Section 5.7

6. If $f(t) = t - 1$ and $g(t) = t - 2$, then

$$\begin{aligned}
f * g(t) &= \int_0^t f(u)g(t-u)\,du \\
&= \int_0^t (u-1)(t-u-2)\,du \\
&= \int_0^t (ut - u^2 - u - t + 2)\,du \\
&= \left.\frac{u^2 t}{2} - \frac{u^3}{3} - \frac{u^2}{2} - tu + 2u\right|_0^t \\
&= \frac{t^3}{2} - \frac{t^3}{3} - \frac{t^2}{2} - t^2 + 2t \\
&= \frac{t^3}{6} - \frac{3t^2}{2} + 2t.
\end{aligned}$$

12. If $f(t) = \cos t$ and $g(t) = t^2$, then

$$\begin{aligned}
f * g(t) &= \int_0^t f(u)g(t-u)\,du \\
&= \int_0^t (\cos u)(t-u)^2\,du.
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\int (\cos u)(t-u)^2\,du &= (\sin u)(t-u)^2 \\
&\quad + 2\int (\sin u)(t-u)\,du.
\end{aligned}$$

Integrating by parts again

$$\begin{aligned}
&= (\sin u)(t-u)^2 \\
&\quad + 2\left[(-\cos u)(t-u) - \int \cos u\,du\right], \\
&= (\sin u)(t-u)^2 - 2(\cos u)(t-u) - 2\sin u.
\end{aligned}$$

Thus,

$$\begin{aligned}
f * g(t) &= \big[(\sin u)(t-u)^2 - 2(\cos u)(t-u) \\
&\qquad - 2\sin u\big]_0^t \\
&= -2\sin t + 2t.
\end{aligned}$$

However,

$$\begin{aligned}
\mathcal{L}\{f * g(t)\}(s) &= \mathcal{L}\{-2\sin t + 2t\}(s), \\
&= \mathcal{L}\{-2\sin t\}(s) + \mathcal{L}\{2t\}(s), \\
&= \frac{-2}{s^2+1} + \frac{2}{s^2}, \\
&= \frac{2}{s^2(s^2+1)}.
\end{aligned}$$

18. Note the expression

$$\begin{aligned}
\frac{1}{s^2 - 3s} = \frac{1}{s(s-3)} &= \frac{1}{s}\cdot\frac{1}{s-3}, \\
&= F(s)G(s).
\end{aligned}$$

We have the transform pairs

$$\begin{aligned}
F(s) &= \frac{1}{s} \Longleftrightarrow f(t) = 1, \\
G(s) &= \frac{1}{s-3} \Longleftrightarrow g(t) = e^{3t}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2-3s}\right\}(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\
&= f * g(t), \\
&= \int_0^t f(u)g(t-u)\,du, \\
&= \int_0^t e^{3(t-u)}\,du, \\
&= -\frac{1}{3}e^{3(t-u)}\Big|_0^t, \\
&= -\frac{1}{3} + \frac{1}{3}e^{3t}, \\
&= -\frac{1}{3}\left(1 - e^{3t}\right).
\end{aligned}$$

22. The expression

$$\begin{aligned}
\frac{1}{(s+1)(s^2+4)} &= \frac{1}{s+1}\cdot\frac{1}{s^2+4}, \\
&= F(s)G(s).
\end{aligned}$$

We have transform pairs

$$\begin{aligned}
F(s) &= \frac{1}{s+1} \Longleftrightarrow f(t) = e^{-t}, \\
G(s) &= \frac{1}{2}\cdot\frac{2}{s^2+4} \Longleftrightarrow g(t) = \frac{1}{2}\sin 2t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} \\
&\quad = \mathcal{L}^{-1}\{F(s)G(s)\}(t), \\
&\quad = f * g(t), \\
&\quad = \int_0^t f(u)g(t-u)\,du, \\
&\quad = \frac{1}{2}\int_0^t e^{-u}\sin 2(t-u)\,du.
\end{aligned}$$

Two integrations by parts show that

$$\begin{aligned}
&\int e^{-u}\sin(2t-2u)\,du \\
&\quad = \frac{1}{5}e^{-u}\left[2\cos(2t-2u) - \sin(2t-2u)\right].
\end{aligned}$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\}(t)$$
$$= \frac{1}{10}e^{-u}\left[2\cos(2t-2u) - \sin(2t-2u)\right]\Big|_0^t$$
$$= \frac{1}{5}e^{-t} + \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t.$$

Section 8.1

2. The dimension is 2 since there are two equations and two unknown functions. The system is autonomous since there is no explicit dependence of the right hand sides on the independent variable.

8. If $x(t) = (1+t)e^{-t}$ and $y(t) = -te^{-t}$, then

$$\begin{aligned} x' &= ((1+t)e^{-t})' \\ &= -(1+t)e^{-t} + e^{-t} \\ &= -te^{-t} \\ &= y, \end{aligned}$$

so the first equation is satisfied. Further,

$$y' = (-te^{-t})' = te^{-t} - e^{-t},$$

and

$$\begin{aligned} -x - 2y &= -(1+t)e^{-t} - 2(-te^{-t}) \\ &= te^{-t} - e^{-t}, \end{aligned}$$

and the second equation is satisfied. Finally,

$$\begin{aligned} x(0) &= (1+0)e^{-0} = 1 \\ y(0) &= -0e^{-0} = 0, \end{aligned}$$

and the initial conditions are satisfied.

Section 9.1

2. If

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)(2-\lambda). \end{aligned}$$

Thus, $\lambda = 2$ is a repeated eigenvalue of algebraic multiplicity 2.

53. If

$$V = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix},$$

then

$$\begin{aligned} VDV^{-1} &= \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 3 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 3 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 10 \\ 0 & -2 \end{pmatrix} \\ &= A. \end{aligned}$$

54. If

$$A = \begin{pmatrix} 6 & 0 \\ 8 & -2 \end{pmatrix},$$

then a computer reveals the following eigenvalue-eigenvector pairs.

$$-2 \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad 6 \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the matrices

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix}$$

diagonalize matrix A. That is $A = VDV^{-1}$.