{I}_{k}\hat{\mathcal{D}}^{f_i}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.163} $$

$$ {}^{I}_{k}\hat{\mathcal{D}}^{red,f_{j'}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \times [0,1] \to {}^{I}_{k}\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \tag{4.164} $$

These are not quite homeomorphisms, for the same reasons as earlier. However, the map is a homeomorphism top strata and any (open) stratum on which $p_2 \neq 0$, and has fiber an interval on the (closed locus of) strata where $p_2 = 0$. (If $k = 1$, p_2 by definition is never zero). Since the strata where $p_2 = 0$ is codimension ≥ 2, we conclude that

Lemma 4.11. *The image of a (n Eilenberg-Zilber) product of a fundamental chain for* ${}^{I}_{k}\check{\mathcal{D}}^{red}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2)$ *and* $[0,1]$ *define a fundamental chain for* ${}^{I}_{k}\check{\mathcal{D}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2)$.

We call this chain a *product-like chain* for the map (4.161), and fix such product-like chains for (4.161) - (4.164).

For any of these moduli spaces, by restricting a map to its outgoing boundary segments, using the length parametrisation in the target and the image of the point p as the initial point, we obtain maps

$$ \mathrm{ev} : {}^{I}_{k}\check{\mathcal{D}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \to \mathcal{L}M \tag{4.165} $$

$$ \mathrm{ev} : {}^{I}_{k}\hat{\mathcal{D}}^{f_i}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \to \mathcal{L}M \tag{4.166} $$

$$ \mathrm{ev} : {}^{I}_{k}\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2) \to \mathcal{L}M. \tag{4.167} $$

Proposition 4.7. *For* $k > 0$ *and* X *equal to* ${}^{I}_{k}\check{\mathcal{D}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2)$, ${}^{I}_{k}\hat{\mathcal{D}}^{f_i}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2)$, *or* ${}^{I}_{k}\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2}(q, \vec{x}^1, \vec{x}^2)$, *with the chosen product-like chains,*

$$ \mathrm{ev}_*([X]) = 0. \tag{4.168} $$

in the normalized singular chain complex.

Proof. Let Y denote the corresponding space from the left hand sides of (4.161) - (4.164), so that there is a map $i : Y \times [0,1] \to X$, and a map $p : X \to Y$ such that the composition coincides with the projection to the first factor: $Y \times [0,1] \to Y$. We also note that the evaluation $\mathrm{ev} : X \to \mathcal{L}M$ factors as

$$ X \xrightarrow{p} Y \xrightarrow{\bar{\mathrm{ev}}} \mathcal{L}M \tag{4.169} $$

Note now that

$$ \mathrm{ev}_*([X]) = \mathrm{ev}_*(i_*([Y] \times [[0,1]])) = \bar{\mathrm{ev}}_*(p_* i_*([Y] \times [[0,1]])). \tag{4.170} $$

Reference degenerate chains lemma

But $p \circ i$ is the projection $\pi : Y \times [0,1] \to Y$, and by Lemma CITE, $\pi_*([Y] \times [[0,1]]) = 0$. $\square$

Corollary 4.2.

$$\mathrm{ev}_*[{}^I_k\check{\mathcal{D}}_{d_1,d_2}(q,\vec{x}^1,\vec{x}^2)] = \begin{cases} 0 & k>0 \\ \mathcal{A}^{(0)}(\mathbf{P}^d(x_d,\dots,x_1)) & k=0,\ d_1=0 \\ \mathcal{A}^{(1)}(\mathbf{P}^{d_1}(x_{j+1},\dots,x_d,\dots,x_i),\mathbf{P}^{d_2}(x_{i+1},\dots,x_j)) & \end{cases} \tag{4.171}$$

$$\mathrm{ev}_*[{}^I_k\hat{\mathcal{D}}^{f_i}_{d_1,d_2}(q,\vec{x}^1,\vec{x}^2)] = \begin{cases} 0 & k>0 \\ e\otimes\mathcal{A}^{(0)}(\mathbf{P}^d(x_d,\dots,x_1)) & k=0,\ d_1=0 \\ 0 & k=0, d_1>0 \end{cases} \tag{4.172}$$

$$\mathrm{ev}_*[{}^I_k\hat{\mathcal{D}}^{f_{j'}}_{d_1,d_2}(q,\vec{x}^1,\vec{x}^2)] = 0 \ \textit{if}\ k\geq 1. \tag{4.173}$$

Remark 4.3. (4.173) *only describes the case $k \geq 1$ because it is the only case which will appear as a codimension 1 boundary of later moduli spaces.*

4.4.6 Annuli with cyclic homotopies

In this section, we will define and study operations controlled by three (families of) moduli spaces of annuli with *cyclic homotopies*. Our main goal will be to realize the following compatibility of Floer-theoretic and topological maps, on the level of S^1-equivariant homologies:

Proposition 4.8. *The following diagram is homotopy-commutative, up to an overall sign of $(-1)^{n(n+1)/2}$:*

$$\begin{tikzcd} \mathrm{CC}^-_*(CW^*_b(T^*_qM)) \arrow[r, "\tilde{\mathbf{P}}_\#"] \arrow[d, "\widetilde{\mathcal{OC}}"] & \mathrm{CC}^-_*(C_{-*}(\Omega_qM)) \arrow[d, "\tilde{\mathcal{A}}^{nu}"] \\ SC^{-,*}_{S^1,b}(T^*M) \arrow[r, "\widetilde{\mathcal{CL}}"] & \tilde{C}^{S^1}_{-,n-*}(\mathcal{L}M) \end{tikzcd} \tag{4.174}$$

First,

$${}_k\check{\mathcal{C}}_d \tag{4.175}$$

is the moduli space of annuli S with one marked point p_{out} on one boundary component, d boundary (incoming) punctures $z_1,\dots,z_d$ on the other boundary component, and k interior auxiliary marked points $p_1,\dots,p_k$, such that

$$\begin{gathered}\text{The triple } (S,p_{out},p_1) \text{ is biholomorphic to a standard annulus} \\ S' = \{z\in C | 1\leq z\leq r\},\ \text{for some } r, \text{ with } p_{out}=-1,\ p_1\in\mathbb{R}^+,\end{gathered} \tag{4.176}$$

We formally define $p_{k+1} = z_d$, so that there is an identification

$${}_0\check{\mathcal{C}}_d \cong \check{\mathcal{C}}^-_d. \tag{4.177}$$

Call the S' satisfying (4.176) equipped with the images of various marked points

$$(p_{out}, p_1, \dots, p_k, z_1, \dots, z_d)$$

the *standard representative* of a given element, and call the associated r the *modular parameter*. Fix a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

$$\begin{gathered} f(s) \in [1, s]; \\ \text{for } r \gg 0,\ f(s) = s - \frac{1}{2}. \end{gathered} \tag{4.178}$$

Now, on the standard representative of S, we further require that the interior marked points satisfy an ordering condition

$$1 < |p_1| \leq |p_2| \leq \cdots \leq |p_k| \leq f(r). \tag{4.179}$$

where f is the function defined above.

The Deligne-Mumford compactifications of (4.175) (along with (4.188) and (4.204)) allow for the usual boundary bubbling, but allows interior marked points to go past one another, and land on the outgoing boundary component (one can think of it as the usual Deligne-Mumford compactification where certain strata are collapsed, or Deligne-Mumford compactify first, and then equip the compactification with the additional points $p_1, \dots, p_k$). As a result, ${}_k\overline{\check{\mathcal{C}}}_d$ has codimension 1 boundary covered by the images of the natural inclusions of the following spaces:

$${}^{i,i+1}_{k}\overline{\check{\mathcal{C}}}_d \tag{4.180}$$

$${}^{0}_{k}\overline{\check{\mathcal{C}}}_d \tag{4.181}$$

$${}_{k-1}\overline{\check{\mathcal{C}}}^{S^1}_d \tag{4.182}$$

$${}_{k}\overline{\check{\mathcal{C}}}_{d_1} \times \mathcal{R}_{d_2},\ d_1 + d_2 = d - 1 \tag{4.183}$$

$$\overline{\mathcal{H}}_r \times {}_{k-r}\overline{\check{\mathcal{R}}}^{-1}_d \tag{4.184}$$

$${}^{I}_{k}\overline{\mathcal{D}}_{d_1,d_2},\ d_1 + d_2 = d,\ I \subset \{2, \dots, k\} \tag{4.185}$$

Above, (4.180) describes the locus where $|p_i| = |p_{i+1}|$, (4.181) describes the locus where $|p_1| = 1$ lies on the outgoing boundary component, (4.182) is the region $|p_k| = f(r)$ is maximal (a space we examine in detail below), (4.183) is the usual boundary bubbling, and (4.184) and (4.185) describe the broken limits as the modular parameter r approaches ∞ and 1 respectively.

Associated to the stratum (4.180) where p_i and p_{i+1} have coincident magnitudes, there is a forgetful map

$$\check{\pi}_i : {}^{i,i+1}_{k}\overline{\check{\mathcal{C}}}_d \to {}_{k-1}\overline{\check{\mathcal{C}}}_d \tag{4.186}$$

which forgets the point p_{i+1}; this forgetful map has fiber S^1.

Similarly, on the stratum (4.181) when p_1 lies on the outgoing boundary segment, there is a forgetful-type map

$$\check{\pi}_0 : {}^{0}_{k}\overline{\check{\mathcal{C}}}_d \to {}_{k-1}\overline{\check{\mathcal{C}}}_d \tag{4.187}$$

which forgets p_1 and p, and markes a new point p' on the outgoing boundary segment which is now "opposite from" p_2.

Next,

$$ {}_k\mathcal{C}_d^{S^1} \tag{4.188} $$

is the same moduli space of annuli as above, except now there are $k+1$ interior auxiliary marked points satisfying the same ordering condition with the norm of p_{k+1} maximal:

$$ 1 < |p_1| \leq |p_2| \leq \cdots \leq |p_k| \leq |p_{k+1}| = f(r). \tag{4.189} $$

Recording the position of p_{k+1} on the circle of the standard representative gives a diffeomorphism

$$ {}_k\mathcal{C}_d^{S^1} \cong {}_k\check{\mathcal{C}}_d \times S^1. \tag{4.190} $$

which extends to a diffeomorphism of compactifications. Explicitly enumerating the boundary strata, we see that the Deligne-Mumford compactification of (4.188) is covered by the images of natural inclusions of the following spaces:

$$ {}_k^{i,i+1}\overline{\mathcal{C}}_d^{S^1} \tag{4.191} $$

$$ {}_k^{0}\overline{\mathcal{C}}_d^{S^1} \tag{4.192} $$

$$ {}_k^{k,k+1}\overline{\mathcal{C}}_d^{S^1} \tag{4.193} $$

$$ {}_k\overline{\mathcal{C}}_{d_1}^{S^1} \times \overline{\mathcal{R}}^{d_2},\ d_1 + d_2 = d-1 \tag{4.194} $$

$$ \overline{\mathcal{H}}_r \times {}_{k-r}\overline{\mathcal{R}}_d^{S^1} \tag{4.195} $$

$$ {}_k^{I}\overline{\mathcal{D}}_{d_1,d_2}^{S^1},\ d_1 + d_2 = d,\ I \subset \{2, \ldots, k\} \tag{4.196} $$

where (4.191) and (4.192) denote once more the locus where $|p_i| = |p_{i+1}|$ for $i < k$ and $|p_1| = 1$ respectively, (4.193) denotes the stratum where $|p_k| = |p_{k+1}| = f(r)$, and (4.194), (4.195), and (4.196) denote boundary bubbling and the broken limits as the modular paramter $r \to \infty$ and 1 respectively.

Note that as in the case of open-closed maps, there is a *free* $\mathbb{Z}_d$ action generated by

$$ \kappa : {}_k\overline{\mathcal{C}}_d^{S^1} \to {}_k\overline{\mathcal{C}}_d^{S^1} \tag{4.197} $$

which cyclically permutes the labels of the boundary punctures $z_1, \ldots, z_d$; explicitly κ changes the label z_i to $z_{i+1 \text{ mod d}}$. It is easy to see that for instance, κ sends ${}_k\mathcal{C}_d^{S^1_{i,i+1}}$ to ${}_k\mathcal{C}_d^{S^1_{i+1,i+2}}$.

Once more, we record some forgetful maps from the boundary strata that will be of interest. First, associated to the stratum (4.191) where p_i and p_{i+1} have coincident magnitudes, there is a forgetful map

$$ \pi_i^{S^1} : {}_k^{i,i+1}\overline{\mathcal{C}}_d^{S^1} \to {}_{k-1}\overline{\mathcal{C}}_d^{S^1} \tag{4.198} $$

which forgets the point p_{i+1}; this forgetful map has fiber S^1. Similarly, on the stratum (4.192) when p_1 lies on the outgoing boundary segment, there is a forgetful-type map

$$\pi_0^{S^1} : {}_k^0\overline{\mathcal{C}}_d^{S^1} \to {}_{k-1}\overline{\mathcal{C}}_d^{S^1} \tag{4.199}$$

which forgets p_1 and p, and markes a new point p' on the outgoing boundary segment which is now "opposite from" p_2. Finally, on the stratum (4.193), there is a forgetful map

$$\pi_k^{S^1} : {}_k^{k,k+1}\overline{\mathcal{C}}_d^{S^1} \to {}_{k-1}\overline{\mathcal{C}}_d^{S^1} \tag{4.200}$$

which forgets p_{k+1}.

For an element $A \in {}_k\mathcal{C}_d^{S^1}$, we say that p_{k+1} *points at a boundary point* z_i if, on the standard representative of A, we have that $\arg(p_{k+1}) = \arg(z_i)$. The locus where p_{k+1} points at z_i is denoted

$${}_k\mathcal{C}_d^{S^1_i} \tag{4.201}$$

Similarly, we say that p_{k+1} *points between* z_i *and* z_{i+1} *(modulo d)* if on the standard representative, $\arg(p_{k+1})$ falls between $\arg(z_i)$ and $\arg(z_{i+1})$ (with respect to the counterclockwise relative cyclic ordering). The locus where p_{k+1} *points between* z_i *and* z_{i+1} is denoted

$${}_k\mathcal{C}_d^{S^1_{i,i+1}}. \tag{4.202}$$

Although the moduli space (4.202) sits inside (4.188) as a submanifold, we note (as in the case of moduli spaces appearing in Section 3.3.3) that the compactification of ${}_k\mathcal{C}_d^{S^1_{i,i+1}}$ inside ${}_k\overline{\mathcal{C}}_d^{S^1}$ is *not a sub-manifold with corners.* A short explanation is this: at a stratum consisting of the main component attached to a disc bubble containing z_i and z_{i+1} has bubbled off also has fixed p_{k+1}, so should be codimension 2. However, at this stratum, there is no chart parametrized by $[0,\infty)^2$ that recovers all possible nearby loci (for instance, if the first gluing coordinate parametrized gluings of the bubble containing z_i and z_{i+1} to the main component, we observe that there is no 1-dimensional family of gluings parametrized by $(0,\lambda)$).

In order to obtain an operation parametrized by (a compactification of) ${}_k\mathcal{C}_d^{S^1_{i,i+1}}$, we work with a different model: for $k \geq 0$, define

$${}_k\hat{\mathcal{C}}_d^i \tag{4.203}$$

to be the moduli space of annuli S with:

- one marked point p_{out} on one boundary component, and d boundary (incoming) punctures and an auxiliary marked point on the other boundary component, labeled $z_1, \ldots, z_d, z_f$ respectively, with $z_1, \ldots, z_d$ labels in counter-clockwise order, and z_f between $z_{i \text{ mod d}}$ and $z_{i+1 \text{ mod } d}$; and
- k interior marked points $p_1, \ldots, p_k$, such that (S, p_{out}, p_1) satisfies the same conformal constraint (4.176) (where we now formally define $p_{k+1} = z_f$ to extend this constraint to the case $k = 0$).

As before, with respect to the standard representative, the interior marked points also satisfy an ordering condition (4.179).

