Solution to HW2.

Section 2.6

10. With $P = 1 - y\sin x$ and $Q = \cos x$, we see that

$$\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x},$$

so the equation is exact. We solve by setting

$$F(x, y) = \int P(x, y)\,dx = \int (1 - y\sin x)\,dx$$
$$= x + y\cos x + \phi(y).$$

To find ϕ, we differentiate

$$Q(x, y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y).$$

Thus $\phi' = 0$, so we can take $\phi = 0$. Hence the solution is $F(x, y) = x + y\cos x = C$.

14. Not exact.

26. $\mu(x) = 1/x^2$,

$$F(x, y) = \frac{xy^2/2 - y}{x} = C$$

Section 2.7

4. The right hand side of the equation is $f(s, \omega) = \omega\sin\omega + s$, which is continuous in the whole plane. $\partial f/\partial\omega = \sin\omega + \omega\cos\omega$ is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

10. The y-derivative of the right hand side $f(t, y) = ty^{1/2}$ is $ty^{-1/2}/2$ which is not continuous at $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.

26. The equation $x' = f(t, x)$ satisfies the hypotheses of the uniqueness theorem. Notice that $x_1(\pi/2) = x_2(\pi/2) = 0$. If they were both solutions $x' = f(t, x)$ near $t = \pi/2$, then by the uniqueness theorem they would have to be equal everywhere. Since they are not, they cannot both be solutions of the differential equation.

Section 2.8

16. (a) The right hand side of the equation is $f(t, x) = (x - 1)\cos t$. Thus $\partial f/\partial x = \cos t$, and $\max|\partial f/\partial x| = \max|\cos t| = 1$. Hence Theorem 7.15 predicts that $|x(t) - y(t)| \le |x(0) - y(0)|e^{|t|}$.

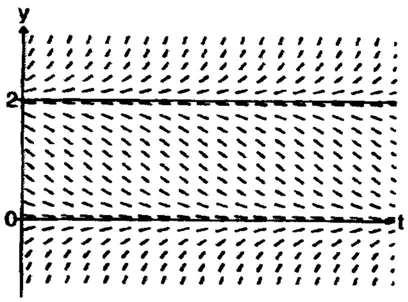
(b) The equation is separable and linear, and the solutions are $x(t) = 1 - e^{\sin t}$ and $y(t) = 1 - 9e^{\sin t}/10$. Hence the separation is $x(t) - y(t) = e^{\sin t}/10$. Since $\sin t \le |t|$, we see that

$$|x(t) - y(t)| = e^{\sin t}/10 \le e^{|t|}/10 = |x(0) - y(0)|e^{|t|}.$$

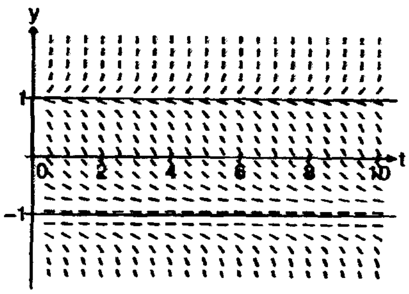
(c) Since $\sin t < |t|$ except at $t = 0$, we have $|x(t) - y(t)| < e^{|t|}/10$, except at $t = 0$.

Section 2.9

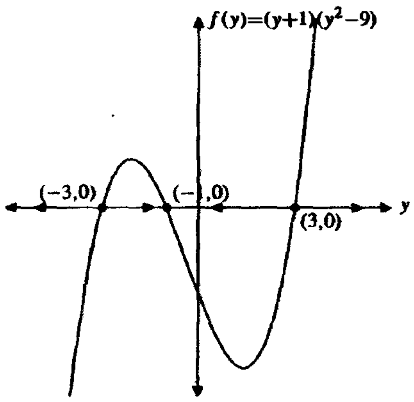
8. Note that the graph of $f(y)$ intercepts the y-axis at $y = 0$ and $y = 2$. Consequently, $y = 0$ and $y = 2$ are equilibrium points ($f(0) = 0$ and $f(2) = 0$) and $y(t) = 0$ and $y(t) = 2$ are equilibrium solutions, shown in the following figure. The solution $y = 0$ is asymptotically stable and $y = 2$ is unstable.



9. Since $f(y)$ has zeros at $y = -1$ and $y = 1$, these are equilibrium points. Correspondingly, $y(t) = -1$ and $y(t) = 1$ are equilibrium solutions, and are plotted in the following figure. Both are unstable.

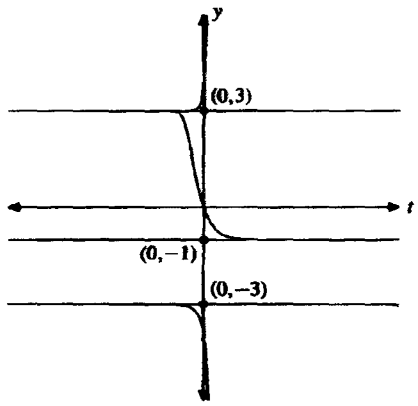


20. (i) In this case, $f(y) = (y+1)(y^2-9)$ factors as $f(y) = (y+1)(y-3)(y+3)$, whose graph is shown in the next figure.



