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MORSE THEORY AND CLASSIFYING SPACES

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§1 Introduction and the statement of the main result

Floer's work in symplectic geometry [2], and instanton homology [3] are two striking new examples of the use of the classical techniques of Morse theory in infinite dimensional settings. The first of these is based on a version of Morse theory on the (free) loop space LM of a symplectic manifold M. The instanton homology of a homology 3-sphere Σ is defined using a similar version of Morse theory on the space of connections on the trivial principal bundle $\Sigma \times SU(2)$.

This paper arose from an attempt to identify and understand the underlying algebraic topological aspects of Floer theory. In studying the homotopy theoretic aspects of this type of infinite dimensional Morse theory, one is naturally led to re-examine finite dimensional Morse theory. The purpose of this paper is to describe a method of processing the data provided by finite dimensional Morse theory in a way that generalises naturally to these infinite dimensional settings. In a sequel we will describe how this method allows us to associate to the data provided by Floer theory suitable spaces whose homotopy types yield invariants which include and generalise the various forms of Floer homology.

The method also gives a particularly clean way of viewing finite dimensional Morse theory. The idea is to associate to a Morse function $f : M \to \mathbb{R}$ on a closed Riemannian manifold M a category $\mathcal{C}_f$ whose objects are the critical points of f. The morphisms between two critical points a and b are, in a natural sense, "piecewise flow lines" of the gradient flow of f which connect a to b. Given a piecewise flow line connecting critical points a and b and one connecting critical points b and c there is an obvious way of joining them to get a piecewise flow line connecting a to c. This defines the composition law in the category $\mathcal{C}_f$.

The goal of this paper is to show how to explicitly recover the topology of M from the category $\mathcal{C}_f$. More precisely, given a topological category $\mathcal{C}$ one can construct its classifying space $B\mathcal{C}$, see [7]. We will describe the classifying space $B\mathcal{C}_f$ in detail in §3 but for now recall that it is a simplicial space whose k-simplices are parameterized by the space of k-tuples of composable morphisms in $\mathcal{C}_f$.

The main theorem of this paper is the following.

Theorem. *Let $f : M \to \mathbb{R}$ be a Morse function defined on a closed Riemannian manifold M. Then associated to f is a topological category $\mathcal{C}_f$ whose objects are the critical points of f and whose space of morphisms between critical points a and b is the space $\bar{\mathcal{M}}(a,b)$ of piecewise flow lines, of the gradient flow of f, joining a to b.*

(1) If f is a generic Morse function (one whose gradient flow satisfies the Morse-Smale transversality condition) there is a homeomorphism

$$B\mathcal{C}_f \cong M.$$

(2) For any Morse function there is a homotopy equivalence

$$B\mathcal{C}_f \simeq M.$$

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Observe that the classifying space of a category is given by an explicit construction and so the theorem gives a precise, combinatorial method of constructing M as a simplicial space from the critical points and flow lines of a Morse function. A striking feature of the construction is that in the generic case it gives M up to homeomorphism not just up to homotopy type.

We now recall some basic notation and terminology. Let M be a closed Riemannian manifold and let $f : M \to \mathbb{R}$ be a Morse function (that is all the critical points of f are isolated and non-degenerate). Let $\operatorname{grad}(f)$ be the gradient vector field of f. The flow-lines of f are the curves $\gamma(t)$ in M which satisfy the differential equation

$$\frac{d\gamma}{dt} = -\operatorname{grad}(f). \tag{1.1}$$

If γ is a flow-line then $\gamma(t)$ converges to critical points of f as $t \to \pm\infty$ and we define

$$s(\gamma) = \lim_{t\to-\infty} \gamma(t), \qquad e(\gamma) = \lim_{t\to\infty} \gamma(t).$$

For any point $x \in M$, let γ_x be the unique flow line satisfying the initial condition $\gamma_x(0) = x$. For a critical point a we denote the stable manifold and unstable manifolds of a by $W^s(a)$ and $W^u(a)$, that is

$$\begin{aligned} W^s(a) &= \{x \in M : e(\gamma) = a\} \\ W^u(a) &= \{x \in M : s(\gamma) = a\}. \end{aligned}$$

It is a standard fact from Morse theory that the unstable manifold is diffeomorphic to an open disk of dimension λ_a and the stable manifold is diffeomorphic to an open disk of dimension $n - \lambda_a$ where $n = \dim M$ and λ_a is the index of a. We use the notation

$$W(a,b) = W^u(a) \cap W^s(b)$$

for the space of points which lie on flow-lines from a to b.

The gradient flow of f satisfies the **Morse-Smale condition** if for any two critical points a and b the manifolds $W^u(a)$ and $W^s(b)$ intersect transversely and a Morse function whose gradient flow satisfies the Morse-Smale condition will be called a **Morse-Smale function**. Given that f is Morse-Smale it follows that $W(a,b)$ is a submanifold of M of dimension $\lambda_a - \lambda_b$.

The space $W(a,b)$ has a free action of $\mathbb{R}$ given by the flow of $\operatorname{grad}(f)$. Indeed if we pick any point r between $f(a)$ and $f(b)$ and set $W^r(a,b)$ to be the submanifold $W(a,b) \cap f^{-1}(r) \subset M$ then there is a natural diffeomorphism

$$W^r(a,b) \times \mathbb{R} \to W(a,b), \qquad (x,t) \mapsto \gamma_x(t).$$

Therefore we may form the quotient space

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}$$

and $\mathcal{M}(a,b)$ is diffeomorphic to $W^r(a,b)$. We refer to this space $\mathcal{M}(a,b)$ is the **moduli space of flow-lines** from a to b. If f is Morse-Smale then $\mathcal{M}(a,b)$ is a smooth manifold of dimension $\lambda_a - \lambda_b - 1$.

We now construct the category $\mathcal{C}_f$ referred to in the above theorem. Since f is strictly decreasing along flow lines it defines a diffeomorphism of the flow line $\gamma(t)$ with the open interval $(f(b), f(a))$ where $s(\gamma) = a$ and $e(\gamma) = b$. This reparameterisez the flow-line as a smooth function

$$\omega : (f(b), f(a)) \to M$$

such that

$$f(\omega(t)) = t.$$

We can extend ω to a smooth function defined on $[f(b), f(a)]$ by setting $\omega(f(b)) = b$ and $\omega(f(a)) = a$. This extended function satisfies the differential equation

$$\frac{d\omega}{dt} = -\frac{\mathrm{grad}(f)}{\|\,\mathrm{grad}(f)\|^2} \tag{1.2}$$

with boundary conditions

$$\omega(f(b)) = b, \qquad \omega(f(a)) = a. \tag{1.3}$$

We define $\bar{\mathcal{M}}(a,b)$ to be the space of all continuous curves in M which are smooth on the complement of the critical points of f and satisfy (1.2) and (1.3). Here, of course, we understand that ω satisfies (1.2) on the complement of the set of critical points of f. This space $\bar{\mathcal{M}}(a,b)$ is topologized as a subspace of the space $\mathrm{Map}([f(b), f(a)], M)$, of all continuous maps with the compact open topology. Note that if ω is any solution of (1.2) and (1.3) then if we remove the points where $\omega(t)$ is a critical point of f each component of ω is geometrically a flow-line but it is parameterized so that $f(\omega(t)) = t$. Thus we use the natural terminology and refer to a curve in $\bar{\mathcal{M}}(a,b)$ as a **piecewise flow-line** from a to b.

It is straightforward to check that $\bar{\mathcal{M}}(a,b)$ is a compact space and it clearly contains $\mathcal{M}(a,b)$. We show in §2 and §6 that if f is Morse-Smale then $\mathcal{M}(a,b)$ is open and dense in $\bar{\mathcal{M}}(a,b)$ and so $\bar{\mathcal{M}}(a,b)$ is a compactification of the moduli space of flow lines $\mathcal{M}(a,b)$.

There is an obvious associative composition law

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c)$$

which is denoted by $\gamma_1 \circ \gamma_2$. We now define the category $\mathcal{C}_f$ as follows:

The objects of $\mathcal{C}_f$. The objects of $\mathcal{C}_f$ are the critical points of f.

The morphisms of $\mathcal{C}_f$. If a and b are distinct critical points of f then the morphisms from a to b are defined to be

$$\mathcal{C}_f(a,b) = \bar{\mathcal{M}}(a,b).$$

The only morphism from a to itself is the identity.