We will use the following shorthand when z_f is between z_d and z_1:

$$ {}_k\hat{\mathcal{C}}_d := {}_k\hat{\mathcal{C}}^0_d = {}_k\hat{\mathcal{C}}^d_d. \tag{4.204}$$

On the open stratum, we note that there is a map, which we call the *auxiliary-rescaling map*

$$\pi^i_f : {}_k\hat{\mathcal{C}}^i_d \to {}_k\mathcal{C}_d^{S^1_{i,i+1}} \tag{4.205}$$

which can be described as follows: given the standard representative S of an element $[S]$ in ${}_k\hat{\mathcal{C}}^i_d$ with modular parameter r, there is a unique interior point p with $|p| = f(r)$ and $arg(p) = arg(z_f)$. $\pi_f([S])$ is the element of ${}_k\mathcal{C}_d^{S^1_{i,i+1}}$ with p_{k+1} equal to this point p and z_f deleted. Of course z_f is not actually forgotten, because it is determined by the position of p_{k+1}; in particular (4.205) is a diffeomorphism.

The (Deligne-Mumford-type) compactification is covered by the images of the natural inclusions of the following codimension 1 strata:

$$ {}^{j,j+1}_{k}\overline{\hat{\mathcal{C}}}_d \tag{4.206}$$

$$ {}^{0}_{k}\overline{\hat{\mathcal{C}}}_d \tag{4.207}$$

$$ {}_{k-1}\overline{\hat{\mathcal{C}}}^{i,S^1}_d \tag{4.208}$$

$$ {}_{k}\overline{\hat{\mathcal{C}}}_{d_1} \times \overline{\mathcal{R}}^{d_2},\ d_1 + d_2 = d - 1 \tag{4.209}$$

$$ {}_{k}\overline{\check{\mathcal{C}}}_{d_1} \times \overline{\mathcal{R}}^{d_2, f_i},\ d_1 + d_2 = d - 1 \tag{4.210}$$

$$ \overline{\mathcal{H}}_r \times {}_{k-r}\overline{\hat{\mathcal{R}}}^{1}_d \tag{4.211}$$

$$ {}^{I}_{k}\overline{\hat{\mathcal{D}}}_{d_1,d_2},\ d_1 + d_2 = d,\ I \subset \{2, \ldots, k\} \tag{4.212}$$

On the stratum (4.208) there is a forgetful map

$$\hat{\pi}_{boundary} : {}_{k-1}\overline{\hat{\mathcal{C}}}^{i,S^1}_d \to {}_{k-1}\overline{\hat{\mathcal{C}}}^{S^1}_d \tag{4.213}$$

which forgets the point z_f (as usual, collapsing any resulting unstable strata). On (4.206), there is another forgetful map

$$\hat{\pi}_j : {}^{j,j+1}_{k}\overline{\hat{\mathcal{C}}}_d \to {}_{k-1}\overline{\hat{\mathcal{C}}}_d \tag{4.214}$$

forgetting the position of p_{j+1}. (We remark that (4.213) can be thought of as the case $j = k$ of (4.214) via the labeling of z_f as "p_{k+1}.") Finally, on the stratum (4.207) when p_1 lies on the outgoing boundary segment, there is a forgetful-type map

$$\hat{\pi}_0 : {}^{0}_{k}\overline{\hat{\mathcal{C}}}^i_d \to {}_{k-1}\overline{\hat{\mathcal{C}}}^i_d \tag{4.215}$$

which forgets p_1 and p, and markes a new point p' on the outgoing boundary segment which is now "opposite from" p_2.

Definition 4.7. *A universal and conformally consistent choice of Floer data for the cyclic homotopy is an inductive choice,* $\mathbf{D}_{H^{cyc}}$ *for every* $k \geq 0$ *and* $d \geq 1$ *of Floer data for every representative* $S_0 \in {}_k\overline{\check{\mathcal{C}}}_d$*,* $S_1 \in {}_k\overline{\mathcal{C}}_d^{S^1}$*, varying smoothly over these moduli spaces, whose restriction to the boundary stratum is conformally equivalent to a product of Floer data coming from lower dimensional moduli spaces. Moreover, near the nodal boundary strata, with regards to gluing coordinates, this choice should agree to infinite order with the Floer data obtained by gluing. Moreover, this choice should satisfy the following additional requirements:*

For $S_0 \in {}_k\overline{\check{\mathcal{C}}}_d$,

At boundary strata of the form (4.180), the Floer datum for S_0 is conformally equivalent to one pulled back from ${}_{k-1}\overline{\check{\mathcal{C}}}_d$ via the forgetful map $\check{\pi}_i$; (4.216)

At boundary strata of the form (4.181), the Floer datum for S_0 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\check{\mathcal{C}}}_d$ via the forgetful map $\check{\pi}_0$. (4.217)

For $S_1 \in {}_k\overline{\mathcal{C}}_d^{S^1}$,

At boundary strata of the form (4.191), the Floer datum for S_1 is conformally equivalent to one pulled back from ${}_{k-1}\overline{\mathcal{C}}_d^{S^1}$ via the forgetful map $\pi_i^{S^1}$; (4.218)

At boundary strata of the form (4.192), the Floer datum for S_1 is conformally equivalent to the one pulled back from ${}_{k-1}\overline{\mathcal{C}}_d^{S^1}$ via the forgetful map $\pi_0^{S^1}$ (4.199); (4.219)

At boundary strata of the form (4.193), the Floer datum for S_1 is conformally equivalent to one pulled back from ${}_{k-1}\overline{\mathcal{C}}_d^{S^1}$ via the forgetful map $\pi_k^{S^1}$; (4.220)

On the codimension-1 locus ${}_k\overline{\mathcal{C}}_d^{S^1_i}$ where p_{k+1} points at z_i, the Floer datum should agree with the pullback of the existing Floer datum for ${}_k\overline{\check{\mathcal{C}}}_d$ under the map which applies κ^{d-i} and then forgets p_{k+1}; (4.221)

The Floer datum should be κ-equivariant, where κ is the map (4.197). (4.222)

For $S_2 \in {}_k\overline{\hat{\mathcal{C}}}{}_d^i$,

$$\text{At boundary strata of the form (4.206), the Floer datum for } S_1 \text{ is conformally equivalent to one pulled back from } {}_{k-1}\overline{\hat{\mathcal{C}}}_d \text{ via the forgetful map } \hat{\pi}_i \text{ (4.214).} \tag{4.223}$$

$$\text{At boundary strata of the form (4.207), the Floer datum on the main component of } S_2 \text{ is conformally equivalent to the one pulled back from } {}_{k-1}\overline{\hat{\mathcal{C}}}_d \text{ via the forgetful map (4.215);} \tag{4.224}$$

$$\text{At boundary strata of the form (4.208), the Floer datum on the main component of } S_2 \text{ is conformally equivalent to the one pulled back from } {}_{k-1}\overline{\check{\mathcal{C}}}{}_d^{S^1} \text{ via the forgetful map (4.213);} \tag{4.225}$$

$$\text{The choice of Floer datum on strata containing } \mathcal{R}^{d,f_i} \text{ components should be constant along fibers of the forgetful map } \mathcal{R}^{d,f_i} \to \mathcal{R}^{d-1}; \tag{4.226}$$

$$\text{The Floer datum on the main component } (S_2)_0 \text{ of } \pi_f(S_2) \text{ should coincide with the Floer datum chosen on } (S_2)_0 \in {}_k\mathcal{C}_d^{S^1_{i,i+1}} \subset {}_k\mathcal{C}_d^{S^1}; \tag{4.227}$$

Need to rewrite the "point at z_i" condition.

Lemma 4.12. *Universal and conformally consistent choices of Floer data for the cyclic homotopy exist.*

Proof. The choices of Floer data at each stage are constractible, so one simply verifies that, for a suitable chosen inductive order, the conditions satisfied by the Floer data at various stages do not contradict each other. □

Given a sequence of chords $\vec{x} = \{x_1, \ldots, x_d\}$, define

$$ {}_k\check{\mathcal{C}}_d(\vec{x}) \tag{4.228}$$

to be the space of maps

$$\{(S, u) : S \in {}_k\check{\mathcal{C}}_d, u : S \to T^*M\} \tag{4.229}$$

mapping the outgoing boundary component to the zero section M and the other boundary components to T_q^*M, with asymptotic condition $\psi^{w_k, S} x_k$ at the kth incoming end, solving Floer's equation using the chosen Floer datum for S: $(du - X_S \otimes \alpha_S)^{0,1} = 0$.

Similarly, we define moduli spaces

$$ {}_k\mathcal{C}_d^{S^1}(\vec{x}) \tag{4.230}$$

$$ {}_k\hat{\mathcal{C}}_d^i(\vec{x}), \ i \in \mathbb{Z}_d \tag{4.231}$$

as parametrized spaces of maps with asymptotics, boundary conditions, and equations exactly as above, though the source surfaces (and corresponding Floer data) are now parametrized by the abstract moduli spaces ${}_k\mathcal{C}_d^{S^1}$ and ${}_k\hat{\mathcal{C}}_d^i$ respectively. As before abbreviate

$$ {}_k\hat{\mathcal{C}}_d(\vec{x}) := {}_k\hat{\mathcal{C}}_d^0(\vec{x}) := {}_k\hat{\mathcal{C}}_d^d(\vec{x}); \tag{4.232} $$

to somewhat streamline the notation, we will state the next Lemma with respect to the space of maps (4.232).

Lemma 4.13. *For generic choices of Floer data $\mathbf{D}_{\mathcal{H}^{cyc}}$, the Gromov bordification of* (4.228), (4.230), *and* (4.232) *are compact manifolds of dimension*

$$ \dim {}_k\overline{\check{\mathcal{C}}}_d(\vec{x}) = d - \sum_{i=1}^d |x_i| + 2k. \tag{4.233} $$

$$ \dim {}_k\overline{\mathcal{C}}_d^{S^1}(\vec{x}) = d - \sum_{i=1}^d |x_i| + 2k + 1. \tag{4.234} $$

$$ \dim {}_k\overline{\hat{\mathcal{C}}}_d(\vec{x}) = d - \sum_{i=1}^d |x_i| + 2k + 1. \tag{4.235} $$

The boundary of of the bordification of (4.228) can be written as a union of codimension-1 strata which are images of the natural inclusions

$$ {}_k^{i,i+1}\overline{\check{\mathcal{C}}}_d(\vec{x}) \tag{4.236} $$

$$ {}_k^0\overline{\check{\mathcal{C}}}_d(\vec{x}) \tag{4.237} $$

$$ {}_{k-1}\overline{\mathcal{C}}_d^{S^1}(\vec{x}) \tag{4.238} $$

$$ {}_k\overline{\check{\mathcal{C}}}_{d_1}(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2}(x;\vec{x}^2),\ d_1 + d_2 - 1 = d,\ d_1, d_2 \geq 1 \tag{4.239} $$

$$ \overline{\mathcal{H}}_r(y_0) \times {}_{k-r}\overline{\check{\mathcal{R}}}_d^1(y_0;\vec{x}) \tag{4.240} $$

$$ {}_k^I\overline{\mathcal{D}}_{d_1,d_2}(\vec{x}'),\ d_1 + d_2 = d,\ I \subset \{2, \ldots, k\} \tag{4.241} $$

Similarly, the boundary of the bordification of (4.230) is covered by the natural inclusions of the following codimension 1 strata

$$ {}_k^{i,i+1}\overline{\mathcal{C}}_d^{S^1}(\vec{x}) \tag{4.242} $$

$$ {}_k^0\overline{\mathcal{C}}_d^{S^1}(\vec{x}) \tag{4.243} $$

$$ {}_k^{k,k+1}\overline{\mathcal{C}}_d^{S^1}(\vec{x}) \tag{4.244} $$

$$ {}_k\overline{\mathcal{C}}_{d_1}^{S^1}(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2}(x;\vec{x}^2),\ d_1 + d_2 - 1 = d,\ d_1, d_2 \geq 1 \tag{4.245} $$

$$ \overline{\mathcal{H}}_r(y_0) \times {}_{k-r}\overline{\mathcal{R}}_d^{S^1}(y_0;\vec{x}) \tag{4.246} $$

$$ {}_k^I\overline{\mathcal{D}}_{d_1,d_2}^{S^1}(\vec{x}),\ d_1 + d_2 = d,\ I \subset \{2, \ldots, k\} \tag{4.247} $$

Finally, the codimension 1 boundary of (4.232) is covered by the natural inclusions of the following codimension 1 strata:

$$ {}_k^{j,j+1}\overline{\hat{\mathcal{C}}}_d(\vec{x}) \tag{4.248} $$

$$ {}_k^{0}\overline{\hat{\mathcal{C}}}_d(\vec{x}) \tag{4.249} $$

$$ {}_k^{k,k+1}\overline{\hat{\mathcal{C}}}_d(\vec{x}) \tag{4.250} $$

$$ {}_k\overline{\hat{\mathcal{C}}}_{d_1}(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2}(\vec{x}^2),\ d_1 + d_2 = d - 1 \tag{4.251} $$

$$ {}_k\overline{\check{\mathcal{C}}}_{d_1}(\vec{x}^1) \times \overline{\mathcal{R}}^{d_2,f_i}(\vec{x}^2),\ d_1 + d_2 = d - 1 \tag{4.252} $$

$$ \overline{\mathcal{H}}_r(y_0) \times {}_{k-r}\overline{\hat{\mathcal{R}}}^1_d(y_0;\vec{x}) \tag{4.253} $$

$$ {}_k^I\overline{\hat{\mathcal{D}}}_{d_1,d_2}(\vec{x}),\ d_1 + d_2 = d,\ I \subset \{2, \dots, k\}, \tag{4.254} $$

The different asymptotics $\vec{x}^i$ or $\vec{x}'$ associated to some of these strata can be explained as follows:

- *Disk bubbling cases*: In the strata (4.239) and (4.245), x agrees with one of the elements in $\vec{x}^1$, and removing x and inserting $\vec{x}^2$ into $\vec{x}^1$ agrees with $\vec{x}$ up to cyclic reordering.
- $r \to 0$ *degeneration cases*: In the strata (4.241), (4.196), and (4.254), $\vec{x}'$ is a cyclic re-ordering of $\vec{x}$ which has x_d in the final d_2 elements if $k = 0$; otherwise x_d can be anywhere.

Fixing a universal and consistent choice of Floer data for the cyclic homotopy, we note that the manifolds (4.228), (4.230), (4.231) admit relative fundamental chains. Using the description of boundary strata above, we now define inductive choices of compatible (relative) fundamental chains for these moduli spaces. We pay particular attention to fundamental chains on various *degenerate* moduli spaces that arise as strata, meaning spaces that fiber over a lower dimensional space of maps.

Remark 4.4. *In a Morse-style approach to defining operations, degenerate moduli spaces, ones that non-trivially fiber over lower-dimensional spaces, have no zero-dimensional components and therefore induce a zero count. In our situation, operations are defined as evaluations of certain fundamental chains rather than counts of zero-dimensional components, but we would still like degenerate moduli spaces to induce the zero operation. In order to achieve this goal, we choose carefully adapted fundamental chains on these strata that whose evaluations are degenerate in the sense of singular chains, hence zero in the normalized singular chain complex.*

In the base case, note that by Lemmas 4.8 and 4.9 we have already made such a choice on ${}_0\overline{\check{\mathcal{C}}}_d(\vec{x}) \cong \overline{\mathcal{C}}_d(\vec{x})$ and ${}_0\overline{\hat{\mathcal{C}}}_d(\vec{x})$ compatible with chains on boundary strata. Inductively, let us suppose we have made such a compatible series of choices for ${}_{j-1}\overline{\check{\mathcal{C}}}_d(\vec{x})$, ${}_{j-1}\overline{\hat{\mathcal{C}}}^0_d(\vec{x})$, ${}_{j-2}\overline{\mathcal{C}}^{S^1}_d(\vec{x})$ and all of their boundary strata, for all d, all sequences $\vec{x}$ of size d, and all $j < k$ (our convention is that ${}_{-1}\overline{\mathcal{C}}^{S^1}_d(\vec{x})$ is empty).