(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

(iii) The phase line in the second figure indicates that solutions decrease if $y < -3$, increase for $-3 < y < -1$, decrease if $-1 < y < 3$, and increase for $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = -3$, the stable equilibrium solution, $y(t) = -1$, and the unstable equilibrium solution, $y(t) = 3$.



28. We have the equation $x' = f(x) = x(x-1)(x+2)$. The equilibrium points are at $x = 0$, 1, and -2, where $f(x) = 0$. We have $f'(x) = 3x^2 + 2x - 2$. Since $f'(0) = -2 < 0$, $x = 0$ is asymptotically stable. Because $f'(1) = 3 > 0$, $x = 1$ is unstable. Finally, because $f'(-2) = 2 > 0$, $x = -2$ is also unstable.

Section 3.1

2. The equation of the Malthusian model is $P(t) = Ce^{rt}$. Apply the cell counts to solve for r and C. $P(1) = 1000$, so $Ce^r = 1000$, i.e. $r = \ln(1000/C)$. Also, $P(2) = 3000$, so $Ce^{2r} = 3000$, i.e. $r = \frac{1}{2}\ln(3000/C)$. Setting these equal and solving, one obtains $C = 1000/3$. Substituting C into $r = \ln(1000/C)$ gives that $r = \ln 3 \approx 1.0986$. In addition $P(0) = C = 1000/3$.

12. The carrying capacity $K = 20,000$, and the initial condition $P_0 = 1000$, and it is given that $P(10) = 2000$. Using equation (1.13), one obtains

$$2000 = \frac{(20000)(1000)}{1000 + (19000)e^{-10r}}.$$

Solving gives $r = -(\ln(9/19))/10 \approx 0.0747$. After 25 hours, the population is

$$P(25) = \frac{(20000)(1000)}{1000 + (19000)e^{2.5\ln(9/19)}} \approx 5084.$$

Now, find t so that $P(t) = K/2 = 10000$. This gives

$$e^{\frac{1}{10}\ln(\frac{9}{19})t} = \frac{2000 - 1000}{19000}.$$

That is, $t = 10(-\ln(19))/(\ln(9/19)) \approx 39.4055$.

18. We will use days and 1000 individuals as our units. We are given that the carrying capacity is $K = 10$, $P(0) = 1$, and with $t_1 = 2.3/24$,

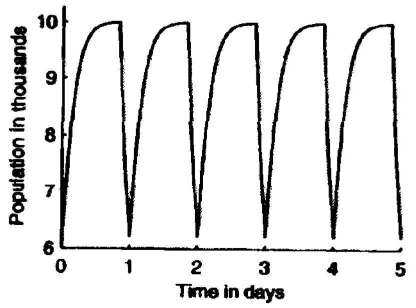
$$2 = P(t_1) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt_1}}.$$

Solving for r, we get $r = 8.4619$.

The harvesting rate during the last four hours of each day is 1500/hr = 36,000/day. Hence the harvesting rate is

$$h(t) = \begin{cases} 0, & \text{in the first 20 hours} \\ 36, & \text{in the last 4 hours} \end{cases}$$

of each day. Using a square wave function to model this in our solver we get the following graph of the solution.



The population at the end of each day is approximately 10,000.

Section 4.1

4. Divide both sides of $ty'' + (\sin t)y' = 4y - \cos 5t$ by t, then rearrange to obtain

$$y'' + \frac{\sin t}{t}y' - \frac{4}{t} = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = (\sin t)/t$, $q(t) = -4/t$, and $g(t) = -(\cos 5t)/t$. Hence, the equation is linear and inhomogeneous.

12. The period of the driving force is 4 seconds. Thus, the circular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ rad/s}.$$

Because the amplitude is $A = 0.25$ m, and the spring is initially displaced 0.25 m downward (remember, upward is negative) by the driving force, the driving force can be described with

$$F(t) = 0.25\cos\frac{\pi t}{2}$$

Now, $m = 5$ kg, $k = 65.3$ N/m, and the damping force is given by $R(v) = -0.125v$. This makes the damping constant $\mu = 0.125$. Thus, the equation

$$my'' + \mu y' + ky = F(t)$$

becomes

$$5y'' + 0.125y' + 65.3y = 0.25\cos\frac{\pi t}{2}$$

From Exercise 10, the initial conditions are $y(0) = -0.36$ and $y'(0) = 0.45$.

14. If $y_1(t) = \cos 2t$, then

$$\begin{aligned} y_1'' + 4y_1 &= (\cos 2t)'' + 4(\cos 2t) \\ &= -4\cos 2t + 4\cos 2t \\ &= 0. \end{aligned}$$

If $y_2(t) = \sin 2t$, then

$$\begin{aligned} y_2'' + 4y_2 &= (\sin 2t)'' + 4(\sin 2t) \\ &= -4\sin 2t + 4\sin 2t \\ &= 0. \end{aligned}$$

Finally, if $y(t) = C_1\cos 2t + C_2\sin 2t$, then

$$\begin{aligned} y'' + 4y &= (C_1\cos 2t + C_2\sin 2t)'' \\ &\quad + 4(C_1\cos 2t + C_2\sin 2t) \\ &= -4C_1\cos 2t - 4C_2\sin 2t \\ &\quad + 4C_1\cos 2t + 4C_2\sin 2t \\ &= 0. \end{aligned}$$