The composition law. The composition law is defined by

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c) \qquad (\gamma_1, \gamma_2) \mapsto \gamma_1 \circ \gamma_2.$$

In fact $\mathcal{C}_f$ is a topological category in the sense that each of the sets $\mathcal{C}_f(a,b)$ comes equipped with a natural topology and the composition law

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c)$$

is continuous. The topological category $\mathcal{C}_f$ has a classifying space $B\mathcal{C}_f$, described in detail in §3.

Our main results are:

(1) There is a homotopy equivalence $M \simeq B\mathcal{C}_f$.
(2) If f is Morse-Smale then there is a homeomorphism $M \cong B\mathcal{C}_f$.

The proof of (1) is reasonably direct and is given in §7. The majority of this paper is taken up in proving (2). The strategy for this proof is to first build a model for M out of the compactified spaces of flow-lines $\bar{\mathcal{M}}(a,b)$ and then to use a "cut and paste" argument to show that this model is in fact homeomorphic to $B\mathcal{C}_f$. The essential ingredients in the argument are an analysis of the ends of the moduli space of flow-lines $\mathcal{M}(a,b)$ and a gluing construction for flow-lines.

This construction only uses the compactified moduli spaces $\bar{\mathcal{M}}(a,b)$ rather than the stable and unstable manifolds. Thus it applies in Floer theory in infinite dimensions where the the stable and unstable manifolds are typically infinite dimensional and so the usual method of constructing a CW-complex using the unstable (or stable) manifolds does not apply. Note that in Floer theory the moduli spaces of flow lines are in fact finite dimensional.

Observe that the category $\mathcal{C}_f$ has a natural filtration defined by the index of the critical points. This induces a filtration of the classifying space and thus a spectral sequence which, in view of our main theorem, converges to the homology of M. We show that the E_1 term of this spectral sequence is the classical Morse chain complex of f. Thus the E_2 term is the homology of M and the spectral sequence collapses. This fact is special to finite dimensions. In the infinite dimensional setting of Floer theory the index filtration yields Floer-type homology groups, which are essentially the E_2 term of the associated spectral sequence.

In [8], Smale defines a partial order $<$ on the set of critical points of a Morse-Smale function as follows: $a < b$ if and only if there is a flow line from a to b. Our category $\mathcal{C}_f$ is related to this partial order in the following way. Let $\mathcal{P}_f$ be the partially ordered set consisting of the critical points of f with the partial ordering $<$. We can regard $\mathcal{P}_f$ as a category with objects the critical points of f and a unique morphism from a to b if $a = b$ or $a < b$. There is an obvious functor $\mathcal{C}_f \to \mathcal{P}_f$. Thus $\mathcal{C}_f$ is a refinement of $\mathcal{P}_f$ which makes sense even if f does not satisfy the Morse-Smale condition.

In [6], Robbin and Salamon consider the simplicial complex defined by the partially ordered set $\mathcal{P}_f$. In our terms this simplicial complex is the classifying space $B\mathcal{P}_f$ of the category $\mathcal{P}_f$. They construct a map, which they refer to as a Lyapunov map,

$$M \to B\mathcal{P}_f.$$

Presumably their map is related to our main theorem in the following way. The functor $\mathcal{C}_f \to \mathcal{P}_f$ induces a map of classifying spaces and we can compose this map with the above homeomorphism

$$M \to B\mathcal{C}_f \to B\mathcal{P}_f.$$

It seems natural to expect that this map is the map constructed by Robbin and Salamon.

The organization of this paper is as follows. In §2 we examine the ends of the moduli space of flows, state the main gluing theorem, and use it to describe a particular combinatorial model $\mathcal{R}_f$ of the manifold. In §3 we discuss the classifying space of $\mathcal{C}_f$ and prove the main theorem by showing that $B\mathcal{C}_f$ is homeomorphic to $\mathcal{R}_f$ and therefore to M. In §4 we study the precise relationship between the simplicial space $B\mathcal{C}_f$ and the usual CW-complex, $C(f) \simeq M$, defined by the Morse function f. In §5 we describe the details of the gluing constructions. The particular approach we take toward this gluing was worked out by Marty Betz and will appear in his thesis [1]. Finally in §6 we prove that $B\mathcal{C}_f$ is homotopy equivalent to M for a general Morse function.

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§2 The ends of the moduli space of flow lines

Throughout this section we assume that M is a closed Riemannian manifold and that $f : M \to \mathbb{R}$ is a Morse-Smale function. In this section we describe the ends of the moduli space of flow-lines $\mathcal{M}(a, b)$ and relate its compactification to the space of piecewise flow-lines described in the last section. We then use this analysis of the ends of $\mathcal{M}(a, b)$ to construct a combinatorial model $\mathcal{R}_f$ for the manifold M. We show that $\mathcal{R}_f$ is homeomorphic to the classifying space $B\mathcal{C}_f$ in the next section.

Using Smale's partial ordering described in §1, we say that a sequence $\mathbf{a} = (a_0, \cdots, a_{l+1})$ of critical points is **ordered** if $a_i > a_{i+1}$ for all i. Given such a sequence we define

$$\begin{aligned} s(\mathbf{a}) &= a_0, \\ e(\mathbf{a}) &= a_{l+1}, \\ l(\mathbf{a}) &= l, \\ \mathcal{M}(\mathbf{a}) &= \mathcal{M}(a_0, a_1) \times \cdots \times \mathcal{M}(a_l, a_{l+1}). \end{aligned}$$

We now describe the ends of the spaces of flow-lines $\mathcal{M}(a, b)$ in terms of a gluing construction.

Theorem 2.1. *There exists an $\varepsilon > 0$ and maps*

$$\mu : (0, \varepsilon] \times \mathcal{M}(a, a_1) \times \mathcal{M}(a_1, b) \longrightarrow \mathcal{M}(a, b),$$

which we write as

$$(t, \gamma_1, \gamma_2) \longrightarrow \gamma_1 \circ_t \gamma_2,$$

such that:

(1) *The map μ satisfies the following associativity law*

$$(\gamma_1 \circ_s \gamma_2) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_2 \circ_t \gamma_3)$$

for all s, $t \leq \varepsilon$.

(2) *Let $\mathbf{a}$ be an ordered sequence with $s(\mathbf{a}) = a$, $e(\mathbf{a}) = b$, and $l(\mathbf{a}) = l$. Then the map*

$$\mu : (0, \varepsilon]^l \times \mathcal{M}(\mathbf{a}) \longrightarrow \mathcal{M}(a, b)$$

defined by

$$(s_1, \ldots, s_l; \gamma_0, \ldots, \gamma_l) \mapsto \gamma_0 \circ_{s_1} \gamma_1 \circ_{s_2} \cdots \circ_{s_l} \gamma_l$$

is a diffeomorphism onto its image.

(3) *Define $\mathcal{K}(a, b) \subset \mathcal{M}(a, b)$ to be*

$$\mathcal{K}(a, b) = \mathcal{M}(a, b) - \bigcup \mu\left((0, \varepsilon)^l \times \mathcal{M}(\mathbf{a})\right)$$

where the union is taken over all ordered sequences $\mathbf{a}$ *with* $s(\mathbf{a}) = a$, $e(\mathbf{a}) = b$, *and* $l(\mathbf{a}) \geq 1$. *Then* $\mathcal{K}(a,b)$ *is compact.*

(4) *There are homeomorphisms*

$$\bar{\mathcal{M}}(a,b) \cong \mathcal{M}(a,b) \cup_{\mu} \bigcup [0,\varepsilon]^l \times \mathcal{M}(\mathbf{a})$$
$$\bar{\mathcal{M}}(a,b) \cong \mathcal{K}(a,b).$$

We discuss this result in §6.

Theorem (2.1) shows that the ends of the moduli space $\mathcal{M}(a,b)$ consist of unions of half-open cubes parameterized by composable sequences of flow lines. The compact space $\mathcal{K}(a,b)$ is formed by removing the associated **open** cubes. The compactification $\bar{\mathcal{M}}(a,b)$ is formed by formally **closing** the cubes or, equivalently, by formally adjoining the piecewise flows as described in the previous section. It follows that $\mathcal{K}(a,b)$ and $\bar{\mathcal{M}}(a,b)$ are homeomorphic.

It also follows from Theorem (2.1) that

$$\lim_{t \to 0} \gamma_1 \circ_t \gamma_2 = \gamma_1 \circ \gamma_2$$

where $\circ$ is the composition of piecewise flow-lines described in §1. In view of this fact we will often use the notation $\circ_0$ for $\circ$.