Note that by the equivariance condition (4.222) composition with κ extends to a map on moduli spaces of maps, giving a homeomorphism of manifolds with corners

$$\kappa : {}_{k-1}\overline{\hat{\mathfrak{C}}}_d^i(\vec{x}) \xrightarrow{\sim} {}_{k-1}\overline{\hat{\mathfrak{C}}}_d^{i+1}(\vec{x}). \tag{4.255}$$

Therefore, for $i \neq 0$, we choose a fundamental chain on ${}_{k-1}\overline{\hat{\mathfrak{C}}}_d^i(\vec{x})$

$$[{}_{k-1}\overline{\hat{\mathfrak{C}}}_d^i(\vec{x})] := \kappa_*^i[{}_{k-1}\overline{\hat{\mathfrak{C}}}_d^0(t^i\vec{x})]. \tag{4.256}$$

Next, we note that by the equivariance condition (4.222) and the choice of Floer datum made (4.227) that the *auxiliary-rescaling maps* (4.205) extend to moduli spaces of maps:

$$\pi_f^i : {}_k\overline{\hat{\mathfrak{C}}}_d^i(\vec{x}) \to {}_k\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \tag{4.257}$$

We choose

$$[{}_{k-1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x})] := \sum_{i=0}^{d-1}(\pi_f^i)_*[{}_{k-1}\overline{\hat{\mathfrak{C}}}_d^i(\vec{x})] = \sum_{i=0}^{d-1}(\pi_f^i)_*\kappa_*^i[{}_{k-1}\overline{\hat{\mathfrak{C}}}_d^0(t^i\vec{x})]. \tag{4.258}$$

Lemma 4.14. *The choice* (4.258) *gives a valid fundamental chain with boundary equal to...*

Using the length parametrisation-induced position of the point p_{i+1} in the target (which eliminates ambiguity) with basepoint the case where p_{i+1} is aligned with p_i, we fix identifications

$${}_k^{i,i+1}\overline{\check{\mathfrak{C}}}_d(\vec{x}) \cong {}_{k-1}\overline{\check{\mathfrak{C}}}_d(\vec{x}) \times S^1 \tag{4.259}$$

$${}_k^{i,i+1}\overline{\hat{\mathfrak{C}}}_d(\vec{x}) \cong {}_{k-1}\overline{\hat{\mathfrak{C}}}_d(\vec{x}) \times S^1 \tag{4.260}$$

$${}_k^{i,i+1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \cong {}_{k-1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \times S^1 \tag{4.261}$$

Similarly, the position of p_1 in the target (with basepoint the case p_1 is aligned with p) gives identifications

$${}_k^0\overline{\check{\mathfrak{C}}}_d(\vec{x}) \xrightarrow{\sim} {}_{k-1}\overline{\check{\mathfrak{C}}}_d(\vec{x}) \times S^1 \tag{4.262}$$

$${}_k^0\overline{\hat{\mathfrak{C}}}_d(\vec{x}) \xrightarrow{\sim} {}_{k-1}\overline{\hat{\mathfrak{C}}}_d(\vec{x}) \times S^1 \tag{4.263}$$

$${}_k^0\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \xrightarrow{\sim} {}_{k-1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \times S^1 \tag{4.264}$$

Finally, the position of z_f (and p_{k+1} respectively) in the target give identifications

$${}_k^{k,k+1}\overline{\hat{\mathfrak{C}}}_d(\vec{x}) \xrightarrow{\sim} {}_k\overline{\check{\mathfrak{C}}}_d(\vec{x}) \times [0,1] \tag{4.265}$$

$${}_k^{k,k+1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \xrightarrow{\sim} {}_{k-1}\overline{\mathfrak{C}}_d^{S^1}(\vec{x}) \times S^1. \tag{4.266}$$

In all of the strata (4.259)-(4.266), the composition of this identification with projection to the first factor is the forgetful map, which extends to a map between moduli spaces of maps by conditions imposed on our Floer data by (4.216), (4.223), (4.218), (4.217), (4.224), (4.219), (4.225), and (4.192).

With respect to the product decompositions we have fixed for these strata, choose a fundamental chain that is the (Eilenberg-Zilber) product of the previously chosen fundamental chain of the first factor and a standard fundamental chain of S^1 (or $[0,1]$).

Having fixed all of these fundamental chains, and fundamental chains for ${}_k^I\overline{\check{\mathcal{D}}}_{d_1,d_2}(\vec{x})$, ${}_k^I\overline{\mathcal{D}}^{S^1}_{d_1,d_2}(\vec{x})$, ${}_k^I\overline{\hat{\mathcal{D}}}_{d_1,d_2}(\vec{x})$ from Section 4.4.5, we construct fundamental chains for ${}_k\overline{\check{\mathcal{C}}}_d(\vec{x})$ and ${}_k\overline{\hat{\mathcal{C}}}_d(\vec{x})$, compatible with existing choices, thereby verifying:

Lemma 4.15. *For each d and each k, and all sequences $\vec{x}$ of size d, there exists an inductive choice of (relative) fundamental chains for ${}_k\overline{\check{\mathcal{C}}}_d(\vec{x})$, ${}_k\overline{\check{\mathcal{C}}}_d$, and ${}_k\overline{\mathcal{C}}_d^{S^1}$ satisfying*

$$\begin{aligned}\partial[{}_k\overline{\check{\mathcal{C}}}_d(\vec{x})] = &\sum_{r,y}(-1)^{(d-1)(n-|y|)}\left([\overline{\mathcal{H}}_r(y)] \times [{}_{k-r}\overline{\check{\mathcal{R}}}^1_d(y,\vec{x})]\right) + (-1)^? [{}_k^0\overline{\check{\mathcal{C}}}_d(\vec{x})] + (-1)^? [{}_{k-1}\overline{\mathcal{C}}_d^{S^1}(\vec{x})] \\ &\pm \sum_{i=1}^{k-1} [{}_k^{i,i+1}\overline{\check{\mathcal{C}}}_d(\vec{x})] + \sum_{\vec{x}^1,\vec{x}^2}\left([{}_k\overline{\check{\mathcal{C}}}_{d_1}(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2}(\vec{x}^2)]\right) + \sum_{d_1,d_2,I,\vec{x}'} (-1)^{\frac{n(n+1)}{2}+\cdots} [{}_k^I\overline{\mathcal{D}}_{d_1,d_2}(\vec{x}')]\end{aligned} \tag{4.267}$$

$$\begin{aligned}\partial[{}_k\overline{\hat{\mathcal{C}}}_d(\vec{x})] = &\sum_{r,y}(-1)^{(d-1)(n-|y|)}\left([\overline{\mathcal{H}}_r(y)] \times [{}_{k-r}\overline{\hat{\mathcal{R}}}^1_d(y,\vec{x})]\right) + (-1)^? [{}_k^0\overline{\hat{\mathcal{C}}}_d(\vec{x})] + (-1)^? [{}_k^{k,k+1}\overline{\hat{\mathcal{C}}}_d(\vec{x})] \\ &\pm \sum_{i=1}^{k-1} [{}_k^{i,i+1}\overline{\hat{\mathcal{C}}}_d(\vec{x})] + \sum_{\vec{x}^1,\vec{x}^2}\left([{}_k\overline{\hat{\mathcal{C}}}_{d_1}(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2}(\vec{x}^2)]\right) + \sum_{\vec{x}^1,\vec{x}^2}\left([{}_k\overline{\check{\mathcal{C}}}_{d_1}(\vec{x}^1)] \times [\overline{\mathcal{R}}^{d_2,f_i}(\vec{x}^2)]\right) \\ &+ \sum_{d_1,d_2,I,\vec{x}'} (-1)^{\frac{n(n+1)}{2}+\cdots} [{}_k^I\overline{\hat{\mathcal{D}}}_{d_1,d_2}(\vec{x}')].\end{aligned} \tag{4.268}$$

A number of these signs do not matter at all, given that the relevant operation will turn out to be zero.

with respect to existing choices of fundamental chains, such that on the boundary strata (4.259)-(4.266)*, the fundamental chain appearing on the RHS is the prescribed product chain, and the fundamental chain on ${}_k\overline{\mathcal{C}}_d^{S^1}(\vec{x})$ appearing on the RHS is the one described in* (4.258)*.*

Proof. The only thing remaining to finish the induction is to check that the chains appearing on the RHS of CITE are closed. This in turn follows from the compatibility of these chains with choices made on lower boundary strata. ☐

More to insert

The reason these product chains need to be compatible with lower dimensional strata is to verify that the sum of boundary chains is closed, so that there exists a fundamental chain on the top stratum with boundary these factors.

$$\mathrm{ev} : {}_k\overline{\check{\mathcal{C}}}_d(\vec{x}) \to \mathcal{L}M \tag{4.269}$$

$$\mathrm{ev} : {}_k\overline{\hat{\mathcal{C}}}_d(\vec{x}) \to \mathcal{L}M \tag{4.270}$$

This induces a map ev on any boundary stratum of maps, for instance ${}_{k-1}\overline{\mathcal{C}}_d^{S^1}(\vec{x})$. We as usual write ev_* for the induced map on (relative) singular chains.

Lemma 4.16. *The evaluation map* ev_* *applied to any chosen fundamental chain of the strata* (4.259)-(4.261) *is zero.*

Proof. Without loss of generality, we consider the case (4.259). With respect to the identification in (4.259), we note that the evaluation factors as ${}_{k-1}\overline{\check{\mathcal{C}}}_d(\vec{x}) \times S^1 \to {}_{k-1}\overline{\check{\mathcal{C}}}_d(\vec{x}) \to \mathcal{L}M$, where the first map is projection to the first factor. By Lemma CITE $\square$

cite fact about singular chains and projection to a factor

Lemma 4.17. *Let Y denote any of the three moduli spaces of maps on the left hand side of* (4.262) - (4.264)*, with X the corresponding reduced space from the right hand side such that $Y \cong X \times S^1$. For our choices of fundamental chains for these strata,* $\mathrm{ev}_*([Y]) = \Delta(\mathrm{ev}_*([X]))$*, where Δ is the BV operator on $\tilde{C}_*\mathcal{L}M$.*

Proof. In all of the aforementioned cases, we note that the evaluation factors as

$$ev_Y : X \times S^1 \xrightarrow{ev_X \times id} \mathcal{L}\mathcal{M} \times S^1 \xrightarrow{m} \mathcal{L}M, \tag{4.271}$$

where m is the usual multiplication on (Moore) free loops. The Lemma is an immediate consequence. $\square$

Lemma 4.18. *The evaluation map applied to any chosen fundamental chain of the strata* (4.265) -(4.266) *is zero.*

Proof. The proof is identical to Lemma 4.16. $\square$

Lemma 4.19. *Something about the Connes' B operator—basically characterizing the operation associated to ${}_k\mathcal{C}_d^{S^1}$ applied to a Hochschild chain x as $\hat{\mathcal{H}}(B^{nu}(x))$.*

Most detailed argument left to insert.

For $k \geq 0$, define

$$\check{\mathcal{H}}_k : \check{\mathrm{CH}}_*(CW_b^*(T_q^*M)) \to \tilde{C}_{-(*-2k)}(\mathcal{L}M) \tag{4.272}$$

as the linear extension of

$$x_d \otimes \cdots \otimes x_1 \mapsto (-1)^? \mathrm{ev}_*([{}_k\overline{\check{\mathcal{C}}}_d^{-}(\vec{x})]) \tag{4.273}$$

Similarly, define

$$\hat{\mathcal{H}}_k : \hat{\mathrm{CH}}_*(CW_b^*(T_q^*M)) \to \tilde{C}_{-(*-2k)}(\mathcal{L}M) \tag{4.274}$$

$$x_d \otimes \cdots \otimes x_1 \mapsto (-1)^? \mathrm{ev}_*([{}_k\overline{\hat{\mathcal{C}}}_d^{-}(\vec{x})]) \tag{4.275}$$

Proposition 4.9. *For each $k \geq 1$, the collection of operations $\{\check{\mathcal{H}}_j\}$, $\{\hat{\mathcal{H}}_j\}$ satisfy the following equations:*

$$\check{\mathcal{H}}_k \circ b + \hat{\mathcal{H}}_{k-1} B^{nu} - d\check{\mathcal{H}}_k - \Delta\check{\mathcal{H}}_{k-1} = \sum_{j=0}^{k} \mathcal{CL}_j \circ \check{\mathcal{OC}}_{k-j} \tag{4.276}$$

$$\hat{\mathcal{H}}_k \circ b' + \check{\mathcal{H}}_k \circ (1-t) - d\hat{\mathcal{H}}_k - \Delta\hat{\mathcal{H}}_{k-1} = \sum_{j=0}^{k} \mathcal{CL}_j \circ \hat{\mathcal{OC}}_{k-j} \tag{4.277}$$

(Remark: there is a different equation satisfied at $k = 0$, with an extra term involving $\mathcal{A} \circ \mathbf{P}$)

Elaborate on remark

Proof. We note that $d\check{\mathcal{H}}_k(\vec{x}) = \mathrm{ev}_*(\partial[(-1)^? \mathrm{ev}_*([{}_k\overline{\check{\mathcal{C}}}_d^{-}(\vec{x})])])$. Now, we apply the inductive formula (4.267), noting that

- by Lemma 4.16, $\mathrm{ev}_*([{}_k^{i,i+1}\overline{\check{\mathcal{C}}}_d(\vec{x})]) = 0$;
- by Lemma 4.17, $\mathrm{ev}_*([{}_k^{0}\overline{\check{\mathcal{C}}}_d(\vec{x})]) = \Delta \mathrm{ev}_*([{}_{k-1}\overline{\check{\mathcal{C}}}_d(\vec{x})]) = \Delta\check{\mathcal{H}}_{k-1}$;
- by Lemma 4.19, $\mathrm{ev}_*([{}_{k-1}\overline{\check{\mathcal{C}}}_d^{S^1}(\vec{x})]) = \hat{\mathcal{H}}_{k-1}(B^{nu}\vec{x})$; and
- by Corollary 4.2, $\mathrm{ev}_*([{}_k^{I}\overline{\mathcal{D}}_{d_1,d_2}(\vec{x}')]) = 0$ for $k \geq 1$;
- ...

This implies the desired formula (4.276). The case of (4.277) is the same, noting that $d\hat{\mathcal{H}}_k(\vec{x}) = \mathrm{ev}_*(\partial[(-1)^? \mathrm{ev}_*([{}_k\overline{\hat{\mathcal{C}}}_d^{-}(\vec{x})])])$ and applying the inductive formula (4.268). The only new phenomena are:

- The fundamental chain of the stratum ${}_{k-1}\overline{\hat{\mathcal{C}}}_d^{S^1}(\vec{x})$ now evaluates to zero, because the Floer data on moduli space is independent of the position of the marked point z_f, and we have chosen fundamental chains compatibly with this degeneracy by Lemma 4.18.
- The presence of the forgetful map means that there are b' and $(1-t)$-type terms instead of b terms. *elaborate.*

Elaborate.

□

Proof of Proposition 4.8. Combining all of the operations in Proposition 4.9, set

$$\widetilde{\mathcal{H}} = \sum_{j=0}^{\infty} (\check{\mathcal{H}}_j \oplus \hat{\mathcal{H}}_j)u^j : \Big(\check{\mathrm{C}}\mathrm{H}_*(CW_b^*(T_q^*M)) \oplus \hat{\mathrm{C}}\mathrm{H}_*(CW_b^*(T_q^*M))[[u]], b^{nu} + uB^{nu}\Big) \longrightarrow \Big(\tilde{C}_{-*}(\mathcal{L}M)[[u]], d + u\Delta\Big). \tag{4.278}$$

Note that the source of $\widetilde{\mathcal{H}}$ is the negative cyclic chain complex of $CW_b^*(T_q^*M)$, and that Proposition 4.9 gives an order-by-order in u verification that $\widetilde{\mathcal{H}}$ satisfies

$$\widetilde{\mathcal{H}} \circ (b^{nu} + uB^{nu}) - (d + u\Delta) \circ \widetilde{\mathcal{H}} = (-1)^{\frac{n(n+1)}{2}} \widetilde{\mathcal{CL}} \circ \widetilde{\mathcal{OC}} - \widetilde{\mathcal{A}} \circ \widetilde{\mathbf{F}}_*. \tag{4.279}$$

(We note that the term $\widetilde{\mathcal{A}} \circ \widetilde{\mathbf{F}}_*$ only contributes to the $u = 0$ coefficient of the expansion, and for instance does not appear at higher coefficients of u.) This establishes the Proposition. □

4.5 The bar and cyclic bar construction, and a proof of Lemma 13

Let G be a topological group. Consider the simplicial inclusion

$$\iota : BG \hookrightarrow N^{cy}(G) \tag{4.280}$$

which on the level of k-simplices is defined by

$$g_1, \cdots, g_k \longrightarrow (g_1 \cdots g_k)^{-1}, g_1, \cdots, g_k \tag{4.281}$$

This defines a simplicial isomorphism of BG with the cyclic subspace of $N^{cy}(G)$ whose k-simplices those $(k+1)$-tuples $h_0, \cdots, h_k \in G^{k+1}$ satisfying $\prod_{i=0}^{k} h_i = 1$. This, in particular gives the geometric realization $|BG|$ an S^1-action and makes the induced map $\iota : |BG| \to |N^{cy}(G)|$ S^1 equivariant. The goal of this section is to show that this action on $|BG|$ is in an appropriate sense homotopically trivial, and that ι can, up to homotopy, be viewed as the inclusion of the constant loops $BG \hookrightarrow LBG$. (See Theorem 18 below.)

The main step in making these ideas precise and proving the corresponding statements is the following.

Theorem 15. *There is an S^1-equivariant map*

$$\Psi_G : |BG| \to L(EG)/G$$

which is a weak S^1-equivariant equivalence, and which makes the following diagram of S^1-equivariant maps commute:

$$\begin{tikzcd} {|BG|} \arrow[r, "\psi", "\simeq"'] \arrow[d, "\iota"'] & L(|EG|)/G \arrow[d, "p"] \\ N^{cy}(G) \arrow[r, "\simeq", "\mathcal{G}"'] & L(|BG|). \end{tikzcd}$$

Note: 1. By a "weak S^1-equivariant equivalence" between two S^1-spaces X and Y we mean a "zig-zag" of S^1-maps, $X \to Y_1 \leftarrow Y_2 \to \cdots \leftarrow Y$ where each of the maps in the zig-zag are S^1-equivariant, and are ordinary weak homotopy equivalences.

2. In the statement of this theorem the G-action on the loop space $L(|EG|)$ is pointwise, and is therefore free. The map $p : L(|EG|)/G \to L(|BG|) = L(|EG|/G)$ is the obvious quotient map. $\mathcal{G} : N^{cy}(G) \to L(|BG|)$ is the Goodwillie equivalence.

Proof. In the statement of the theorem and in the proof, EG is the simplicial space $B(G, G, pt)$. $N^{cy}(G)$ is the usual cyclic bar construction. We begin with the observation that EG can be viewed as the nerve of the topological category $\mathcal{E}_G$, whose space of objects is G, and there is a unique morphism from g_1 to g_2. This morphism can be thought of as $g_1^{-1}g_2$. When viewed this way, composition of morphisms is group multiplication. The entire space of morphisms is thus homeomorphic to $G_1 \times G_2$. The k-simplices of the nerve of $\mathcal{E}_G$ is given by $(k+1)$-tuples $(g, h_1, \cdots, h_k)$ where g represents the initial object, h_1 is the unique morphism from g to gh_1, h_2 is the unique morphism from gh_1 to gh_1h_2, and so on. This identification of the k-simplicies of the nerve of $\mathcal{E}_G$ with G^{k+1} defines an isomorphism of the nerve of $\mathcal{E}_g$ with $B(G, G, pt)$.

Notice that given any topological category $\mathcal{C}$, one can also consider the *cyclic nerve* $N^{cy}(\mathcal{C})$. This is a cyclic space, whose k-simplices are made up of $(k+1)$-tuples of cyclically composable morphisms,

$$c_0 \xrightarrow{\mu_0} c_1 \xrightarrow{\mu_1} c_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} c_k \xrightarrow{\mu_k} c_0$$

Here the c_j's are objects in $\mathcal{C}$, and the μ_j's are morphisms. The k-simplices are topologized so as to be subspaces of iterated fiber products with respect to the source and target maps, which are assumed to be fibrations. The face and degeneracy maps are given by compositions and the insertions of identity morphisms respectively, in the usual way.

For a topological group G, let $\mathcal{B}_G$ denote the topological category with one object, whose endomorphisms are elements of G. Then clearly, $N^{cy}(\mathcal{B}_G) = N^{cy}(G)$, the usual cyclic bar construction of a group.

Now consider the cyclic bar construction on the category $\mathcal{E}_G$. The k-simplices in $N^{cy}(\mathcal{E}_G)$ consist of $(k+1)$-tuples of cyclically composable morphisms,

$$c_0 \xrightarrow{\mu_0} c_1 \xrightarrow{\mu_1} c_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} c_k \xrightarrow{\mu_k} c_0$$

where the object c_0 is an arbitrary element of the group, as are the morphisms $\mu_0, \cdots, \mu_{k_1}$. The other objects would be then given by the products of the group elements

$$c_j = c_0\mu_0 \cdots \mu_{j-1}$$

and the final morphism is given by $\mu_k = (\mu_1 \cdots \mu_k)^{-1}$.

Notice that the group G acts freely and simplicially on $N^{cy}(\mathcal{E}_G)$. The action on the level of k-simplices, is given by

$$g \cdot \left(c_0 \xrightarrow{\mu_0} c_1 \xrightarrow{\mu_1} c_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} c_k \xrightarrow{\mu_k} c_0\right) = gc_0 \xrightarrow{\mu_0} gc_1 \xrightarrow{\mu_1} gc_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} gc_k \xrightarrow{\mu_k} gc_0.$$

The following is now immediate from the cyclic description of BG given in 4.280.

Lemma 16. *There is an isomorphism of cyclic spaces,*

$$\Phi_G : BG \xrightarrow{\cong} N^{cy}(\mathcal{E}_G)/G.$$

Now for any topological category $\mathcal{C}$, consider the S^1 equivariant map to the free loop space,

$$\mathcal{G}_{\mathcal{C}} : |N^{cy}(\mathcal{C})| \to L(|B\mathcal{C}|)$$

defined as the composition

$$\mathcal{G}_{\mathcal{C}} : |N^{cy}(\mathcal{C})| \xrightarrow{\alpha} L(|N^{cy}(\mathcal{C})|) \xrightarrow{Lp} L(|B\mathcal{C}|) \tag{4.282}$$

where α is the adjoint of the S^1-action induced by the cyclic structure, $S^1 \times |N^{cy}(\mathcal{C})| \to |N^{cy}(\mathcal{C})|$, and Lp is induced by the usual simplicial projection map $N^{cy}(\mathcal{C}) \to B\mathcal{C}$. Notice that in the case of $\mathcal{C} = \mathcal{B}_G$, then $\mathcal{G}_{\mathcal{C}}$ is Goodwillie's weak S^1-equivalence,

$$\mathcal{G} : N^{cy}(G) \xrightarrow{\simeq} L(|BG|).$$

Lemma 17. *For $\mathcal{C} = \mathcal{E}_G$, $\mathcal{G}_{\mathcal{E}_G} : |N^{cy}(\mathcal{E}_G)| \to L(|EG|)$ defines a weak S^1-equivalence*

$$\mathcal{G}_{\mathcal{E}_G} : |N^{cy}(\mathcal{E}_G)|/G \xrightarrow{\simeq} L(|EG|)/G.$$

Proof. First notice that the simplicial projection map $p : N^{cy}(\mathcal{E}_G) \to B(\mathcal{E}_G) = B(G, G, pt) = EG$ is, in this case, an isomorphism of simplicial spaces. This is because the projection map is given on the level of k-simplices by

$$p\left(c_0 \xrightarrow{\mu_0} c_1 \xrightarrow{\mu_1} c_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} c_k \xrightarrow{\mu_k} c_0\right) = \left(c_0 \xrightarrow{\mu_0} c_1 \xrightarrow{\mu_1} c_2 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k-1}} c_k\right)$$

Since, as seen above, in the case of $\mathcal{E}_G$ $\mu_k = (\mu_1 \cdots \mu_{k-1})^{-1}$, the space of k-simplices of both $N^{cy}(\mathcal{E}_G)$ and of $B(\mathcal{E}_G)$ are homemorphic to G^{k+1}, and the restriction of p to the k-simplices is a homeomorphism.

Now in the category $\mathcal{E}_G$ each object is initial, so its classifying space $|B\mathcal{E}_G|$ is contractible. Hence $|N^{cy}(\mathcal{E}_G)|$ is contractible. Furthermore, since $\mathcal{G}_{\mathcal{E}_G} : |N^{cy}(\mathcal{E}_G)| \to L(|B(\mathcal{E}_G|) = L(|EG|)$ is a G-equivariant between two contractible spaces with free G-actions. Furthermore these G-actions commute with the S^1-actions. Hence the induced map on the quotient spaces

$$\mathcal{G}_{\mathcal{E}_G} : |N^{cy}(\mathcal{E}_G)|/G \to L(|EG|)/G.$$

is a weak S^1 equivalence. □

In order to complete the proof of Theorem 15 we let

$$\Psi_G = \mathcal{G}_{\mathcal{E}_G} \circ \Phi_G : |BG| \to |N^{cy}(\mathcal{E}_G)|/G \to L(|EG|)/G.$$

The fact that it is a weak S^1-equivalence follows from Lemmas 16 and 17. The fact that the diagram in the statement of the theorem commutes follows from the definitions of the maps involved. □

We now make one more observation.

Theorem 18. *There is a weak S^1-equivalence between $|BG|$ with the action defined by the cyclic structure given in 4.280 with $|BG|$ with the trivial action. Furthermore, with respect to this equivalence, the composition*

$$\iota : |BG| \xrightarrow{\iota} |N^{cy}(G)| \xrightarrow{\mathcal{G}_G} L(|BG|)$$

is homotopic to the inclusion of the constant loops.

Proof. The inclusion of the space of constant loops defines a map

$$\gamma : |BG| = |EG|/G \hookrightarrow L(|EG|)/G$$

which identifies $|BG|$ with the fixed points of the S^1-action of $L(|EG|)/G$. Thus the zig-zag

$$|BG| \xrightarrow{\Psi_G} L(|EG|)/G \xleftarrow{\gamma} |BG|$$

defines a weak S^1 equivalence with the required properties. $\square$

4.6 Comparing the Abouzaid and Goodwillie equivalences

We study in some detail, the Goodwillie's equivalence $\mathcal{G} : N^{cy}(\Omega M) \to LM$ on the level of singular-chains. This will allow us to compare this map with that described by Abouzaid.

In the discussion below ΩX will denote the *Moore* loops of a space X with base point $x_0 \in X$. It consists of pairs (γ, ℓ), where $\ell \geq 0$, and $\gamma : [0, \ell] \to X$ is a path such that $\gamma(0) = \gamma(\ell) = x_0 \in X$. ΩX is a monoid under the usual concatenation pairing $(\gamma_1, \ell_1) \times (\gamma_2, \ell_2) \to (\gamma_1 * \gamma_2, \ell_1 + \ell_2)$. By abuse of notation we often leave out the lengths ℓ.

We begin by recalling the map $\mathcal{G}$. $N^{cy}(\Omega M)$ is the geometric realization of a cyclic space, and hence carries with it a well determined S^1-action,

$$\alpha : S^1 \times N^{cy}(\Omega M) \to N^{cy}(\Omega M).$$

Now let $\pi : N^{cy}(\Omega M) \to B(\Omega M)$ be the natural projection of the cyclic bar construction onto the ordinary bar construction. Then $\mathcal{G} : N^{cy}(\Omega M) \to LM$ is defined to be the adjoint of the composition

$$S^1 \times N^{cy}(\Omega M) \xrightarrow{\alpha} N^{cy}(\Omega M) \xrightarrow{\pi} B(\Omega M) \xrightarrow{\simeq} M.$$

This is clearly an S^1-equivariant map. The last equivalence in this composition is the usual equivalence $f : B(\Omega M) \to M$ defined as follows. Recall

$$B(\Omega M) = \bigcup_{k=1}^{\infty} (\Omega M)^k \times \Delta^k / \sim$$

where Δ^k is the standard k-simplex parameterized by nondecreasing sequences of real numbers, $0 \leq s_1 \leq \cdots \leq s_k \leq 1$. The relations used in gluing this union are given by the usual face and degeneracy map relations. Then f is defined to be the union of the maps

$$f_k : (\Omega M)^k \times \Delta^k \to LM$$
$$((\gamma_1, \ell_1), \cdots, (\gamma_k, \ell_k)) \times (s_1, \cdots, s_k) \to (\gamma_1 * \cdots * \gamma_k)(p_k(\vec{\ell}, \vec{s})) \in M.$$

Here we are using vector notation $\vec{\ell}$ to denote the k-tuple $(\ell_1, \cdots, \ell_k)$ and $\vec{s}$ to denote the k-tuple $(s_1, \cdots, s_k) \in \Delta^k$. The function p_k is given by

$$p_k(\vec{\ell}, \vec{s}) = \sum_{i+1}^{k} \ell_i (1 - s_i). \tag{4.283}$$

Notice that for a fixed k-tuple $\vec{\ell}$, then as $\vec{s}$ varies in Δ^k, $p_k((\vec{\ell},\vec{s})$ takes values in $[0,\ell]$ where $\ell = \ell_1 + \cdots + \ell_k$.

We can now give a more explicit definition of the Goodwillie equivalence $\mathcal{G} : N^{cy}(\Omega M) \to LM$. Again we describe $\mathcal{G}$ as the union of the maps

$$\mathcal{G}_k : (\Omega M)^{k+1} \times \Delta^k \to LM$$

defined by

$$\mathcal{G}_k(((\gamma_0,\ell_0),\cdots,(\gamma_k,\ell_k)),(s_1,\cdots,s_k)) = (\gamma_0 * \cdots * \gamma_k)_{p_k} \tag{4.284}$$

where $(\gamma_0 * \cdots * \gamma_k)_{p_k}$ is the based loop $\gamma_0 * \cdots * \gamma_k$ rotated so that its starting point is at $p_k((\ell_1,\cdots,\ell_k),(s_1,\cdots s_k))$. That is,

$$(\gamma_0 * \cdots * \gamma_k)_{p_k}(t) = (\gamma_0 * \cdots * \gamma_k)(t + \sum_{i=1}^{k} \ell_i(1-s_i)).$$

Now by applying singular chains and the Eilenberg-Zilber construction, there is a well-known equivalence of chain complexes

$$\mathcal{EZ} : CH_*(C_*(\Omega M) \xrightarrow{\simeq} C_*(N^{cy}(\Omega M)). \tag{4.285}$$

Using this and formula 4.284 we have an explicit description of Goodwillie's equivalence of the Hochschild chains to the chains of the free loop space,

$$\mathcal{G}_* : CH_*(C_*(\Omega M) \xrightarrow{\simeq} C_*(LM).$$

We also observe that while we are focused on applying this formula to manifolds, Goodwillie's equivalence applies to any space, X,

$$\mathcal{G}_* : CH_*(C_*(\Omega X)) \xrightarrow{\simeq} C_*(LX). \tag{4.286}$$

This is an equivalence of A_∞ - $C_*(S^1)$-modules. Now as studied above, a singular chain version of Abouzaid's map defines an equivalence of A_∞-$C_*(S^1)$-modules,

$$\mathcal{A}_* : CH_*(C_*(\Omega X)) \xrightarrow{\simeq} C_*(LX)/Q \tag{4.287}$$

where $C_*(LX)/Q$ is a certain quotient of the singular chains $C_*(LX)$ by an acyclic $C_*(S^1)$- subcomplex, so that the quotient map $C_*(LX) \xrightarrow{\pi} C_*(LX)/Q$ is an equivalence of A_∞-$C_*(S^1)$-modules.