The homeomorphism between $\mathcal{K}(a,b)$ and $\bar{\mathcal{M}}(a,b)$ allows us to define the category $\mathcal{C}_f$ in two equivalent (isomorphic) ways. The first way, described in the previous section, is to define the space of morphisms between critical points a and b to be $\bar{\mathcal{M}}(a,b)$, and the composition law is given by

$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c), \qquad (\gamma_1,\gamma_2) \to \gamma_1 \circ_0 \gamma_2.$$

The second is to define the space of morphisms between critical points a and b to be $\mathcal{K}(a,b)$, and this time the composition law is given by

$$\mathcal{K}(a,b) \times \mathcal{K}(b,c) \to \mathcal{K}(a,c), \qquad (\gamma_1,\gamma_2) \to \gamma_1 \circ_\varepsilon \gamma_2.$$

We now use Theorem (2.1) to produce a combinatorial model $\mathcal{R}_f$ of the manifold M. We begin by describing a filtration of the spaces $\bar{\mathcal{M}}(a,b)$. By scaling if necessary, we can assume that the constant ε in the statement of Theorem 2.1 is 1. If $\gamma_1 \in \mathcal{M}(a,a_1)$ and $\gamma_2 \in \mathcal{M}(a_1,b)$, then the parameter $t \in (0,1]$ in the flow $\gamma_1 \circ_t \gamma_2 \in \mathcal{M}(a,b)$ can be viewed as a measure of how close this flow comes to the critical point a_1. This interpretation will become clearer in the proof. Thus the fact that the pairing μ is a diffeomorphism onto its image allows us to view the space $\mathcal{K}(a,b)$ as the space of flows that stay at least 1 away from all critical points other than a and b (in this undefined measure).

Next we look at the curves in $\bar{\mathcal{M}}(a,b)$ which get within distance 1 of at most one intermediate critical point. More generally we can filter the space $\bar{\mathcal{M}}(a,b)$ by saying that a curve in $\bar{\mathcal{M}}(a,b)$ has filtration k if it gets within distance less than 1 of at most k intermediate critical points. We now make this description precise.

For any ordered sequence $\mathbf{a} = (a_0, \dots, a_{l+1})$ of critical points define

$$\mathcal{K}(\mathbf{a}) = \mathcal{K}(a_0, a_1) \times \cdots \times \mathcal{K}(a_l, a_{l+1}).$$

Now define

$$\begin{aligned} \mathcal{K}^{(0)}(a,b) &= \mathcal{K}(a,b) \\ \mathcal{K}^{(k)}(a,b) &= \bigcup_{s(\mathbf{a})=a, e(\mathbf{a})=b, l(\mathbf{a})\leq k} \mu\left([0,1]^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a})\right). \end{aligned}$$

In the definition of $\mathcal{K}^{(k)}(a,b)$ we interpret $\mu\left([0,1]^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a})\right)$ in the case where $l(\mathbf{a}) = 0$, that is $\mathbf{a}$ is the ordered sequence (a,b), to be $\mathcal{K}(a,b)$. Thus

$$\mathcal{K}^{(k-1)}(a,b) \subset \mathcal{K}^{(k)}(a,b),$$

and γ is in $\mathcal{K}^{(k)}(a,b)$ if and only if γ can be decomposed as

$$\gamma = \gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l$$

where $\gamma_i \in \mathcal{K}(a_i, a_{i+1})$, $0 \leq s_i \leq 1$, and $l \leq k$.

Notice that

(1)
$$\bigcup \mathcal{K}^{(k)}(a,b) = \mathcal{M}(a,b).$$

(2)
$$\mathcal{K}^{(k)}(a,b) - \mathcal{K}^{(k-1)}(a,b) \cong \coprod_{s(\mathbf{a})=a, e(\mathbf{a})=b, l(\mathbf{a})=k} [0,1)^k \times \mathcal{K}(\mathbf{a}),$$

and from (1) it follows that the map

$$\coprod_{s(\mathbf{a})=a, e(\mathbf{a})=b} [0,1]^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a}) \longrightarrow \bar{\mathcal{M}}(a,b)$$

defined by

$$(s_1, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \longrightarrow \gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l$$

is onto. Therefore $\bar{\mathcal{M}}(a,b)$ can be recovered by imposing an equivalence relation on the above disjoint union. From (2) it follows that this equivalence relation is generated by

$$(s_1, \dots, s_{i-1}, 1, s_{i+1}, \dots, s_l; \gamma_1, \dots, \gamma_l) \simeq \\ (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_l; \gamma_1, \dots, \gamma_{i-1} \circ_1 \gamma_i, \dots, \gamma_l).$$

Note that the relations only involve the faces of the cubes $[0,1]^{l(\mathbf{a})}$ which do not contain the point $(0, \dots, 0)$. From this argument we draw the following conclusion.

Theorem 2.2.

$$\bar{\mathcal{M}}(a,b) = \coprod_{\mathbf{a}} [0,1]^l \times \mathcal{K}(\mathbf{a})/\simeq$$

The next step is to go from this description of the spaces $\bar{\mathcal{M}}(a,b)$ to one of the manifold M. Recall that, by definition, $\bar{\mathcal{M}}(a,b)$ consists of continuous curves

$$\gamma : [f(b), f(a)] \to M$$

which satisfy (1.2) and (1.3). Thus we get a map

$$\varphi : [f(b), f(a)] \times \bar{\mathcal{M}}(a,b) \to M$$

whose image is the closure of the space $W(a,b) \subset M$.

Let us simplify the notation slightly by writing

$$J_{\mathbf{a}} = [f(e(\mathbf{a})), f(s(\mathbf{a}))], \qquad I^{\mathbf{a}} = [0,1]^{l(\mathbf{a})}.$$

Then the previous observation shows that the map

$$\coprod_{\mathbf{a}} J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a}) \to M$$

defined by

$$(t; s_1, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \longrightarrow (\gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l)(t)$$

is onto. Once more it is not difficult to extract the appropriate equivalence relation on the disjoint union.

Define

$$\mathcal{R}_f = \coprod_{\mathbf{a}} J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a})/\sim \tag{2.3}$$

where the relations $\sim$ are given by

$$\begin{aligned}(t; s_1, \cdots, s_{i-1}, 1, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \\ (t; s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_0, \cdots, \gamma_{i-1} \circ_1 \gamma_i, \cdots, \gamma_l)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}(t; s_1, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \\ \begin{cases} (t; s_1, \cdots, s_{i-1}; \gamma_0, \cdots, \gamma_{i-1}), & \text{if } t \in [f(a_i), f(a_0)] \\ (t; s_{i+1}, \cdots, s_l; \gamma_{i+1}, \cdots, \gamma_l), & \text{if } t \in [f(a_{l+1}), f(a_i)] \end{cases}\end{aligned} \tag{2.5}$$

The map φ respects the equivalence relation $\sim$ so gives a well defined map

$$\mathcal{R}_f \longrightarrow M.$$

An elementary analysis now shows that all the identifications which can take place are consequences of (2.4) and (2.5). This leads to the following theorem.

Theorem 2.6. *The map*

$$\varphi : \mathcal{R}_f \to M$$

is a homeomorphism.

§3 The Classifying space of a Morse function

In this section we continue to assume that $f : M \to \mathbb{R}$ is a Morse-Smale function. The goal of this section is to prove the following theorem.

Theorem 3.1. *There is a natural homeomorphism*

$$\psi : \mathcal{R}_f \to B\mathcal{C}_f.$$

where $\mathcal{R}_f$ is as in (2.3).

In view of Theorem (2.6) this shows that $B\mathcal{C}_f$ is homeomorphic to M.