Our goal is to relate the equivalences $\mathcal{G}_* : CH_*(C_*(\Omega X) \xrightarrow{\simeq} C_*(LX) \xrightarrow{\pi} C_*(LX)/Q$ with $\mathcal{A}_*$ on the level of homotopy fixed points. To be more precise, recall from the last subsection that there is a map, well-defined up to chain homotopy,

$$\iota : C_*(B(\Omega X)) \longrightarrow C_*(N^{cy}(\Omega X))^{hS^1} \xrightarrow[\simeq]{\mathcal{EZ}^{-1}} CH_*(C_*(\Omega X))^{hS^1} \quad = \quad CC_*^-(C_*(\Omega X). \tag{4.288}$$

Combining with the equivalence above, $p_* : C_*(B(\Omega X)) \xrightarrow{\simeq} C_*(X)$, we get two maps, well defined up to chain homotopy, on the level of homotopy fixed points:

$$\mathcal{G}_*^B : C_*(X) \xrightarrow{\iota} CH_*(C_*(\Omega X))^{hS^1} \xrightarrow{\mathcal{G}_*} C_*(LX)^{hS^1} \to (C_*(LX)/Q)^{hS^1} \quad \text{and} \tag{4.289}$$

$$\mathcal{A}_*^B : C_*(X) \xrightarrow{\iota} CH_*(C_*(\Omega X))^{hS^1} \xrightarrow{\mathcal{G}_*} (C_*(LX)/Q)^{hS^1} \tag{4.290}$$

From now on, when it will not cause confusion, we will suppress the Q from the notation for the quotient, and write the targets of these maps as $C_*(LX)^{hS^1}$.

We now prove the following, which as proved above, implies the main Theorem 8.

Theorem 19. *For any connected space X of the homotopy type of a CW-complex, the maps*

$$\mathcal{G}_*^B \quad \textit{and} \quad \mathcal{A}_*^B : C_*(X) \to C_*(LX)^{hS^1}$$

are chain homotopic.

Note: As proved above, the map $\mathcal{G}_*^B : C_*(X) \to C_*(LX)^{hS^1}$ is homotopic to the map induced by the inclusion $X \hookrightarrow LX$ as the constant loops. This theorem therefore shows the Abouzaid map $\mathcal{A}_*^B$ is homotopic to this as well.

Proof. For a simplicial space $Y_\bullet$, let $|Y|_q$ denote the q^{th} skeletal filtration of the geometric realization,

$$|Y|_q = \bigcup_{i=0}^{q} \Delta^q \times Y_q / \sim$$

For a monoid or group G we write $B_q(G)$ and $N_q^{cy}(G)$ to refer to $|BG|_q$ and $|N^{cy}(G)|_q$ respectively.

We first observe the following.

Lemma 20. *When restricted to the one skeleton, the Goodwillie and Abouzaid maps*

$$\mathcal{G}_* \quad \textit{and} \quad \mathcal{A}_* : C_*(N_1^{cy}(\Omega X)) \hookrightarrow CH_*(C_*(\Omega X)) \to C_*(LX)$$

are equal.

Proof. This is immediate from the definitions of the Goodwillie map (4.284) and of the Abouzaid map (reference?) ☐

Recall that the one-skeleton of the Bar construction of any group G is the suspension,

$$B_1(G) = \Sigma(G_+).$$

So in particular we can consider the composition

$$e : \Sigma(\Omega X_+) = B_1(\Omega X) \hookrightarrow B(\Omega X) \xrightarrow[\simeq]{p} X$$

By examining the formula for the equivalence p we see that e is the evaluation map,

$$e(t, \alpha) = \alpha(t) \in X.$$

Lemma 21. *The compositions*

$$\mathcal{G}_*^B \circ e \quad and \quad \mathcal{A}_*^B \circ e : C_*(\Sigma\Omega X) \to C_*(X) \to C_*(LX)^{hS^1}$$

are canonically homotopic.

Proof. From Lemma 20 we know that

$$\mathcal{G}_* \quad and \quad \mathcal{A}_* : C_*(N_1^{cy}(\Omega X)) \hookrightarrow CH_*(C_*(\Omega X)) \to C_*(LX)$$

are equal. This implies that when restricted to the one-skeleton,

$$\mathcal{G}_*^B \quad and \quad \mathcal{A}_*^B : C_*(B_1(\Omega X)) \hookrightarrow C_*(B(\Omega X)) \xrightarrow{\iota} CH_*(C_*(\Omega X)) \to C_*(LX)$$

are equal. As observed above, ι is a $C_*(S^1)$-equivariant map, as are $\mathcal{G}_*$ and $\mathcal{A}_*$. So if we let $\overline{C_*(B_1(\Omega X))}$ denote the $C_*(S^1)$-submodule of $C_*(B(\Omega X))$ generated by the one-skeleton, $C_*(B_1(\Omega X))$ then on the level of homotopy fixed sets, the compositions,

$$\mathcal{G}_*^B \quad and \quad \mathcal{A}_*^B : \overline{C_*(B_1(\Omega X))}^{hS^1} \hookrightarrow C_*(B(\Omega X))^{hS^1} \xrightarrow{\iota} CH_*(C_*(\Omega X))^{hS^1} \to C_*(LX)^{hS^1}$$

are equal.

By Theorem 18 the equivalence $p : B(\Omega X) \xrightarrow{\simeq} X$ induces an equivalence,

$$p_* : C_*(B(\Omega X))^{hS^1} \xrightarrow{\simeq} C_*(X) \otimes C^*(BS^1).$$

Let $q_* : C_*(X) \otimes C^*(BS^1) \to C_*(B(\Omega X))^{hS^1}$ be a homotopy inverse to p_*. Then we have a homotopy commutative diagram,

$$\begin{tikzcd}
\overline{C_*(B_1(\Omega X))}^{hS^1} \arrow[r, hook, "\iota"] & C_*(B(\Omega X))^{hS^1} \\
C_*(\Sigma\Omega X) \otimes C^*(BS^1) \arrow[u, "q_*"] \arrow[r, "e_*"'] & C_*(X) \otimes C^*(BS^1). \arrow[u, "q_*"', "\simeq"]
\end{tikzcd}$$

Let Q_* be a choice of homotopy between the two compositions in this diagram that satisfies the following property. Notice if we compose this diagram with the map $p_* : C_*(B(\Omega X))^{hS^1} \xrightarrow{\simeq} C_*(X)^{hS^1}$, which is an equivalence, then the diagram would commute "on the nose". That is, $p_* \circ \iota \circ q_* = p_* \circ q_* \circ e_*$. Since p_* is an equivalence, we can find a homotopy Q_* between $\iota \circ q_*$ and $q_* \circ e_*$ such that $p_* \circ Q_*$ is *compatible* with the constant homotopy between $p_* \circ \iota \circ q_*$ and $p_* \circ q_* \circ e_*$. By "compatible" we mean the following. If H_1 and H_2 are two homotopies between chain maps f_* and $g_* : C_* \to D_*$, or equivalently null homotopies of $f_* - g_*$, then their difference defines a degree one chain map

$$H_1 - H_2 : C_* \to D_{*+1}.$$

We say that H_1 and H_2 are compatible if $H_1 - H_2$ is null-homotopic.

The fact that $p_* : C_*(B(\Omega X))^{hS^1} \xrightarrow{\simeq} C_*(X)^{hS^1}$ is an equivalence implies that the choice of homotopy Q with this property is unique up to homotopy. That is if Q' is another homotopy with this property, then Q and Q' are compatible.

Then Q_* defines a homotopy between the compositions

$$C_*(\Sigma\Omega X) \xrightarrow{u=id\otimes 1} C_*(\Sigma\Omega X) \otimes C^*(BS^1) \xrightarrow{q_*} \overline{C_*(B_1(\Omega X))}^{hS^1} \xrightarrow{\mathcal{G}_*^B} C_*(LX)^{hS^1}$$

and

$$\mathcal{G}_*^B \circ e_* : C_*(\Sigma\Omega X) \to C_*(X) \to C_*(LX)^{hS^1}.$$

But as observed before this first composition is equal to the same composition with $\mathcal{A}_*^B$ replacing $\mathcal{G}_*^B$:

$$C_*(\Sigma\Omega X) \xrightarrow{u=id\otimes 1} C_*(\Sigma\Omega X) \otimes C^*(BS^1) \xrightarrow{q_*} \overline{C_*(B_1(\Omega X))}^{hS^1} \xrightarrow{\mathcal{A}_*^B} C_*(LX)^{hS^1}$$

And again, the homotopy Q_* defines a homotopy between this composition and

$$\mathcal{A}_*^B \circ e_* : C_*(\Sigma\Omega X) \to C_*(X) \to C_*(LX)^{hS^1}.$$

Thus Q_* defines a homotopy between $\mathcal{A}_*^B \circ e_*$ and $\mathcal{G}_*^B \circ e_*$, thus proving the lemma. $\square$

Remark. The chain homotopy in the proof of this lemma is induced from the "canonical" homotopy Q_* between $q_* \circ e_* : C_*(\Sigma\Omega X) \to C_*(X) \otimes C^*(BS^1) \to C_*(B(\Omega X))^{hS^1}$ and $C_*(\Sigma\Omega X) \hookrightarrow \overline{C_*(B_1(\Omega X))}^{hS^1} \hookrightarrow C_*(B(\Omega X))^{hS^1}$. By "canonical" we mean unique up to homotopy satisfying the property described in the proof of the lemma. When we compose the first composition with $\mathcal{G}_*$ or $\mathcal{A}_* : C_*(B(\Omega X))^{hS^1} \to C_*(LX)^{hS^1}$ we get $\mathcal{G}_*^B \circ e_*$ and $\mathcal{A}_*^B \circ e_*$ respectively. When we compose the second of these compositions with $\mathcal{G}_*$ or $\mathcal{A}_*$ we get equal maps, as shown above. This is how Q_* defines a canonical homotopy, which we call $\mathcal{K}_X$ between $\mathcal{A}_*^B \circ e_*$ and $\mathcal{G}_*^B \circ e_*$.

Using this lemma one can quickly prove Theorem 19 for a certain class of spaces X.

Lemma 22. *If X is a suspension space, say $X \simeq \Sigma Y$, then Theorem 19 holds.*

Proof. Consider the evaluation map in the case of a suspension space, $e : \Sigma\Omega\Sigma Y \to \Sigma Y$. This map has a section, $j : \Sigma Y \to \Sigma\Omega\Sigma Y$ defined to be the suspension of the inclusion $\nu : Y \hookrightarrow \Omega\Sigma Y$ which is given by the adjoint of the identity $\Sigma Y \to \Sigma Y$. That is, $\nu(y) : S^1 \to \Sigma Y$ is given by $\nu(y)(t) = (t, y)$. j is a section of e in the sense that $e \circ j : \Sigma Y \to \Sigma\Omega\Sigma Y \to \Sigma Y$ is equal to the identity.

Thus on the chain level

$$\mathcal{G}_*^B = \mathcal{G}_*^B \circ (e \circ j)_* = \mathcal{G}_*^B \circ e_* \circ j_* : C_*(\Sigma Y) \to C_*(L\Sigma Y)^{hS^1}.$$

But from Lemma 21 we know that

$$\mathcal{G}_*^B \circ e_* \simeq \mathcal{A}_*^B \circ e_*.$$

So

$$\mathcal{G}_*^B = \mathcal{G}_*^B \circ e_* \circ j_* \simeq \mathcal{A}_*^B \circ e_* \circ j_* = \mathcal{A}_*^B.$$

$\square$

This lemma implies that Theorem 19 holds for spheres, and more generally, wedges of spheres. This will be important in an inductive argument that we will give to prove that Theorem 19 holds for all connected CW-complexes.

Our inductive setup will be the following. We consider the difference

$$D_X = \mathcal{G}_*^B - \mathcal{A}_*^B : C_*(X) \to C_*(LX)^{hS^1}.$$

We wish to prove that D_X is null homotopic for any connected CW-complex X. Notice that given any null-homotopy $\mathcal{H}_X$, then $\mathcal{H}_X \circ e$ is a null homotopy of $D_X \circ e = \mathcal{G}_*^B \circ e - \mathcal{A}_*^B \circ e : C_*(\Sigma\Omega X) \to C_*(X) \to C_*(LX)^{hS^1}$. However by Lemma 21 we already know that this map is null homotopic with canonical null homotopy $\mathcal{K}_X$ of $D_X \circ e$.

We now prove Theorem 19 by induction on the dimension of the CW-complex using the following inductive assumption.

Inductive Assumption: For any connected CW-complex X of dimension $\leq n$, there is a null homotopy $H_X : C_*(c(X)) \to C_*(LX)^{hS^1}$ of D_X such that $\mathcal{H}_X \circ e$ is compatible with $\mathcal{K}_X$.

Note. Here $C_*(c(X))$ is the singular chains of the cone on X. Clearly a null-homotopy of a chain map with source $C_*(X)$ is equivalent to an extension of that map to $C_*(c(X))$.

We now check that the inductive assumption holds for $n = 1$. Since any connected one-dimensional CW-complex is homotopic to a suspension space, $X \simeq \Sigma Y$, where Y is a discrete set. Lemma 22 then implies that the map D_X is null homotopic. We take as our null homotopy, the composition,

$$\mathcal{H}_X : C_*(c(\Sigma Y)) \xrightarrow{j_*} C_*(c(\Sigma\Omega\Sigma(Y_+) \xrightarrow{\mathcal{K}_X} C_*(LX)^{hS^1}.$$

To verify that $\mathcal{H}_X$ is compatible with $\mathcal{K}_X$, we must verify that the difference between the null homotopies

$$\mathcal{H}_X \circ e : C_*(c(\Sigma\Omega\Sigma(Y_+) \xrightarrow{e_*} C_*(c(\Sigma Y)) \xrightarrow{j_*} C_*(c(\Sigma\Omega\Sigma(Y_+) \xrightarrow{\mathcal{K}_X} C_*(LX)^{hS^1}$$

and

$$\mathcal{K}_X : C_*(c(\Sigma\Omega\Sigma(Y_+) \to C_*(LX)^{hS^1}$$

defines a map $\mathcal{K}_X - \mathcal{H}_X \circ e : C_{*-1}(\Sigma\Omega\Sigma(Y_+) \to C_*(LX)^{hS^1}$ which is null homotopic. That is, to complete the verification of the inductive assumption for $n = 1$we need to show that the null homotopies $\mathcal{K}_X \circ j_* \circ e_*$ is compatible with $\mathcal{K}_X$. We will actually show that this compatibility holds for any suspension space (i.e not only one-dimensional spaces).

Lemma 23. *Let Y be any CW complex and let $X = \Sigma Y$. Then the null homotopies $\mathcal{K}_X$ and $\mathcal{K}_X \circ j_* \circ e_*$ of the composition $C_*(\Sigma\Omega X) \xrightarrow{e} C_*(X) \xrightarrow{D_X} C_*(LX)^{hS^1}$ are compatible.*

Proof. Let Q_* be the canonical homotopy between $q_* \circ e_* \circ u_* : C_*(\Sigma\Omega X) \to C_*(X) \xrightarrow{u_*} C_*(X) \otimes C^*(BS^1) \to C_*(B(\Omega X))^{hS^1}$ and $C_*(\Sigma\Omega X) \hookrightarrow \overline{C_*(B_1(\Omega X))}^{hS^1} \hookrightarrow C_*(B(\Omega X))^{hS^1}$ as in the proof of Lemma 21. When one composes the first composition with $\mathcal{G}_*$ or $\mathcal{A}_* : C_*(B(\Omega X))^{hS^1} \to C_*(LX)^{hS^1}$ we get $\mathcal{G}^B_* \circ e_*$ and $\mathcal{A}^B_* \circ e_*$ respectively. When we compose the second of these compositions with $\mathcal{G}_*$ or $\mathcal{A}_*$ we get equal maps, as shown above. So $\mathcal{K}_X$ is a combination of the homotopy $\mathcal{G}_* \circ Q_* \circ u_*$ between $\mathcal{G}^B_* \circ e_*$ and $\mathcal{G}_* \circ q_* \circ e_* = \mathcal{A}_* \circ q_* \circ e_* \circ u_*$, and the homotopy $\mathcal{A}_* \circ Q^-_* \circ u_*$ from $\mathcal{A}_* \circ q_* \circ e_* \circ u_*$ to $\mathcal{A}^B_* \circ e_*$. Here Q^-_* is the reverse of the homotopy Q_*. Thus if we prove that the homotopies $\mathcal{G}_* \circ Q_* \circ u_*$ and $\mathcal{G}_* \circ Q_* \circ u_* \circ j_* \circ e_*$ are compatible, as are the homotopies $\mathcal{A}_* \circ Q_* \circ u_*$ and $\mathcal{A}_* \circ Q_* \circ u_* \circ j_* \circ e_*$, then we will know that the homotopies $\mathcal{K}_X \circ j_* \circ e_*$ and $\mathcal{K}_X$ are compatible. This is what we now prove.

Now since p_* and q_* are homotopy inverse to each other, the homotopy $\mathcal{G}_* \circ Q_*$ is compatible with the homotopy $\mathcal{G}_* \circ q_* \circ p_* \circ Q_*$, and similarly the homotopy $\mathcal{A}_* \circ Q^-_*$ is compatible with the homotopy $\mathcal{A}_* \circ q_* \circ p_* \circ Q^-_*$. So it suffices to prove that $\mathcal{G}_* \circ q_* \circ p_* \circ Q_* \circ u_*$ is compatible with $\mathcal{G}_* \circ q_* \circ p_* \circ Q_* \circ u_* \circ j_* \circ e_*$, and similarly with $\mathcal{A}_*$ replacing $\mathcal{G}_*$ in the above compositions. In particular it suffices to show that the homotopies $p_* \circ Q_* \circ u_*$ and $p_* \circ Q_* \circ u_* \circ j_* \circ e_*$ compatible.

By the defining property of the homotopy Q, we know that $p_*Q_*u_*j_*e_*$ is compatible with the constant homotopy of $p_*q_*e_*u_*j_*e_*$. But $e_*u_*j_*e_* : C_*(\Sigma\Omega\Sigma Y) \to C_*(\Sigma Y) \to C_*(\Sigma\Omega\Sigma Y) \to C_*(\Sigma\Omega\Sigma Y) \otimes C^*(BS^1) \to C_*(\Sigma Y) \otimes C^*(BS^1)$ is equal to $u_*e_* : C_*(\Sigma\Omega\Sigma Y) \to C_*(\Sigma Y) \to C_*(\Sigma Y) \otimes C^*(BS^1)$. This is because j_* is a section of e_* (i.e $e_*j_* : C_*(\Sigma Y) \to C_*(\Sigma\Omega\Sigma Y) \to C_*(\Sigma Y)$ is the identity.) But u_*e_* equals e_*u_*. That is, the following diagram commutes:

$$\begin{array}{ccc}
C_*(\Sigma\Omega\Sigma Y) & \xrightarrow{e_*} & C_*(\Sigma Y) \\
{\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle u_*} \\
C_*(\Sigma\Omega\Sigma Y) \otimes C^*(BS^1) & \xrightarrow[e_*]{} & C_*(\Sigma Y) \otimes C^*(BS^1)
\end{array}$$

Thus we have shown that $p_*q_*e_*u_*j_*e_* = p_*q_*u_*e_* = p_*q_*e_*u_*$. This means that the homotopy $p_*Q_*u_*j_*e_*$ is compatible with the constant homotopy of $p_*q_*e_*u_*$. But by the defining property of the homotopy Q_* this constant homotopy is compatible with the homotopy $p_*Q_*u_*$. Thus the homotopies $p_*Q_*u_*j_*e_*$ and $p_*Q_*u_*$ are compatible. □

We now assume that our inductive assumptions hold for any connected CW-complex of dimension $\leq n$, with $n \geq 1$, and with that assumption we prove that it holds for $n+1$-dimensional complexes. Let $X = X^{(n+1)}$ be a connected, $(n+1)$-dimensional CW complex. Let $X^{(n)}$ denote its n-skeleton. Consider the cofibration sequence of chain complexes,

$$C_*(\bigvee_{I_n} S^n) \xrightarrow{\alpha_*} C_*(X^{(n)}) \xrightarrow{\iota_*} C_*(X)$$

where $\alpha : \bigvee_{I_n} S^n \to X^{(n)}$ is the attaching map of the $(n+1)$-cells.

Consider the following commutative diagram of chain complexes:

$$\begin{tikzcd}
C_*(\bigvee_{I_n} S^n) \arrow[d, "j_*"'] & & & C_*(\bigvee_{I_n} S^{n+1}) \arrow[d, "j_*"'] \\
C_*(\Sigma\Omega(\bigvee_{I_n} S^n)) \arrow[r, "\alpha_*"'] \arrow[d, "e_*"'] & C_*(\Sigma\Omega X^{(n)}) \arrow[r, "\iota_*"'] \arrow[d, "e_*"'] & C_*(\Sigma\Omega X) \arrow[r] \arrow[d, "e_*"'] & C_*(\Sigma\Omega X \cup_\iota c(\Sigma\Omega X^{(n)}) \arrow[d, "e_*"'] \\
C_*(\bigvee_{I_n} S^n) \arrow[r, "\alpha_*"'] \arrow[d, "D_{\bigvee_{I_n} S^n}"'] & C_*(X^{(n)}) \arrow[r, "\iota_*"'] \arrow[d, "D_{X^{(n)}}"'] & C_*(X) \arrow[r] \arrow[d, "D_X"'] & C_*(\bigvee_{I_n} S^{n+1}) \\
C_*(L(\bigvee_{I_n} S^n))^{hS^1} \arrow[r, "\alpha_*"'] & C_*(LX^{(n)})^{hS^1} \arrow[r, "\iota_*"'] & C_*(LX)^{hS^1} &
\end{tikzcd} \tag{4.291}$$

Here the maps labelled α_* are all induced by the attaching map α in obvious ways.

By induction we a null homotopy $\mathcal{H}_{X^{(n)}} : C_*(c(X^{(n)})) \to C_*(LX^{(n)})$ of $D_{X^{(n)}}$ satisfying our inductive assumption. We abbreviate $\mathcal{H}_{X^{(n)}}$ by $\mathcal{H}_n$. By assumption, the null homotopies $\mathcal{H}_n \circ e_*$ and $\mathcal{K}_{X^{(n)}}$ are compatible. So $\mathcal{H}_n e_* \alpha_*$ is compatible with $\mathcal{K}_{(\bigvee_{I_n} S^n)}$ as null homotopies of $D_{X^{(n)}} e_* \alpha_* : C_*(\bigvee_{I_n} S^n) \to C_*(LX^{(n)})^{hS^1}$.

Consider the null homotopy $\mathcal{K}_{(\bigvee_{I_n} S^n)} \circ j_*$ of $D_{\bigvee_{I_n} S^n} \circ e_* \circ j_* = D_{\bigvee_{I_n} S^n}$ since $j_* e_* = id$. We now define a null homotopy

$$\mathcal{H}_X : C_*(c(X)) \to C_*(LX)^{hS^1}$$

in the following way. The obstruction to defining $\mathcal{H}_X$ is a map $o(\mathcal{H}_n) : \bigvee_{I_n} S^{n+1} \to C_*(LX)^{hS^1}$. This is obtained by identifying this space with the mapping cone

$$\bigvee_{I_n} S^{n+1} \simeq X \cup_\iota c(X^{(n)}),$$

and defining $o(\mathcal{H}_n)$ to be the natural extension of $\mathcal{H}_n$

$$o(\mathcal{H}_n) : C_*(X \cup_\iota c(X^{(n)}) \to C_*(LX)^{hS^1}$$

which is given by D_X on $C_*(X)$ and $\mathcal{H}_n$ on $C_*(c(X^{(n)})$.

Notice that the composition $o(\mathcal{H}_n) \circ e_* : C_*(\Sigma\Omega X \cup_\alpha c(\Sigma\Omega X^{(n)})) \to C_*(LX)^{hS^1}$ is null homotopic because that is the obstruction to the null homotopy $\mathcal{H}_n \circ e$ extending to a null homotopy of $D_X \circ e_*$. However since $ch_n \circ e_*$ is compatible with $\mathcal{K}_{X^{(n)}}$, which in turn extends to the null homotopy $\mathcal{K}_X$, this gives a null homotopy of the obstruction $o(\mathcal{H}_n) \circ e_*$.

By the commutativity of the above diagram, the section $j : \bigvee_{I_n} S^{n+1} \to \Sigma\Omega(\bigvee_{I_n} S^{n+1}))$ defines a section

$$j_* : \bigvee_{I_n} S^{n+1} \to \Sigma\Omega X \cup_\alpha c(\Sigma\Omega X^{(n)}),$$

as pictured in that diagram.

If we compose the null-homotopy of $o(\mathcal{H}_n) \circ e_*$ induced by $\mathcal{K}_X$ with j_* we get a null homotopy of $o(\mathcal{H}_n) \circ e_* \circ j_* = o(\mathcal{H}_n)$. This induces a null homotopy we call $\mathcal{H}_X$. What remains is to check that $\mathcal{H}_X \circ e_*$ is compatible with $\mathcal{K}_X$. Notice by construction, $\mathcal{H}_X \circ e_*$ is compatible with $\mathcal{K}_X \circ j_* \circ e_*$. But by Lemma 23 $\mathcal{K}_X \circ j_* \circ e_*$ is compatible with $\mathcal{K}_X$. This completes the inductive step, and therefore completes the proof of the theorem. $\square$

A Appendices

A.1 Products of singular chains and a quotient complex

A.1.1 Eilenberg-Zilber maps

Let Δ^n denote the standard n-simplex

$$\Delta^n = \{(t_0, \ldots, t_n) | t_j \geq 0, \sum_i t_i = 1\}.$$

For $i = 0, \ldots, n$, there are *co-face maps*

$$\begin{aligned} \delta^i : \Delta^{n-1} &\to \Delta^n \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \end{aligned} \tag{A.1}$$

and for $i = 1, \ldots, n+1$, *co-degeneracies*

$$\begin{aligned} s^i : \Delta^{n+1} &\to \Delta^n \\ (t_0, \ldots, t_{n+1}) &\mapsto (t_0, \ldots, t_{i-2}, t_{i-1} + t_i, t_{i+1}, \ldots, t_{n+1}) \end{aligned} \tag{A.2}$$

Let C_* denote the normalized singular chains functor. Generators of $C_k(X)$ are singular simplices $\sigma : \Delta^k \to X$, modulo those in the image of a degeneracy, and the differential of a singular simplex σ is the sum of restrictions to various faces:

$$d\sigma = \sum_{i=0}^{k} d_i \sigma = \sum_{i=0}^{k} (-1)^i \sigma|_{\delta^i(\Delta^{k-1})} \tag{A.3}$$

Recall that for X and Y a pair of spaces, the *Eilenberg-Zilber map*

$$EZ = C_p(X) \otimes C_q(Y) \to C_{p+q}(X \times Y) \tag{A.4}$$

$$\tag{A.5}$$

gives a prescription of how to cut apart a product of simplices into higher dimensional simplices (this generalizes the familiar *prism operator* @@CITE-HATCHER, the case $p = 1$ or $q = 1$).

The Eilenberg-Zilber map, which we will denote simply by $\times$ outside of this Appendix, satisfies the following properties:

- EZ is strictly (chain-level) associative.
- EZ is functorial in X and Y.
- On a generator $\alpha \otimes \beta$ where $|\alpha| = 0$, $EZ(\alpha \otimes \beta) = \beta$, extended linearly (so $EZ((\sum n_i \sigma_i) \otimes \beta) = (\sum n_i)\beta$, where $\sigma_i \in C_0(X)$ are any generators.

Remark A.1. *A popular method for constructing EZ via the method of acyclic models. CITE. However, the resulting formula is only a priori homotopy-associative; one needs to use an explicit formula involving e.g., shuffle products CITE to guarantee strict associativity.*

Definition A.1. *An element of $C_*(X \times Y)$ is called* **product-like** *for (α, β), if it is equal to $EZ(\alpha \otimes \beta)$ for $\alpha \in C_*(X)$, $\beta \in C_*(Y)$.*

Lemma A.1. *Suppose $f : X \times Y \to Z$ is a map that factors through projection to Y, and $\sigma \in C_*(X \times Y)$ is product-like for (α, β) with $deg(\alpha) > 0$. Then, $f_*\sigma$ is 0 on the chain level.*

Proof. The statement that f factors through projection means also that f factors through a map $X \times Y \to pt \times Y$, i.e., a product of maps $\pi \times id$. Now, by naturality note that $f_*(EZ(\alpha \otimes \beta)) = \bar{f}_*(EZ(\pi \times id)_*(\alpha \otimes \beta)) = \bar{f}(EZ(\pi_*\alpha \otimes \beta))$, but now $\pi_*\alpha = 0$ unless $|\alpha| = 0$, as normalized singular chains of a point is concentrated in degree zero. $\square$

By strict associativity, there is an unambiguous extension of EZ to a map involving multiple factors, which we will also call EZ:

$$EZ : C_{i_1}(X_1) \otimes \cdots \otimes C_{i_k}(X_k) \to C_{i_1 + \cdots + i_k}(X_1 \times \cdots \times X_k) \tag{A.6}$$

In particular, the EZ map can be used to view maps from products of simplices as elements of the singular chain complex.

Definition A.2. *The* **universal singular simplex in dimension** *k is the element of $C_k(\Delta^k)$ given by the identity map*

$$\eta_k : \Delta^k \xrightarrow{id} \Delta^k \tag{A.7}$$

Definition A.3. *A* **product singular chain**, *or* **prod-singular chain** *in X is a map*

$$\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to X. \tag{A.8}$$

The **singularization** *of σ is the element of $C_{i_1 + \cdots + i_k}(X)$ given by*

$$sing(\sigma) := \sigma_\sharp(EZ(\eta_{i_1} \otimes \cdots \otimes \eta_{i_k})) \tag{A.9}$$

where $\sigma_\sharp : C_*(\Delta^{i_1} \times \cdots \times \Delta^{i_k}) \to C_*(X)$ is the induced map on singular chains.

In the definition above, we assume each $i_r > 0$. Alternatively, we identify $\Delta^{i_1} \times \cdots \times \Delta^{i_{r-1}} \times \Delta^0 \times \Delta^{i_{r+1}} \cdots \times \Delta^{i_k}$ with $\Delta^{i_1} \times \cdots \times \Delta^{i_{r-1}} \times \Delta^{i_{r+1}} \times \cdots \times \Delta^{i_k}$.

Given a prod-singular chain $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k}$, and a pair (r, s) with $r \in \{1, \ldots, k\}$ and $s \in \{0, 1, \ldots, i_r\}$, we can restrict σ to the sth face on the rth factor to obtain another prod-singular chain

$$ {}_r d_s(\sigma) = \sigma|_{(id^{r-1}\times\delta^s\times id^{k-r})(\Delta^{i_1}\times\cdots\times\Delta^{i_{r-1}}\times\Delta^{i_r-1}\times\Delta^{i_{r+1}}\times\cdots\times\Delta^{i_k})}. \tag{A.10}$$

Naturality of the Eilenberg-Zilber map implies that

Lemma A.2. *For a prod-singular chain $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to X$, singularization is compatible with taking boundary maps, in the sense that*

$$ d(sing(\sigma)) = \sum_{r=1}^{k} \sum_{s=0}^{i_r} (-1)^{r+s} sing({}_r d_s \sigma). \tag{A.11}$$

Remark A.2. *One can think of cubical chains as a special case of prod-singular chains where all $i_j = 1$. The above Lemma implies that singularization gives a natural chain map (inducing an isomorphism) from cubical chains to singular chains.*

Here is an explicit formula for the Eilenberg-Zilber map: recall that for a singular simplex $\sigma : \Delta^n \to X$, the *degeneracies* $s_i(\sigma) : \Delta^{n+1} \to X$ for $i = 1, \ldots, n+1$ are given by

$$ s_i(\sigma) = \sigma \circ s^i, \tag{A.12}$$

where s^i are the codengeneracies (A.2). Now, for $\alpha \in C_j(X)$, $\beta \in C_k(Y)$, one can define

$$ EZ(\alpha \otimes \beta) = \sum_{(\gamma,\eta)\in S_{j,k}} (-1)^{\epsilon(\gamma,\eta)} (s_{\gamma(j)} \cdot \cdots \cdot s_{\gamma(1)}\alpha, s_{\eta(k)} \cdots s_{\eta(1)}\beta) \tag{A.13}$$

where $S_{j,k}$ is the set of (j, k) shuffles and $\epsilon : S_{j+k} \to \{\pm 1\}$ is the sign homomorphism. (Regarding the notation, recall that the output of EZ is a singular chain in $\sigma : \Delta^{k+l} \to X \times Y$, which can be represented as a tuple $(\pi_X \sigma, \pi_Y \sigma)$). CITE[Malm for source, but also Felix, Halperin, Thomas?] Citation needed

Let $\tau : X \times Y \to Y \times X$ be the permutation of factors, which induces a chain map $\tau_\sharp : C_*(X \times Y) \to C_*(Y \times X)$. Similarly, let $\bar{\tau} : C_j(X) \otimes C_k(Y) \to C_k(Y) \otimes C_l(X)$ be the algebraic map which permutes factors.