Recall from [7] that the classifying space of $\mathcal{C}_f$ is given by

$$B\mathcal{C}_f = \coprod_{\mathbf{a}} \Delta^{l(\mathbf{a})+1} \times \bar{\mathcal{M}}(\mathbf{a}) / \sim$$

where Δ^n is the standard n-simplex. The identifications $\sim$ are given by the following rules. If $t \in \Delta^{l(\mathbf{a})}$ and $x \in \bar{\mathcal{M}}(\mathbf{a})$, then

$$(t, d_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{l(\mathbf{a})+2}$ and $x \in \bar{\mathcal{M}}(\mathbf{a})$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

Here

(1) $\delta_i : \Delta^n \to \Delta^{n+1}$ is the inclusion of the i-th face;
(2) $\sigma_j : \Delta^{n+1} \to \Delta^n$ is the j-th degeneracy, given by projecting linearly onto the j-th face;
(3) $d_i : \bar{\mathcal{M}}(a_0, \dots, a_{l+1}) \to \bar{\mathcal{M}}(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{l+1})$ is given by

$$d_i(\gamma_0, \dots, \gamma_l) = \begin{cases} (\gamma_1, \cdots, \gamma_n) & \text{for } i = 0 \\ (\gamma_0, \cdots, \gamma_i \circ \gamma_{i+1}, \cdots \gamma_n) & \text{for } 1 \leq i \leq l \\ (\gamma_0, \cdots, \gamma_{l-1}) & \text{for } i = l. \end{cases}$$

(4) $s_j : \bar{\mathcal{M}}(a_0, \dots, a_{l+1}) \to \bar{\mathcal{M}}(a_0, \dots, a_j, a_j, \dots, a_{l+1})$ is given by

$$s_j(\gamma_0, \cdots, \gamma_l) = (\gamma_0, \cdots \gamma_j, 1, \gamma_{j+1}, \cdots, \gamma_l)$$

Recall that $\mathcal{R}_f$ is the union of spaces of the form

$$J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a})$$

where $\mathbf{a} = (a_0, \cdots, a_{l+1})$ is an ordered sequence of critical points. Recall also that the spaces $\mathcal{K}(a_{i-1}, a_i)$ are homeomorphic to the compactified spaces $\bar{\mathcal{M}}(a_{i-1}, a_i)$ in such a way that the composition in the category corresponds to $\circ_1$. Thus the construction of $\mathcal{R}_f$ is very similar to that of $B\mathcal{C}_f$, the main difference is that $\mathcal{R}_f$ is constructed from the cubes $J_{\mathbf{a}} \times I^{\mathbf{a}}$ whereas $B\mathcal{C}_f$ is constructed from simplices. The main point in the argument to prove Theorem (3.1) is to show that the equivalence relations used to define $\mathcal{R}_f$ can be imposed in two steps; the first step turns the cubes into simplices and the second step imposes the identifications among the simplices that make up $B\mathcal{C}_f$.

Proof of 3.1. First we look at the image of a single cube

$$J_{\mathbf{a}} \times I^{\mathbf{a}} = J_{\mathbf{a}} \times I^{\mathbf{a}} \times (\gamma_0, \cdots, \gamma_l),$$

where $\mathbf{a}$ is the ordered sequence $(a_0, \ldots, a_{l+1})$, in the quotient space $\mathcal{R}_f$. For each i with $0 \leq i \leq l+1$ define

$$\mathbf{a}_i = (a_0, \cdots a_{i-1}, a_{i+1}, \cdots a_{l+1})$$

and a map

$$\partial_i : J_{\mathbf{a}_i} \times I^{l-1} \longrightarrow J_{\mathbf{a}} \times I^l$$

by the formula

$$\partial_i(t; s_1, \cdots, s_{l-1}) = \begin{cases} (t; 0, s_1, \cdots, s_{l-1}), & \text{if } i = 0 \\ (t; s_1, \cdots, s_{i-1}, 1, s_i, \cdots, s_{l-1}), & \text{for } 1 \leq i \leq l \\ (t; s_1, \cdots, s_{l-1}, 0), & \text{if } i = l+1. \end{cases}$$

Now consider the spaces

$$J_{\mathbf{a}} \times I^{l-1} / \sim$$

where we make the following list of identifications: If $1 \leq i \leq l$, so that $J_{\mathbf{a}} = J_{\mathbf{a}_i}$, then

$$\begin{aligned} (3.2) \quad & (t; s_1, \cdots s_{i-1}, 0, s_{i+1}, \cdots, s_l) \\ & \sim \begin{cases} (t; s_1, \cdots, s_{i-1}, 0, s'_{i+1}, \cdots s'_l) & \text{if } f \in [f(a_i), f(a_0)] \\ (t; s'_1, \cdots, s'_{i-1}, 0, s_{i+1}, \cdots, s_l), & \text{if } t \in [f(a_{l+1}, f(a_i)]; \end{cases} \end{aligned}$$

if $i = 0, l+1$ then

$$(3.3) \qquad \begin{aligned} (t; 0, s_2, \cdots, s_l) \sim (t; 0, s'_2, \cdots s'_l) \quad & \text{if } t \in [f(a_1), f(a_0)] \\ (t; s_1, \cdots, s_{l-1}, 0) \sim (t; s'_1, \cdots, s'_{l-1}, 0) \; & \text{if } t \in [f(a_{l+1}, a_l)]; \end{aligned}$$

finally

$$(3.4) \qquad \begin{aligned} (f(a_{l+1}); s_1, \cdots, s_l) &\sim (f(a_{l+1}); s'_1, \cdots, s'_l) \\ (f(a_0; s_1, \cdots, s_l) &\sim (f(a_0); s'_1, \cdots, s'_l). \end{aligned}$$

It is straightforward to check that if two points in $J_{\mathbf{a}} \times I^{\mathbf{a}}$ are identified then they have the same image in $\mathcal{R}_f$. So we can construct $\mathcal{R}_f$ from the spaces $J_{\mathbf{a}} \times I^{\mathbf{a}} / \sim$. However the space $J_{\mathbf{a}} \times I^l / \sim$ is naturally homeomorphic to an $(l+1)$-simplex, and using these homeomorphisms the map ∂_i corresponds to the map δ_i, that is the inclusion of the i-th face. More precisely, we have the following combinatorial result, whose verification is straightforward.

Lemma 3.5. *There are homeomorphisms*

$$h_{\mathbf{a}} : J_{\mathbf{a}} \times I^{\mathbf{a}}/\sim \;\to \Delta^{l+1}$$

which make the following diagrams commute

$$\begin{array}{ccc} J_{\mathbf{a}} \times I^{\mathbf{a}}/\sim & \xrightarrow{h_{\mathbf{a}}} & \Delta^{l+1} \\ {\scriptstyle \partial_i}\uparrow & & \uparrow{\scriptstyle \delta_i} \\ J_{\mathbf{a}_i} \times I^{\mathbf{a}_i}/\sim & \xrightarrow[h_{\mathbf{a}_i}]{} & \Delta^l. \end{array}$$

At this stage we have used up the first relations (2.5) in the definition (2.4) of $\mathcal{R}_f$. Now we impose the relations (2.4) to get the following result. Once more the proof is straightforward.

Lemma 3.6. *There is a homeomorphism*

$$\mathcal{R}_f \cong \coprod_{\mathbf{a}} \Delta^{l(\mathbf{a})+1} \times \mathcal{K}(\mathbf{a})/\sim$$

where, in the coordinates for Δ^n given by

$$\Delta^n = \left\{ (s_1, \dots, s_n) \in \mathbb{R}^n : 0 \leq s_i \leq 1, \text{ and } \sum_{i=1}^n s_i \leq 1 \right\},$$

$$(s_0, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim$$
$$\begin{cases} (s_1, \cdots s_l; \gamma_1, \cdots, \gamma_l) & \text{if } i = 0 \\ (s_0, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_{i-1} \circ_1 \gamma_i, \cdots \gamma_l) & \text{if } 1 \leq i \leq l-1 \\ (s_0, \cdots, s_{l-1}; \gamma_0, \cdots, \gamma_{l-1}) & \text{if } i = l \end{cases}$$

We can now complete the proof of Theorem (3.1). First we must regard the category $\mathcal{C}_f$ as the category with spaces of morphisms $\mathcal{K}(a, b)$ and composition law defined by $\circ_1$. Now recall that we are assuming that the sequence $\mathbf{a}$ is strictly ordered, that is $\mathbf{a} = (a_0, \dots, a_{l+1})$ with $a_i > a_{i+1}$ using Smale's partial ordering. If we now compare the model for $\mathcal{R}_f$ given by Lemma (3.6) with the definition of $B\mathcal{C}_f$ we see that the difference is that in Lemma (3.6) we have used the space of **non-degenerate** simplices rather than the space of all simplices. Thus using Lemma (3.6) we have constructed a map

$$\mathcal{R}_f \to B\mathcal{C}_f.$$

Now using the following two properties of $\mathcal{C}_f$

(1) the only morphism in $\mathcal{C}_f$ from the object a to itself is the identity,
(2) if $\alpha_1, \beta_1 : a \to b$, $\alpha_2, \beta_2 : b \to c$ are morphisms in $\mathcal{C}_f$ such that

$$\alpha_2 \circ \alpha_1 = \beta_2 \circ \beta_1$$

then it follows that

$$\alpha_1 = \beta_1, \qquad \alpha_2 = \beta_2,$$

one simply verifies that the map

$$\mathcal{R}_f \to B\mathcal{C}_f.$$

is a bijection with an obviously continuous inverse. $\square$

We end this section with an illustrative example. Let M be the torus $M = S^1 \times S^1$. Let $f : M \to \mathbb{R}$ be the height function of the torus which is tilted slightly off its vertical axis. That is, think of the torus standing on the floor at an angle slightly less than $\pi/2$ with the floor. (We cannot use the usual height function on the vertical torus since it does not satisfy the Morse-Smale transversality condition.) See Diagram (3.7) for the flow defined by the height function on the tilted torus.