Lemma A.3. *The diagram*

$$\begin{tikzcd} C_j(X) \otimes C_k(Y) \arrow[r, "EZ"] \arrow[d, "\bar{\tau}"] & C_{j+k}(X \times Y) \arrow[d, "\tau_\sharp"] \\ C_k(Y) \otimes C_j(X) \arrow[r, "EZ"] & C_{j+k}(Y \times X) \end{tikzcd} \tag{A.14}$$

commutes up to an overal sign of $(-1)^{jk}$.

Proof. There is an abstract isomorphism between sets of shuffles $S_{j,k} \cong S_{k,j}$. This isomorphism changes the sign homomorphism precisely by a factor of $(-1)^{jk}$. $\square$

Lemma A.4. *Let*

$$\sigma : \Delta^a \times \Delta^b \to X \tag{A.15}$$

and

$$\sigma' = \sigma \circ \tau : \Delta^b \times \Delta^a \to X, \tag{A.16}$$

where $\tau : \Delta^a \times \Delta^b \to \Delta^b \times \Delta^a$ *is the isomorphism exchanging factors. Then,* $sing(\sigma) = (-1)^{ab} sing(\sigma')$.

Proof. Denoting by $\eta_i : \Delta^i \to \Delta^i$ the universal simplex, we have that

$$\begin{aligned}
sing(\sigma) &= \sigma_\sharp(EZ(\eta_a \otimes \eta_b)) && \text{(A.17)}\\
&= \sigma_\sharp(EZ(\bar{\tau}(\eta_b \otimes \eta_a))) && \text{(A.18)}\\
&= (-1)^{ab}\sigma_\sharp(\tau_\sharp(EZ(\eta_b \otimes \eta_a))) && \text{(A.19)}\\
&= (-1)^{ab}\sigma'_\sharp(EZ(\eta_b \otimes \eta_a)) && \text{(A.20)}\\
&= (-1)^{ab} sing(\sigma'), && \text{(A.21)}
\end{aligned}$$

where the third equality used the previous Lemma A.3. $\square$

A.1.2 A convenient quotient of singular chains

Definition A.4. *An* **abstract generalized prism** *is a product of simplices of the form*

$$\Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \tag{A.22}$$

A **singular generalized prism** *in* X *is a map from an abstract generalized prism to* X

$$\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X \tag{A.23}$$

Definition A.5. *Let* σ_1 *and* σ_2 *be singular generalized prisms in* X *with the same domain, that is*

$$\sigma_i : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X,\ i = 1, 2. \tag{A.24}$$

We say σ_1 *and* σ_2 **fit together** *if for all* $\mathbf{s} \in \Delta^{i_1} \times \cdots \times \Delta^{i_k}$*,*

$$\sigma_1(\mathbf{s}, 1) = \sigma_2(\mathbf{s}, 0). \tag{A.25}$$

Given a pair σ_1, σ_2 of singular generalized prisms that fit together, and a function $f : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to I$ such that

$$\begin{aligned}
f^{-1}(0) &\subset \{\mathbf{s} | \sigma_1(\mathbf{s}, t) \text{ is independent of } t\} && \text{(A.26)}\\
f^{-1}(1) &\subset \{\mathbf{s} | \sigma_2(\mathbf{s}, t) \text{ is independent of } t\} && \text{(A.27)}
\end{aligned}$$

we can define an f*-gluing*

$$\sigma_1 \sharp_f \sigma_2 : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X \tag{A.28}$$

$$\sigma_1 \sharp_f \sigma_2(\mathbf{s}, t) = \begin{cases} \sigma_1(\mathbf{s}, \frac{t}{f(\mathbf{s})}) & t \leq f(s) \\ \sigma_2(\mathbf{s}, \frac{t - f(\mathbf{s})}{1 - f(\mathbf{s})}) & \text{otherwise,} \end{cases} \tag{A.29}$$

where $\mathbf{s}$ denotes the coordinate on $\Delta^{i_1} \times \cdots \times \Delta^{i_k}$. The conditions (A.26)-(A.27) guarantee that this gluing extends continuously to the case $f(\mathbf{s}) \in \{0, 1\}$. This prodedure is essentially reversible: given a singular generalized prism $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X$, and a function $f : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to I$, we can define the two singular generalized prisms obtained by cutting along the graph of f and rescaling accordingly:

$$\begin{aligned} \sigma_1^f(\mathbf{s}, t) &= \sigma(\mathbf{s}, f(\mathbf{s})t), \\ \sigma_2^f(\mathbf{s}, t) &= \sigma(\mathbf{s}, f(\mathbf{s}) + (1 - f(\mathbf{s}))t). \end{aligned} \tag{A.30}$$

Recall now that for any singular generalized prism σ in X, its singularization $sing(\sigma)$ is an element of $C_*(X)$.

Lemma A.5. *The subgroup*

$$D^1_*(X) = \bigoplus sing(\sigma_1) + sing(\sigma_2) - sing(\sigma_1 \sharp_f \sigma_2) \tag{A.31}$$

where the direct sum is taken over all singular generalized prisms which fit together, is a contractible subcomplex of $C_*(X)$.

Proof. First, we verify $D^1_*(X)$ is a subcomplex. By Lemma A.2, for a singular generalized prism $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X$,

$$d(sing(\sigma)) = \sum_{r=1}^{k} \sum_{s=0}^{i_r} (-1)^{r+s} sing({}_r d_s \sigma) + (-1)^{k+1} (sing({}_{k+1} d_0 \sigma) - sing({}_{k+1} d_1 \sigma)). \tag{A.32}$$

Now if σ_1 and σ_2 glue together via f to σ, then ${}_r d_s \sigma_1$ and ${}_r d_s \sigma_2$ glue together via

$$f|_{(id^{r-1} \times \delta^s \times id^{k-r})(\Delta^{i_1} \times \cdots \times \Delta^{i_{r-1}} \times \Delta^{i_r - 1} \times \Delta^{i_{r+1}})}$$

to ${}_r d_s \sigma$, for $r \leq k$. So for $r \leq k$, ${}_r d_s(\sigma_1 + \sigma_2 - \sigma_1 \sharp \sigma_2) \in D^1_*(X)$.

For the last factor $r = k + 1$, the pairs $sing({}_{k+1} d_1 \sigma_1)$ and $sing({}_{k+1} d_0 \sigma_2)$ cancel, $sing({}_{k+1} d_0 \sigma_1)$ and $sing({}_{k+1} d_0 \sigma_1 \sharp_f \sigma_2)$ cancel, and $sing({}_{k+1} d_1 \sigma_2)$ and $sing({}_{k+1} d_1 \sigma_1 \sharp_f \sigma_2)$ cancel, as the relevant prod-singular chains are equal.

Given a pair (σ, f), denote by σ^f the generator of $D^1_*(X)$ of the form $sing(\sigma_1^f) + sing(\sigma_2^f) - sing(\sigma)$; we note that every generator admits such a representation. The above discussion shows that for a generator

$$
\begin{aligned}
d\sigma^f &= \sum_{r=1}^{k}\sum_{s=0}^{i_r}(-1)^{r+s} sing({}_r d_s \sigma_1^f) + sing({}_r d_s \sigma_2^f) - sing({}_r d_s \sigma) \\
&= \sum_{r=1}^{k}\sum_{s=0}^{i_r}(-1)^{r+s}({}_r d_s \sigma)^{{}_r d_s f}
\end{aligned} \tag{A.33}
$$

where in the second expression we think of f as a prod-singular chain in I.

Now consider the map(s)

$$P : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \times I \to \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \tag{A.34}$$

$$(\mathbf{s}, t_1, t_2) \mapsto \begin{cases} (\mathbf{s}, t_1 + t_2) & t_1 + t_2 \leq 1 \\ (\mathbf{s}, 1) & \text{otherwise} \end{cases} \tag{A.35}$$

and denote by

$$Q : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to \Delta^{i_1} \times \cdots \times \Delta^{i_k} \tag{A.36}$$

the projection to the first k factors. Define

$$H(\sigma^f) = (\sigma \circ P)^{f \circ Q}. \tag{A.37}$$

We check that

$$dH(\sigma^f) = \sum_{r=1}^{k+1}\sum_{s=0}^{i_r}(-1)^{r+s}({}_r d_s(\sigma \circ P))^{{}_r d_s (f \circ Q)} \tag{A.38}$$

and

$$Hd(\sigma^f) = \sum_{r=1}^{k}\sum_{s=0}^{i_r}(-1)^{r+s}(({}_r d_s(\sigma)) \circ P)^{({}_r d_s f) \circ Q} \tag{A.39}$$

Using the fact that for $r \leq k$,

$$(({}_r d_s(\sigma)) \circ P)^{({}_r d_s f) \circ Q} = ({}_r d_s(\sigma \circ P))^{{}_r d_s (f \circ Q)} \tag{A.40}$$

and for $r = k+1$,

$$(-1)^{k+1}({}_{k+1} d_s(\sigma \circ P))^{{}_{k+1} d_s (f \circ Q)} = \begin{cases} (-1)^{k+1}\sigma^f & s = 0 \\ 0 \text{ (degenerate)} & s = 1. \end{cases} \tag{A.41}$$

So $d(\tilde{H}) + \tilde{H}d = id$, where $\tilde{H}(\sigma^f) = (-1)^{k+1}H(\sigma^f)$. i.e., $\tilde{H}$ gives the desired contracting homotopy. □

Let

$$\tau_s : \Delta^{i_1} \times \cdots \times \Delta^{i_s} \times \Delta^{i_{s+1}} \times \cdots \times \Delta^{i_k} \to \Delta^{i_1} \times \cdots \times \Delta^{i_{s-1}} \times \Delta^{i_{s+1}} \times \Delta^{i_s} \times \Delta^{i_{s+2}} \cdots \times \Delta^{i_k} \tag{A.42}$$

be the isomorphism that exchanges the s and $s+1$st factors.

Lemma A.6. *If $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \to X$, then $sing(\sigma \circ \tau_s) = (-1)^{i_s \cdot i_{s+1}} sing(\sigma)$.*

Proof. This follows the same argument as Lemma A.4. $\square$

Remark A.3. *Unlike the case of cubical chains in which one has to quotient to identify chains such as σ with $(-1)^{i_s \cdot i_{s+1}} \sigma \circ \tau_s$ as in [3]*§C, this equality comes for free by singularizing.*

Lemma A.7. *If X admits an S^1 action, then $D^1(X)$ is preserved under the resulting $C_*(S^1)$ action.*

Proof. If $m : S^1 \times X \to X$ denotes the multiplication, then the induced multiplication Δ by $[S^1]$ admits the following chain level description: for $\alpha : \Delta^i \to X$, denote by

$$\begin{aligned} \tilde{\alpha} : \Delta^i \times I &\to X \\ (\mathbf{s}, t) &\mapsto m(e^{2\pi i t}, \alpha(\mathbf{s})). \end{aligned} \tag{A.43}$$

and $\Delta(\alpha) := sing(\tilde{\alpha})$. Now, if $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X$ is a singular generalized prism, denote by

$$\begin{aligned} \tilde{\sigma} : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \times I &\to X \\ (\mathbf{s}, t_1, t_2) &\mapsto m(e^{2\pi i t_2}, \sigma(\mathbf{s}, t_1)). \end{aligned} \tag{A.44}$$

(by naturality, this is compatible with (A.43) in the sense that $sing(\tilde{\sigma}) = \sum_i n_i sing(\tilde{\alpha_i})$, where $sing(\sigma) = \sum_i n_i \alpha_i$). By Lemma A.6, $sing(\tilde{\sigma}) = -sing(\bar{\sigma})$, where $\bar{\sigma}$ is $\tilde{\sigma}$ composed with the map that changes the last two factors

$$\begin{aligned} \bar{\sigma} : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \times I &\to X \\ (\mathbf{s}, t_1, t_2) &\mapsto m(e^{2\pi i t_1}, \sigma(\mathbf{s}, t_2)). \end{aligned} \tag{A.45}$$

But now, it is clear that if σ_1 and σ_2 fit together using f to $\sigma_1 \sharp_f \sigma_2$, then $\bar{\sigma}_1$ and $\bar{\sigma}_2$ fit together to $\overline{\sigma_1 \sharp_f \sigma_2}$. $\square$

Going forward, denote by $\tilde{C}_*(X)$ the quotient $C_*(X)/D^1_*(X)$ and call the projection

$$pr_* : C_*(X) \to \tilde{C}_*(X). \tag{A.46}$$

We have shown:

Proposition A.1. *pr_* is a homology isomorphism, and if X admits an S^1 action, then the induced operator $\Delta : C_*(X) \to C_*(X)$ descends to the quotient $\tilde{C}_*(X)$.*

A.1.3 A further quotient

Definition A.6. *Let $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X$ be a singular generalized prism. The last-factor reversal of σ is the singular generalized prism*

$$\begin{aligned} \bar{\sigma} : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I &\to X \\ (\mathbf{s}, t) &\mapsto \sigma(\mathbf{s}, 1 - t) \end{aligned} \tag{A.47}$$

Lemma A.8. *The subgroup*

$$D^2_*(X) = \bigoplus sing(\sigma_1) + sing(\bar{\sigma_1}) \tag{A.48}$$

where the direct sum is taken over all singular generalized prisms, is a contractible subcomplex of $C_*(X)$.

Proof. First we verify $D^2_*(X)$ is a subcomplex. Given a singular generalized prism $\sigma : \Delta^{i_1} \times \cdots \times \Delta^{i_k} \times I \to X$, the differential applied to $sing(\sigma)$ is as in (A.32). Now, let us analyze the terms in the sum $d(sing(\sigma) + sing(\bar{\sigma})$. Let us note that for $r \le k$, ${}_r d_s \sigma$ and ${}_r d_s \bar{\sigma}$ differ by last-factor reversal, so their sum lies in D^2_*. For $r = k+1$, note that ${}_{k+1}d_0\sigma = {}_{k+1}d_1\bar{\sigma}$ and ${}_{k+1}d_1\sigma = {}_{k+1}d_0\bar{\sigma}$. So the relevant terms in the sum cancel. Thus, $d(sing(\sigma) + sing(\bar{\sigma}) \in D^2_*(X)$.

INSERT MORE ∎

Take a further quotient by cyclic permutations.

INSERT SECTION ABOUT PERMUTING COORDINATES; CALL THE SUBCOMPLEX

$$D^3_*(X) = \bigoplus \sigma + \sigma \circ \phi \tag{A.49}$$

A.2 Modules over $C_*(S^1)$ versus $H_*(S^1)$.

Let us take the explicit model $S^1 = \mathbb{R}/\mathbb{Z}$. The multiplication $m : S^1 \times S^1 \to S^1$ sending $(s,t) \mapsto s + t \pmod 1$ induces a map on singular chains

$$m_* : C_*(S^1 \times S^1) \to C_*(S^1), \tag{A.50}$$

which, composed with the Eilenberg-Zilber map, gives the standard *Pontryagin product* on $C_*(S^1)$:

$$p : C_*(S^1) \otimes C_*(S^1) \stackrel{EZ}{\to} C_*(S^1 \times S^1) \stackrel{m_*}{\to} C_*(S^1). \tag{A.51}$$

Since p is a chain map, it induces the homology level multiplication $[p] : H_*(S^1) \otimes H_*(S^1) \to H_*(S^1)$ leading to the identification $H_*(S^1) \cong \mathbb{K}[\beta]/\beta^2$, where $\beta = [S^1]$.

In a similar fashion, given a space X with an S^1 action $S^1 \times X \to X$, the Pontryagin product $p : C_*(S^1) \otimes C_*(X)$ produces a module structure of $C_*(X)$ over $C_*(S^1)$.

It is well known that $C_*(S^1)$ is *formal*, e.g., equivalent via a zig-zig of dga quasi-isomorphisms or single A_∞ quasi-isomorphism to $H_*(S^1)$, so the category of *dg* modules over $C_*(S^1)$ is equivalent to *dg* modules over $H_*(S^1)$. The goal of this (nearly tautalogous) section is to realize the equivalence explicitly, for a space X with S^1 action. Namely, we will show that $\tilde{C}_*(X)$, where $\tilde{C}_*(X)$ is the quotient of the singular chains functor defined above, is a dg module over $H_*(S^1)$, equivalent to $C_*(X)$ as dg module over $C_*(S^1)$ under the equivalence $H_*(S^1) \simeq C_*(S^1)$. (so in particular, computations of homotopy fixed points, etc. agree).

The first thing to do is to produce an explicit equivalence $H_*(S^1) \sim C_*(S^1)$. There are many choices of chain map (where the left hand side has zero differential) inducing a homology isomorphism, such as:

$$\begin{aligned} \iota : H_*(S^1) &\to C_*(S^1) \\ 1 &\mapsto \gamma : \Delta^0 \to S^1,\ pt \mapsto 0 \text{ (mod 1)} \\ \beta &\mapsto \sigma : I \to S^1,\ t \mapsto t \text{ (mod 1)} \end{aligned} \tag{A.52}$$

However, this map is not an algebra map! To see this, note that the while $p(\gamma \otimes \gamma) = \gamma$, the product of two copies of σ is exact but not strictly 0.

If we compose with projection $C_*(S^1) \to \tilde{C}_*(S^1)$, this deficiency goes away:

Lemma A.9. *The inclusion $\iota : H_*(S^1) \to \tilde{C}_*(S^1)$ is an algebra map.*

Proof. Let $\sigma = \iota(\beta)$ denote the map $I :\to S^1$ sending t to t (mod 1). We need to show that $p(\sigma \otimes \sigma) = 0$ on the chain level. We note first that

$$\begin{aligned} \kappa = m \circ (\sigma \times \sigma) : I \times I &\to S^1 \\ (s,t) &\mapsto s + t \text{ (mod 1)} \end{aligned} \tag{A.53}$$

gives a prod-singular chain (in fact, a cubical chain) on S^1, which by construction has the property that $sing(\kappa) = p(\sigma \otimes \sigma)$. It suffices to show that $sing(\kappa) = 0$ as a singular chain.

Let $\tau : I \times I \to I \times I$ be the map that changes factors. By Lemma A.6, $sing(\kappa) = -sing(\kappa \circ \tau)$.

Denote by ρ the prod-singular chain which reverses the direction of the second factor of $\kappa \circ \tau$, so $\rho = \overline{\kappa \circ \tau}$ in the language of the previous section. Explicitly,

$$\begin{aligned} \rho : I \times I &\to X \\ (s,t) &\mapsto t + (1 - s) \cong t - s \bmod 1 \end{aligned} \tag{A.54}$$

In the complex $\tilde{C}_*(X)$ (where we have quotiented by $D^2_*(X)$), $sing(\kappa) = -sing(\kappa \circ \tau) = sing(\rho)$. Explicitly,

$$sing(\rho) = \rho_\sharp EZ(\eta_1 \otimes \eta_1) \tag{A.55}$$

$$= \rho_\sharp((s_1\eta_1, s_2\eta_1) - (s_2\eta_1, s_1\eta_1)) \tag{A.56}$$

where $\eta_1 : I \to I$ denotes the identity singular chain. In coordinates, we write $\eta_1(t_0, t_1) = t_0$, where on the source we use $I = \Delta^1 = \{(t_0, t_1) | t_0 + t_1 = 1\}$ and on the target we think of $I = [0, 1]$.

Now the singular simplex $\sigma_1 = (s_1\eta_1, s_2\eta_1) : \Delta^2 \to I \times I$ is the map

$$\begin{aligned} (t_0, t_1, t_2) &\mapsto (\eta_1 \circ s^1(t_0, t_1, t_2), \eta_1 \circ s^2(t_0, t_1, t_2)) \\ &\mapsto (\eta_1(t_0 + t_1, t_2), \eta_1(t_0, t_1 + t_2)) \\ &= (t_0 + t_1, t_0) \end{aligned} \tag{A.57}$$

Note that $\rho \circ \sigma_1 : (t_0, t_1, t_2) \mapsto -t_1 \cong t_0 + t_2 \mod 1$. After pre-composing with a cyclic permutation of the vertices (which involves adding a representative of $D^3_*(X)$, $\rho \circ \sigma_1$ is a standard degeneracy, hence zero.

Similarly, $\rho \circ \sigma_2 : (s_2\eta_1, s_1\eta_2) : \Delta^2 \to I \times I \to S^1$, which is the map

$$\begin{aligned}(t_0, t_1, t_2) &\mapsto \rho(\eta_1 \circ s^2(t_0, t_1, t_2), \eta_1 \circ s^1(t_0, t_1, t_2)) \\ &= \rho(\eta_1(t_0, t_1 + t_2), \eta_1(t_0 + t_1, t_2)) \\ &= \rho(t_0 + t_1, t_0) = t_1 \cong t_0 + t_2 \mod 1,\end{aligned} \tag{A.58}$$

is trivial for the same reason. □

We have a zig-zag of quasi-equivalences of dgas:

$$H_*(S^1) \xrightarrow{\iota} \tilde{C}_*(S^1) \xleftarrow{pr} C_*(S^1) \tag{A.59}$$

where the map pr denotes projection. Pull-back induces equivalences of dg module categories

$$C_*(S^1)\text{–mod} \xleftarrow{\iota^*} \tilde{C}_*(S^1)\text{–mod} \xrightarrow{pr^*} C_*(S^1)\text{–mod} \tag{A.60}$$

If X is a space with S^1 action, then the quasi-isomorphism $pr : C_*(X) \to \tilde{C}_*(X)$ is a map of $C_*(S^1)$ modules. Now, $\tilde{C}_*(X)$ is equal to $pr^*\tilde{C}_*(X)$, the pull back of the same chain complex with the induced $\tilde{C}_*(X)$ action under the projection $C_*(S^1) \to \tilde{C}_*(S^1)$.

Separtely, one could consider $\tilde{C}_*(X)$ as an $H_*(S^1)$ module, using the map $\iota : H_*(S^1) \to \tilde{C}_*(S^1)$. The equivalences (A.60) imply that

Corollary A.1. *Let X denote a space with S^1 action. Then $\tilde{C}_*(X)$ carries a $H_*(S^1)$ action given by*

$$\begin{aligned}1 \cdot \sigma &= \sigma \\ \beta \cdot \sigma &= p([S^1], \sigma)\end{aligned} \tag{A.61}$$

Need to define p somewhere.

This action is is equivalent under (A.60) *to $C_*(X)$ with its usual $C_*(S^1)$ action, in particular has the same homotopy fixed points, orbits, etc.*

A.3 Koszul duality, Calabi-Yau properties, and duality of field theories

The goal of this section is to describe how Koszul duality is reflected in cCY and sCY structures. We will then prove a duality relationship between the field theories that Koszul dual cCY and sCY algebras and categories represent.

We begin with the following strengthening of the notion of a smooth differential graded algebra.

Definition 3. *An augmented differential graded algebra A s said to be "strongly smooth" if it has the following two properties:*

1. *A is smooth. That is, A is a perfect $A \otimes A^{op}$-module.*

2. *The ground field k is a perfect A-module.*

Lemma 24. *Let X be a finite, connected, based space. That is, X is equipped with a basepoint $x_0 \in X$ and the pair (X, x_0) is weakly homotopy equivalent to a based finite CW-complex. Then if k is any field, the singular chains of the based loop space $C_*(\Omega X; k)$ is strongly smooth.*

Proof. We think of the based loop space ΩX as maps $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1) = x_0$. Consider the fibration

$$e : \Omega X \to X \quad \text{defined by}$$
$$\alpha \to \alpha(1/2).$$

The fiber of e is $\Omega X \times \Omega X$. The inclusion of the fiber $\iota : \Omega X \times \Omega X \to \Omega X$ is given, up to homotopy by the Pontrjagin loop product (concatenation of loops). This fibration therefore realizes the derived equivalence of chain complexes

$$C_*(\Omega X) \otimes^L_{C_*(\Omega X) \otimes C_*(\Omega X)^{op}} k \simeq C_*(X).$$

In this equivalence all chain complexes are taken with coefficients in the field k. More strongly, using the techniques of twisting (co)chains [9], this fibration yields an equivalence of the chains of the total space $C_*(\Omega X)$ with a chain complex which additively is the tensor product of the chains of the base with the chains of the fiber, $C_*(X) \otimes C_*(\Omega X) \otimes C_*(\Omega X)^{op}$. By this we mean that the tensor product of these chain complexes may have an exotic differential. Furthermore this equivalence is as $C_*(\Omega X) \otimes C_*(\Omega X)^{op}$-modules. Thus $C_*(\Omega X)$ is equivalent to a free $C_*(\Omega X) \otimes C_*(\Omega X)^{op}$-module generated by $C_*(X)$. Since X is finite, $C_*(X)$ is equivalent to a finite chain complex. This proves the lemma. $\square$

One of the main goals of this section is to prove the foliowing.

Theorem 25. *Let A be a strongly smooth differential graded algebra over a field k. Suppose B is a differential graded algebra which is Koszul dual to A. That is,*

$$B \simeq Rhom_A(k,k) \quad \textit{and} \quad A \simeq Rhom_B(k,k).$$

Then A is a sCY-algebra if and only if B is a cCY-algebra.

Proof. We first make the following observations.

- k is a perfect $A \otimes A^{op}$-module. The reason this is true is that since k is a perfect A-module, it has a finite, free, acyclic resolution $C_* \to k$. Then the tensor product of this resolution with itself, $C_* \otimes C_* \to k$ is a finite, free, acyclic resolution of k as an $A \otimes A^{op}$-module.

- B is compact. The reason this is true is that since k is a perfect A-module, the quotient $k \otimes^L_A k$ is a perfect k-module. But this module is equivalent to the dual of $Rhom_A(k,k) = B$, and hence B is a perfect k-module.

Now assume that A is a sCY-algebra. Then there is a cycle in the negative cyclic chains, $\tilde{z} \in CC^-(A)$ so that if z is the image of $\tilde{z}$ under the projection $p : CC^-(A) \to CH_*(A,A)$, then z is a "cotrace", in the sense that the cap product

$$\cap z : Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \to A$$

is an equivalence of $A \otimes A^{op}$-modules. Therefore the map

$$\cap\, z \otimes 1 : Rhom_{A\otimes A^{op}}(A, A \otimes A^{op}) \otimes_{A\otimes A^{op}} k \to A \otimes_{A\otimes A^{op}} k \tag{A.62}$$

is an equivalence. But since k is smooth over A, the left hand side is equivalent to $Rhom_{A\otimes A^{op}}(A,k)$, which in turn is equivalent to $Rhom_A(k,k) \simeq B$. The right hand side is $A \otimes_{A\otimes A^{op}} k$ which is equivalent to $k \otimes_A k$ which is equivalent to the linear dual, $B^* = Hom_k(B,k)$. Thus the map in (A.62) defines an equivalence which we call Φ_z

$$\Phi_z : B \xrightarrow{\simeq} B^*.$$

To understand this equivalence, recall that $z \in CH_*(A,A)$. By the Koszul duality of A and B, the Hochschild chains of A are naturally equivalent to the dual of the Hochschild chains of B: $CH_*(A,A) \simeq Hom_k(CH_*(B,B),k) = Hom_k(B \otimes^L_{B\otimes B^{op}} B, k)$ see [25]. So we can view z as a $B \otimes B^{op}$ - invariant homomorphism

$$z : B \otimes B \to k.$$

The homomorphism Φ_z, which is defined in terms of the cap product with z viewed as an element of $A \otimes^L_{A\otimes A^{op}} A$, is the adjoint of z when viewed as a $B \otimes B^{op}$ - equivariant homomorphism $z : B \otimes B \to k$. Since Φ_z is an equivalence this means that z is a cotrace map. That is $z : B \otimes B \to k$ is homotopy nonsingular.

Finally recall that $z \in CH_*(A,A)$ is in the image of $\tilde{z} \in CC^-(A)$. Consider the homotopy commutative diagram

$$\begin{array}{ccc}
CC^-(A) & \xrightarrow{p} & CH_*(A,A) \\
\simeq\downarrow & & \downarrow = \\
Rhom_{E(\Delta)}(k, CH_*(A,A)) & \longrightarrow & CH_*(A,A) \\
\simeq\downarrow & & \downarrow\simeq \\
Rhom_{E(\Delta)}(k, Hom(CH_*(B,B),k)) & \longrightarrow & Hom_k(CH_*(B,B),k) \\
\cong\downarrow & & \downarrow = \\
Rhom_{E(\Delta)}(CH_*(B,B),k) & \xrightarrow[\phi]{} & Hom_k(CH_*(B,B),k)
\end{array}$$

where $E(\Delta)$ is the exterior algebra on one generator Δ in dimension 1, and the bottom horizontal map ϕ forgets the $E(\Delta)$-equivariance. The lifting of $z \in Hom_k(CH_*(B,B),k)$ to $Rhom_{E(\Delta)}(CH_*(B,B),k)$ in this diagram defines a factorization of $z : CH_*(B,B) \to k$ through the cyclic chains, $CC_*(B) = CH_*(B,B) \otimes^L_{E(\Delta)} k$. Since we already observed that B is compact, this structure then defines the Calabi-Yau (CY) structure on B.

The converse is proved in a similar way. Suppose that B is a CY-algebra. Such a structure is given by a map

$$\tilde{w} : CC_*(B) \simeq CH_*(B,B) \otimes^L_{E(\Delta)} k \to k$$

such that the composition

$$w : CH_*(B,B) \xrightarrow{p} CC_*(B) \simeq CH_*(B,B) \otimes^L_{E(\Delta)} k \xrightarrow{\tilde{w}} k$$

is a trace map. That is, the induced map $w : B \otimes B \to B \otimes^L_{B \otimes B^{op}} B \to k$ is homotopy nonsingular, in the sense that its adjoint

$$\bar{w} : B \to B^* \tag{A.63}$$

is an equivalence.

Now $\tilde{w} \in Rhom_{E(\Delta)}(CH_*(B,B),k) = Rhom_{E(\Delta)}(k, Hom_k(CH_*(B,B),k))$, which by Koszul duality is equivalent to $Rhom_{E(\Delta)}(k, CH_*(A,A)) \simeq CC^-(A)$. So $\tilde{w}$ defines an element, which by abuse of notation we call by the same name, $\tilde{w} \in CC^-(A)$. Its projection to $w \in CH_*(A,A)$ is again defined by Koszul duality, $w \in Hom_k(CH_*(B,B),k) \simeq CH_*(A,A)$. Since we observed that A is smooth, to prove that A is sCY, we need only show that the cap product

$$\cap w : Rhom_{A \otimes A^{op}}(A, A \otimes A^{op}) \to A$$

is an equivalence. This argument essentially reverses the argument given above. Namely, recall that $B = Rhom_A(k,k) \simeq Rhom_{A \otimes A^{op}}(A,k)$. And therefore $B^* \simeq A \otimes^L_{A \otimes A^{op}} k$. With respect to these equivalences, the map $\bar{w}$ given in (A.63), is given by the cap product

$$\cap w : Rhom_{A \otimes A^{op}}(A,k) \xrightarrow{\simeq} A \otimes^L_{A \otimes A^{op}} k.$$

Since A is smooth, this can be rewritten as

$$\cap w : Rhom_{A \otimes A^{op}}(A, A \otimes A^{op}) \otimes_{A \otimes A^{op}} k \xrightarrow{\simeq} (A \otimes^L_{A \otimes A^{op}} A \otimes A^{op}) \otimes_{A \otimes A^{op}} k.$$

So we can conclude that taking the cap product with $w \in CH_*(A,A)$ defines an $A \otimes A^{op}$ - equivariant homomorphism

$$\cap w : Rhom_{A \otimes A^{op}}(A, A \otimes A^{op}) \to A \otimes^L_{A \otimes A^{op}} A \otimes A^{op},$$

which, after applying the derived tensor product $\_ \otimes^L_{A \otimes A^{op}} k$ is an equivalence. The only way this can happen is if $\cap w : Rhom_{A \otimes A^{op}}(A, A \otimes A^{op}) \to A \otimes^L_{A \otimes A^{op}} A \otimes A^{op} \simeq A$ is an equivalence of $A \otimes A^{op}$-modules. This proves that A is sCY. $\square$

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