Diagram (3.7)

There are four critical points: a (index 2), b, c (index 1), and d (index 0). As the figure shows, the moduli spaces $\mathcal{M}(a,b)$, $\mathcal{M}(a,c)$, $\mathcal{M}(b,d)$, and $\mathcal{M}(c,d)$ each consist of two distinct points. We denote these flows by α_i, β_i, γ_i, and δ_i respectively. A point on the torus not lying on any of these flows is on a flow in $\mathcal{M}(a,d)$. The space $\mathcal{M}(a,d)$ is 1-dimensional; it is the disjoint union of four open intervals. Thus $\mathcal{K}(a,d) \cong \bar{\mathcal{M}}(a,b)$ is the disjoint union of four closed intervals. If the torus is viewed in the usual way as a square with opposite sides identified, then the flow can be drawn as in Diagram (3.8).

DIAGRAM (3.8)

Now consider the simplicial description of the classifying space $B\mathcal{C}_f$. The vertices correspond to the objects of the category $\mathcal{C}_f$, that is the critical points, and so there are four vertices. There is one 1-simplex (interval) for each morphism (flow line), glued to the vertices corresponding to the start and end of the flow. The points in $\mathcal{K}(a, d) \cong \bar{\mathcal{M}}(a, d)$ index a 1-parameter family of 1-simplices attached to the vertices labelled a and d. There is a 2-simplex for every pair of composable flows. There are eight such pairs (coming from the four points in each of the product moduli spaces $\mathcal{M}(a, b) \times \mathcal{M}(b, d)$ and $\mathcal{M}(a, c) \times \mathcal{M}(c, d)$.) A 2-simplex labelled by a pair of flows, say (α, β) will have its three faces identified with the 1-simplices labelled by α, β, and $\alpha \circ_1 \beta$ respectively. Diagram (3.9) illustrates the resulting decomposition of the classifying space and gives an explicit example of the homeomorphism between the classifying space of $\mathcal{C}_f$ and the underlying manifold M.

DIAGRAM (3.9)

§4 Relationship with the Morse complex

One of the classical results of Morse theory is that if M is a closed manifold and $f : M \to \mathbb{R}$ is a Morse function, then M is homotopy equivalent to a CW-complex $C(f)$ with one cell of dimension r for each critical point of index r. The associated cellular chain complex is the **Morse complex**

$$\cdots \xrightarrow{\partial} C_r(f) \xrightarrow{\partial} C_{r-1}(f) \xrightarrow{\partial} \cdots$$

where $C_r(f)$ is the free abelian group generated by the critical points of index r. The homology of the Morse complex is $H_*(M)$. In this section we show how to obtain the Morse complex from the classifying space $B\mathcal{C}_f$. We assume that we have chosen a metric in M such that the gradient flow of f satisfies the Morse-Smale transersality condition.

For each integer $k \geq 0$ let $\mathcal{C}_f^k$ be the full subcategory of $\mathcal{C}_f$ whose objects are the critical points a of f with $\lambda_a \leq k$. Here, as in §1, λ_a is the index of the critical point a. The term full means that the space of morphisms between any two objects a and b in the subcategory $\mathcal{C}_f^k$ is the same as the space of morphisms in $\mathcal{C}_f$. On the level of classifying spaces we get a filtration

$$B\mathcal{C}_f^0 \subset B\mathcal{C}_f^1 \subset \cdots \subset B\mathcal{C}_f^k \subset B\mathcal{C}_f^{k+1} \subset \cdots \subset B\mathcal{C}_f \cong M.$$

We refer to this filtration as the **index filtration**. The index filtration gives a spectral sequence converging to $H_*(M)$, which we refer to as the **index spectral sequence**.

We use the notation Crit_r for the set of critical points of f with index r.

Theorem 4.1.

(1) *There is a homotopy equivalence*

$$B\mathcal{C}_f^k / B\mathcal{C}_f^{k-1} \simeq \bigvee_{a \in \mathrm{Crit}_k} S_a^k.$$

(2) *In the index spectral sequence*

$$E_1^{r,s} = H_{r+s}(B\mathcal{C}_f^r, B\mathcal{C}_f^{r-1}) = \begin{cases} \bigoplus_{Crit_r} \mathbb{Z} & \text{if } s = 0 \\ 0 & \text{if } s > 0. \end{cases}$$

and the E_1 term

$$\cdots \xrightarrow{d_1} E_1^{r,0} \xrightarrow{d_1} E_1^{r-1,0} \xrightarrow{d_1} \cdots$$

is the Morse complex

$$\cdots \xrightarrow{\partial} C_r(f) \xrightarrow{\partial} C_{r-1}(f) \xrightarrow{\partial} \cdots .$$

It follows from Theorem (4.1) that

$$E_2^{r,s} = \begin{cases} H_r(M) & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{cases}$$

and the spectral sequence collapses at E_2.

To prove Theorem (4.1) we recall some standard material from Morse theory on how the function f gives rise to a CW-decomposition of M , see [4].

Let $f : M \to \mathbb{R}$ be a Morse function and, for $t \in \mathbb{R}$, let

$$M^t = f^{-1}(-\infty, t] \subset M.$$

Let $c \in \mathbb{R}$ be a critical value and let $a_1, \cdots, a_k$ be the critical points with $f(a_i) = c$. Let λ_i be the index of a_i. Choose $\varepsilon > 0$ be such that c is the only critical value in the open interval $(c - \varepsilon, c + \varepsilon)$. As in §1, let $W^s(a_i) \cong \mathbb{R}^{n-\lambda_i}$ and $W^u(a_i) \cong \mathbb{R}^{\lambda_i}$ be the stable and unstable manifolds of a_i.

Theorem 4.2. *The inclusion of the subspace*

$$M^{c-\varepsilon} \cup W^u(a_1) \cup \cdots \cup W^u(a_k) \hookrightarrow M^{c+\varepsilon}$$

is a strong deformation retract.

This theorem allows one to define, in the obvious way, a CW-complex $C(f)$ homotopy equivalent to the manifold M whose cells correspond to the unstable manifolds of the critical points of f. See [4] for details.

We now proceed with the proof of Theorem (4.1). Recall that the homeomorphism $B\mathcal{C}_f \cong M$ is defined via two homeomorphisms:

$$\mathcal{R}_f \to M, \qquad \mathcal{R}_f \to B\mathcal{C}_f$$

where, see (2.4), (2.5) and (2.6),

$$\mathcal{R}_f = \bigsqcup_{\mathbf{a}} J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a}) / \sim .$$

Define the index filtration of $\mathcal{R}_f$ by letting $\mathcal{R}_f^k$ be as above except that the union is taken over sequences $\mathbf{a}$ such that

$$\lambda_{a_0} \leq k$$

where a_0 is the starting point $s(\mathbf{a})$ of the sequence $\mathbf{a}$. The proof of Lemma (3.1) shows that

$$\mathcal{R}_f^k \cong B\mathcal{C}_f^k.$$

Thus it is enough to prove Theorem (4.1) with $B\mathcal{C}_f^k$ replaced by $\mathcal{R}_f^k$. We choose to do this is because the homeomorphism

$$\varphi : \mathcal{R}_f \to M$$

is explicitly given. In particular, the restriction of φ to $J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a})$ is given by

$$\varphi(t; s_1, \cdots, s_l; \gamma_0 \cdots, \gamma_l) = (\gamma_0 \circ_1 \cdots \circ_{s_l} \gamma_l)(t).$$

Here recall that given the ordered sequence of critical points $\mathbf{a} = (a_0, \cdots, a_{l+1})$ then $J_{\mathbf{a}} = [f(a_{l+1}, f(a_0)]$.

To make things clearer, assume that f has exactly one critical point a_0 of index k. Then the space $\mathcal{R}_f^k / \mathcal{R}_f^{k-1}$ is a quotient of

$$\coprod_{s(\mathbf{a})=a_0} J_{\mathbf{a}} \times I^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a})$$

where the disjoint union is taken over all ordered sequences of critical points $\mathbf{a}$ with starting point $s(\mathbf{a})$ equal to the critical point a_0. In general the disjoint union would be taken over all sequences that start at any critical point of index k.

The image of this disjoint union, under φ, is precisely the set of all points in M that lie on a piecewise flow-line which starts at the critical point a_0. This is the closure $\bar{W}^u(a_0)$ of the unstable manifold of a_0. On the other hand, the (point-set) boundary of $\bar{W}^u(a_0)$ is given by

$$\partial \bar{W}^u(a_0) = \bar{W}^u(a_0) - W^u(a_0) = \bigcup_b W^u(b)$$

where the union is taken over critical points b with $\mathcal{M}(a_0, b) \neq \emptyset$. But each such critical point b must satisfy $\lambda_b \leq k - 1$and so it follows that $\bar{W}^u(a_0) - W^u(a_0)$ is in the image of $\mathcal{R}_f^{k-1}$ under φ. Thus we obtain homeomorphisms

$$\begin{aligned} \mathcal{R}_f^k - \mathcal{R}_f^{k-1} &\cong W^u(a_0) &&\cong \mathbb{R}^k \\ \mathcal{R}_f^k / \mathcal{R}_f^{k-1} &\cong \bar{W}^u(a_0)/\partial \bar{W}^u(a_0) &&\cong S^k. \end{aligned}$$

This proves the first statement of Theorem (4.1).

The d_1 differential in the index spectral sequence is computed by studying the homotopy class of the attaching map

$$S^{k-1} \to \mathcal{R}_f^{k-1} \to \mathcal{R}_f^{k-1}/\mathcal{R}_f^{k-2} \simeq \bigvee_{\mathrm{Crit}_{k-1}} S^{k-1}.$$

The above identification of the strata $\mathcal{R}_f^q - \mathcal{R}_f^{q-1}$ of the index filtration of $\mathcal{R}_f$ with the union of the unstable manifolds of the critical points of index q, together with the procedure for constructing the CW-complex $C(f)$ from the deformations in Theorem (4.2), see [4], shows that this attaching map is the relative attaching map in the CW-complex $C(f)$ associated to the Morse function f. The rest of Theorem (4.1) now follows.

The index spectral sequence arises from a topological filtration of $M = B\mathcal{C}_f$, so we get corresponding spectral sequences for any (generalized) homology theory. However in the general setting one would **not** expect the spectral sequence to collapse.

§5 The Gluing Theorem

In this section we discuss the proof of Theorem (2.1). So as in the statement of that theorem we will assume throughout this section that $f : M \to \mathbb{R}$ is a Morse-Smale function on a closed Riemannian manifold M.

As discussed in §2, this theorem describes the ends of the moduli space $\mathcal{M}(a, b)$ in terms of piecewise flow lines. Of course this description is classical and essentially follows from the transitivity property proved by Smale in [8]. This property says that if a, b, and c are critical points with $a > b$ and $b > c$, then $a > c$. That is, if there is a flow from a to b, and one from b to c, then there is a flow from a to c. Indeed it was shown in [8] that there is a flow from a to c which comes arbitrarily close to the piecewise flow determined by the flows from a to b and from b to c. More precisely it was shown in [8] that the closure of the space $W(a, b)$ of points lying on flows between a and b is given by

$$\bar{W}(a, b) = \bigcup_{a \geq \alpha > \beta \geq b} W(\alpha, \beta).$$

This result can certainly be used to describe the ends of the moduli space $\mathcal{M}(a, b)$. However for our purposes, we need something stronger, the existence of a pairing

$$\mu : (0, \varepsilon] \times \mathcal{M}(a, a_1) \times \mathcal{M}(a_1, b) \to \mathcal{M}(a, b), \qquad (t, \gamma_1, \gamma_2) \longrightarrow \gamma_1 \circ_t \gamma_2$$

that satisfies properties (1)-(4) of Theorem (2.1). A particularly important property is the associative property (1),

$$(\gamma_1 \circ_s \gamma_2) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_2 \circ_t \gamma_3).$$

We outline a proof of this theorem; full details are to be found in [1]. There are two main steps, the first is to establish the existence of local diffeomorphisms μ and the second that they can be chosen to satisfy the associativity condition. The existence of the diffeomorphisms $\mu : (0, \varepsilon] \times \mathcal{M}(a, b) \times \mathcal{M}(b, c) \to \mathcal{M}(a, c)$ follows from an analysis of the space of flow lines $\mathcal{M}(a, c)$ that pass near the intermediate critical point b. This is established using the following two lemmas both of which are proved by arguments similar to those in [8] involving the Morse Lemma, the Unstable Manifold Theorem, and standard transversality arguments.

Let p be a critical point of f having index λ_p and let a and b be critical points with $a > p > b$. Let $c = f(p)$ be the corresponding critical value. The first lemma describes the structure of $W^u(a)$ and $W^s(b)$ in a neighbourhood of p and the second lemma refines this to describe the local structure of the moduli space.

Lemma 5.1. *There exists a neighborhood $V(p)$ of p in M, a real number $\delta > 0$ and natural diffeomorphisms*

$$(W^u(a) \cap V(p)) \cap f^{-1}(c - \delta) \cong \mathcal{M}(a, p) \times \mathbb{R}^{\lambda(p)}$$
$$(W^s(b) \cap V(p)) \cap f^{-1}(c + \delta) \cong \mathbb{R}^{n-\lambda(p)} \times \mathcal{M}(p, b).$$

Let $V(p)$ and $\delta > 0$ be as in the Lemma (5.1). Let $V_0(p)$ be the complement of the axes (i.e the stable and unstable manifolds of p) in $V(p)$,

$$V_0(p) = V(p) - (W^u(p) \cup W^s(p)) \cong V(p) - \left(D^{\lambda(p)}(\delta) \times \{0\} \cup \{0\} \times D^{n-\lambda(p)}(\delta)\right).$$

Now for $-\delta \leq t \leq \delta$, let $V_0^t(p) = V_0(p) \cap f^{-1}(t)$. Similarly let $\mathcal{M}^t(a,b) = W(a,b) \cap f^{-1}(t)$. The next result describes $\mathcal{M}^t(a,b) \cap V(p)$. This is the part of the moduli space $\mathcal{M}(a,b)$ that comes close to the critical point p. Notice that this intersection equals $\mathcal{M}^t(a,b) \cap V_0(p) = W(a,b) \cap V_0^t(p)$. In this lemma we view the moduli spaces $\mathcal{M}(a,p)$ and $\mathcal{M}(p,b)$ as submanifolds of the spheres $S^{\lambda_p - 1}$ and $S^{n-\lambda_p-1}$ respectively, and let $\mathrm{cl}(\mathcal{M}(a,p))$ and $\mathrm{cl}(\mathcal{M}(p,b))$ be their closures.

Lemma 5.2. *There is a natural embedding*

$$\mathcal{M}^\delta(a,b) \cap V(p) \hookrightarrow S^{\lambda_p-1} \times S^{n-\lambda_p-1} \times [0, \sqrt{\delta})$$

whose point-set boundary is given by $\mathrm{cl}(\mathcal{M}(a,p)) \times \mathrm{cl}(\mathcal{M}(p,b)) \times 0$. *Moreover there is a subneighborhood* $N(p) \subset V(p)$ *of* p *in* M, *an* $\varepsilon > 0$, *and a diffeomorphism*

$$\mathrm{cl}(\mathcal{M}(a,p)) \times \mathrm{cl}(\mathcal{M}(p,b)) \times [0,\varepsilon] \xrightarrow{\cong} \mathrm{cl}(\mathcal{M}^\delta(a,b) \cap N(p))$$

that extends the inclusion of the boundary $\mathrm{cl}(\mathcal{M}(a,p)) \times \mathrm{cl}(\mathcal{M}(p,b)) \times 0$.

We now discuss how this lemma is used to prove Theorem (2.1). Lemma (5.2) allows us to find an $\varepsilon > 0$ and embeddings

$$\mu : \mathcal{M}(a, a_1) \times \cdots \times \mathcal{M}(a_m, b) \times (0, \varepsilon] \to \mathcal{M}(a,b).$$

This can be done for any sequence of critical points $a > a_1 > \cdots > a_m > b$. We will show that these embeddings can be chosen to satisfy the associativity property (1) in Theorem (2.1) below. In any case the existence of these embeddings (however they are chosen) allows us to verify the remainder of Theorem (2.1) as follows.

The fact that $\mathcal{M}(a,b) \cup_\mu \bigcup [0,\varepsilon]^l \times \mathcal{M}(\mathbf{a})$ is homeomorphic to the space $\bar{\mathcal{M}}(a,b)$ of piecewise flow lines follows immediately from Lemma (5.2).

Define $\mathcal{K}(a,b)$ as in statement (3) of Theorem (2.1). The fact that $\bar{\mathcal{M}}(a,b)$ and $\mathcal{K}(a,b)$ are homeomorphic also follows from Lemma (5.2). To obtain $\mathcal{K}(a,b)$ we must remove spaces of the form $\mathcal{M}(a,b) \cap N(p)$ from $\mathcal{M}(a,b)$, where p is an intermediate point, $a > p > b$. Now assume the neighborhood $N(p)$ is a ball of radius $\varepsilon > 0$ about p, and let $B(p)$ be a concentric ball of of radius $\varepsilon_1 < \varepsilon$ with closure $\bar{B}(p)$. By Lemma (5.2)

$$\mathcal{M}(a,b) \cap (N(p) - B(p)) \cong \mathcal{M}(a,p) \times \mathcal{M}(p,b) \times [\varepsilon_1, \varepsilon)$$

and so

$$\begin{aligned}
\mathcal{M}(a,b) - N(p) &\cong \mathcal{M}(a,b) - B(p) \\
&= (\mathcal{M}(a,b) - N(p)) \cup (\mathcal{M}(a,b) \cap (N(p) - B(p))) \\
&\cong \left(\mathcal{M}(a,b) - \bar{B}(p)\right) \cup_\mu \mathcal{M}(a,p) \times \mathcal{M}(p,b) \times [\varepsilon_1, \varepsilon) \\
&\cong \mathcal{M}(a,b) \cup_\mu \mathcal{M}(a,p) \times \mathcal{M}(p,b) \times [0, \varepsilon).
\end{aligned}$$

These homeomorphisms give the homeomorphism between $\mathcal{K}(a,b)$ and $\bar{\mathcal{M}}(a,b)$.

The fact that $\mathcal{K}(a,b)$ is compact follows from the the fact it can be identified with $\bar{\mathcal{M}}(a,b)$, the space of piecewise flow lines from a to b, which is a compact subspace of $\mathrm{Map}([f(b),f(a)],M)$.

We are done with the proof of Theorem (2.1) except for associativity. We need to choose the embeddings

$$\mu : (0,\varepsilon] \times \mathcal{M}(a,p) \times \mathcal{M}(p,b) \longrightarrow \mathcal{M}(a,b)$$
$$(t,\gamma_1,\gamma_2) \longrightarrow \gamma_1 \circ_t \gamma_2$$

so that they satisfy the required associativity law

$$(\gamma_1 \circ_s \gamma_t) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_t \circ_t \gamma_3).$$

To prove that such embeddings exist we use induction on the relative index $\lambda(a,b) = \lambda_a - \lambda_b$, or equivalently the dimension of the moduli space $\mathcal{M}(a,b)$. So assume that the appropriate associative embeddings have been constructed for all moduli spaces of dimension $< m = \dim(\mathcal{M}(a,b))$. For a set of critical points $a_1, \cdots, a_k$ with $a > a_1 > \cdots > a_k > b$ let $\mathcal{M}(a,b)_{a_1,\cdots,a_k} \subset \mathcal{M}(a,b)$ be those flows that pass through all the neighborhoods $N(a_1), \cdots, N(a_k)$ occuring in Lemma (5.2). Notice that $\mathcal{M}(a,b)_p = \mathcal{M}(a,b) \cap N(p)$ and so Lemma (5.2)

$$\mathcal{M}(a,b)_{a_1,\cdots,a_k} \cong \mathcal{M}(a,a_1) \times \cdots \times \mathcal{M}(a_k,b) \times (0,\varepsilon)^k,$$

For a critical point p with $a > p > b$ we will construct homeomorphisms

$$\mu = \mu_p : \mathcal{M}(a,p) \times \mathcal{M}(p,b) \times (0,\varepsilon) \to \mathcal{M}(a,b)_p$$

so that the appropriate associativity rules hold. Actually we will construct them on the level of compactifications

$$\mu_p : \bar{\mathcal{M}}(a,p) \times \bar{\mathcal{M}}(p,b) \times [0,\varepsilon] \to \bar{\mathcal{M}}(a,b)_p.$$

Assume that for critical points q with $p > q > b$ the homeomorphisms

$$\mu_q : \bar{\mathcal{M}}(a,q) \times \bar{\mathcal{M}}(q,b) \times [0,\varepsilon] \xrightarrow{\cong} \bar{\mathcal{M}}(a,b)_q$$

have been defined so that

$$\mu_q(x, \mu_r(y,z,s), t) = \mu_r(\mu_q(x,y,t), z, s) \in \mathcal{M}(a,b)_{q,r}$$

whenever $q > r > b$. This inductive assumption forces the definition of μ_p when restricted to

$$\big(\bar{\mathcal{M}}(a,p) \times \nu(\partial(\bar{\mathcal{M}}(p,b)) \times [0,\varepsilon]\big) \cup \big(\bar{\mathcal{M}}(a,p) \times \bar{\mathcal{M}}(p,b) \times 0\big)$$

where $\nu(\partial(\bar{\mathcal{M}}(p,b)))$ is a collar of the (point-set) boundary $\partial\bar{\mathcal{M}}(p,b) = \bar{\mathcal{M}}(p,b) - \mathcal{M}(p,b)$. By the inductive assumptions, we have already chosen a homeomorphism

$$\mu : \bigcup \mathcal{M}(\mathbf{a}) \times [0,\varepsilon]^l \xrightarrow{\cong} \nu(\partial(\bar{\mathcal{M}}(p,b))$$

where the union is taken over all ordered sequences of critical points $\mathbf{a}$ of length $l \geq 1$ having $s(\mathbf{a}) = p$ and $e(\mathbf{a}) = b$.

On this subspace μ_p is defined as follows. On the space $\bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times\{0\}$, μ_p is the inclusion of the boundary of the embedding $\mathcal{M}(a,b)\cap N(p) \hookrightarrow S^{\lambda_p-1}\times S^{n-\lambda_p-1}\times[0,\sqrt{\delta})$ given in Lemma (5.2). On an element

$$(x,y,t) \in \bar{\mathcal{M}}(a,p) \times \nu(\partial(\bar{\mathcal{M}}(p,b)) \times [0,\varepsilon]$$

$\mu_p(x,y,t)$ is defined as follows. Since $y \in \nu(\partial(\bar{\mathcal{M}}(p,b))$, y can be expressed in the form $y = \mu_q(u,v,s)$ for some critical point q with $p > q > b$ and $(u,v,s) \in \bar{\mathcal{M}}(p,q)\times\bar{\mathcal{M}}(q,b)\times[0,\varepsilon]$. We then define

$$\mu_p(x,y,t) = \mu_q(\mu_p(x,u,t),v,s). \tag{5.3}$$

This makes sense because the right hand side has already been defined by the inductive assumptions. Moreover by the associativity properties in the inductive assumptions this definition is independent of how $y \in \nu(\partial(\bar{\mathcal{M}}(p,b))$ is represented in the image of a μ map. Notice that the definition in (5.3) is forced upon us in order to satisfy the associativity property.

We therefore have a well defined embedding

$$\mu_p : \big(\bar{\mathcal{M}}(a,p)\times\nu(\partial(\bar{\mathcal{M}}(p,b))\times[0,\varepsilon]\big) \cup \big(\bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times\{0\}\big) \hookrightarrow \bar{\mathcal{M}}(a,b)_p. \tag{5.4}$$

Let

$$\mu_p : \bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times[0,\varepsilon] \xrightarrow{\cong} \bar{\mathcal{M}}(a,b)_p$$

be any homeomorphism that extends the embedding (5.4). (Recall that as observed before lemma 6.4 implies that $\bar{\mathcal{M}}(a,b)_p$ is diffeomorphic to $\bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times[0,\varepsilon]$). The existence of such a homeomorphism is equivalent to the existence of a self-homeomorphism of $\bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times[0,\varepsilon]$ that extends the identity on $\bar{\mathcal{M}}(a,p)\times\bar{\mathcal{M}}(p,b)\times\{0\}$ and a particular given level preserving homeomorphism on $\bar{\mathcal{M}}(a,p)\times\nu(\partial(\bar{\mathcal{M}}(p,b))\times[0,\varepsilon]$. By using a standard collaring argument it is easy to see that such a self homeomorphism exists.

Notice that by (5.3) the definition of μ_p satisfies the inductive assumptions and so the proof of Theorem (2.1)is complete.

§6 The classifying space of a Morse function

The goal of this section is to prove the second part of the main theorem which is restated as follows.

Theorem 6.1. *Let M be a closed Riemannian manifold and $f : M \to \mathbb{R}$ a Morse function. Then there is a homotopy equivalence $B\mathcal{C}_f \simeq M$.*

Note that in this theorem we are not assuming that f satisfies the Morse-Smale transversality condition. The proof uses only general constructions with categories and classifying spaces. Thus it is independent of the previous sections, in particular of the glueing theorem in §5.

Let $\mathcal{C}$ be a small category. Following [5] define a subdivision of $\mathcal{C}$, $\mathrm{sd}(\mathcal{C})$ as follows:

Objects. The objects of $\mathrm{sd}(\mathcal{C})$ are the morphisms in $\mathcal{C}$.

Morphisms. Let $\gamma_1 : a_1 \to b_1$ and $\gamma_2 : a_2 \to b_2$ be objects in $\mathrm{sd}(\mathcal{C})$. A morphism from $\gamma_1 \to \gamma_2$ consists of a pair of morphisms in $\mathcal{C}$

$$\alpha : a_1 \to a_2, \qquad \beta : b_2 \to b_1$$

such that the following diagram commutes

$$\begin{tikzcd} a_1 \arrow[r, "\gamma_1"] \arrow[d, "\alpha"'] & b_1 \\ a_2 \arrow[r, "\gamma_2"'] & b_2 . \arrow[u, "\beta"'] \end{tikzcd}$$

The composition law is the natural one.

Thus in the case of $\mathcal{C}_f$ there is a nontrivial morphism between two piecewise flow lines, γ_1 and γ_2, if γ_1 can be written in the form

$$\gamma_1 = \alpha \circ \gamma_2 \circ \beta.$$

If such a decomposition exists it is unique and so if there is a morphism in $\mathrm{sd}(\mathcal{C}_f)$ from γ_1 to γ_2 then it is unique.

The following lemma is proved in [5].

Lemma 6.2. *There is a natural homotopy equivalence*

$$B\mathcal{C} \simeq B\,\mathrm{sd}(\mathcal{C}).$$

For technical reasons we need to enlarge the category $\mathrm{sd}(\mathcal{C}_f)$. Define $\tilde{\mathrm{sd}}(\mathcal{C}_f)$ as follows:

Objects. The objects of $\tilde{\mathrm{sd}}(\mathcal{C}_f)$ are pairs (γ, x) where γ is a piecewise flow line and x is a point on γ.

Morphisms. There are no morphisms from (γ_1, x_1) to (γ_2, x_2) unless $x_1 = x_2$. If $x_1 = x_2$ then the morphisms from (γ_1, x_1) to (γ_2, x_2) are the same as the morphisms from γ_1 to γ_2 in the category $\mathrm{sd}(\mathcal{C}_f)$.

There is an obvious functor

$$\Psi : \tilde{\mathrm{sd}}(\mathcal{C}_f) \to \mathrm{sd}(\mathcal{C}_f)$$

given by forgeting the preferred point. This functor induces a map of classifying spaces

$$B\Psi : B\,\tilde{\mathrm{sd}}(\mathcal{C}_f) \to B\,\mathrm{sd}(\mathcal{C}_f).$$

Lemma 6.3. *The map $B\Psi : B\,\tilde{\mathrm{sd}}(\mathcal{C}_f) \to B\,\mathrm{sd}(\mathcal{C}_f)$ is a homotopy equivalence.*

Proof. The map induced by the forgetful functor Ψ on the space of chains of composable morphisms of length n is a fibration with contractible fibres and therefore a homotopy equivalence. Standard machinery from the theory of simplicial spaces now shows that $B\Psi$ is a homotopy equivalence. $\square$

¿From the manifold M construct a category $\underline{M}$ whose objects are the points of M and whose morphisms consist only of the identity maps. Thus there are no morphisms in $\underline{M}$ from x to y if $x \neq y$. It is clear from the construction of the classifying space that

$$B\underline{M} = M.$$

There is a functor $\Theta : \tilde{\mathrm{sd}}(\mathcal{C}_f) \to \underline{M}$ defined by sending (γ, x) to x and this induces a map of classifying spaces

$$B\Theta : B\,\tilde{\mathrm{sd}}(\mathcal{C}_f) \to B\underline{M} = M.$$

In view of Lemma (6.3), to prove Theorem (6.1) it suffices to show that this map is a homotopy equivalence. To do this we will construct an explicit homotopy inverse. As in §1, given $x \in M$ let γ_x to be the flow line through x. Then the assignment

$$x \mapsto (\gamma_x, x)$$

defines a functor

$$\Gamma : \underline{M} \to \tilde{\mathrm{sd}}(\mathcal{C}_f).$$

Theorem 6.4. *The induced maps*

$$B\Theta : B\,\tilde{\mathrm{sd}}(\mathcal{C}_f) \to B\underline{M} = M, \qquad B\Gamma : M = B\underline{M} \to B\,\tilde{\mathrm{sd}}(\mathcal{C}_f)$$

are inverse homotopy equivalences.

Proof. It is obvious that $\Theta \circ \Gamma$ is the identity functor of $\underline{M}$. To prove that the composite $B\Gamma \circ B\Theta$ is homotopic to the identity map of $B\,\tilde{\mathrm{sd}}(\mathcal{C}_f)$ it is sufficient, by [7], we construct a natural transformation from the identity functor to the composite $\Gamma \circ \Theta$.

The composite $\Gamma \circ \Theta$ sends the object (ω, x) to (γ_x, x) and it sends every morphism to the identity morphism. Since x is a point on ω it follows that the flow line γ_x is a segment of the piecewise flow line ω. Therefore there is a unique decomposition of ω as

$$\omega = \alpha \circ \gamma_x \circ \beta$$

where α and β are piecewise flow lines. Note that if x is a crtitical point of f then x is a fixed point of the flow and the above equation simply means that $e(\alpha) = s(\beta) = x$ and $\omega = \alpha \circ \beta$. The pair (α, β) give a morphism

$$(\alpha, \beta) : (\omega, x) \to (\gamma_x, x)$$

in the category $\tilde{\mathrm{sd}}(\mathcal{C}_f)$ and we define

$$\mathcal{N}(\omega, x) = (\alpha, \beta) : (\omega, x) \to (\gamma_x, x).$$

We now check that $\mathcal{N}$ gives a natural transformation of the identity functor to the functor $\Gamma \circ \Theta$. To do this suppose that $(\delta, \varepsilon) : (\omega_1, x_1) \to (\omega_2, x_2)$ is a morphism in $\tilde{\mathrm{sd}}(\mathcal{C}_f)$. Then it must follow that

$$\omega_1 = \delta \circ \omega_2 \circ \varepsilon, \qquad x_1 = x_2.$$

Let us write x for the point $x_1 = x_2$. Then as above we get unique decompositions

$$\omega_1 = \alpha_1 \circ \gamma_x \circ \beta_1, \qquad \omega_2 = \alpha_2 \circ \gamma_x \circ \beta_2.$$

Therefore it follows that

$$\omega_1 = \alpha_1 \circ \gamma_x \circ \beta_1 = \delta \circ \alpha_2 \circ \gamma_x \circ \beta_2 \circ \varepsilon.$$

By uniqueness it follows that $\alpha_1 = \delta \circ \alpha_2$, $\beta_1 = \beta_2 \circ \varepsilon$ and therefore

$$\begin{aligned} \mathcal{N}(\omega_1, x_1) &= (\alpha_1, \beta_1) \\ &= (\delta \circ \alpha_2, \beta_2 \circ \varepsilon) \\ &= (\delta, \varepsilon) \circ \mathcal{N}(\omega_2, x_2). \end{aligned}$$

Since the functor $\Gamma \circ \Theta$ sends every morphism to the identitiy morphism the following diagram commutes

$$\begin{tikzcd} (\omega_1, x_1) \arrow[r, "{(\delta,\varepsilon)}"] \arrow[d, "{\mathcal{N}(\omega_1,x_1)}"'] & (\omega_2, x_2) \arrow[d, "{\mathcal{N}(\omega_2,x_2)}"] \\ \Gamma(\Theta(\omega_1, x_1)) \arrow[r, "{\Gamma(\Theta(\delta,\varepsilon))}"'] & \Gamma(\Theta(\omega_2, x_2)). \end{tikzcd}$$

Thus $\mathcal{N}$ is indeed a natural transformation from the identity functor to $\Gamma \circ \Theta$ and so the composite $B\Gamma \circ B\Theta$ is homotopic to the identity.

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