1. (15 points) a. Using the method of undetermined coefficients, find the general solution to the differential equation

$$y'' - 3y' - 10y = 3e^{-2t}.$$

First we want a particular solution ate^{-2t}. We have

$$3e^{-2t} = y'' - 3y' - 10y = a \cdot e^{-2t} \cdot (-7)$$

So $a = -3/7$; i.e., $y_p(t) = -\frac{3}{7} t e^{-2t}$.

We also know that the general solution for the corresponding homogeneous equation is $C_1 e^{5t} + C_2 e^{-2t}$.

Therefore, the general solution for this inhomogeneous equation is

$$y(t) = -\frac{3}{7} te^{-2t} + C_1 e^{5t} + C_2 e^{-2t}$$

b. Use the Laplace transform to solve the second order initial value problem

$$y'' - y' - 2y = e^{2t}, \quad y(0) = -1, \quad y'(0) = 0.$$

Making the Laplace transform, we have

$$(s^2 Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) - 2Y(s) = \frac{1}{s-2}$$

i.e., $(s^2 - s - 2) Y(s) = \frac{1}{s-2} - (s-1)$

$$Y(s) = \frac{1}{(s-2)^2(s+1)} - \frac{s-1}{(s-2)(s+1)} = \frac{1}{3}\frac{1}{(s-2)^2} - \frac{4}{9}\frac{1}{s-2} - \frac{5}{9}\frac{1}{s+1}$$

Therefore, the solution is

$$y(t) = \frac{1}{3} te^{2t} - \frac{4}{9} e^{2t} - \frac{5}{9} e^{-t}.$$

2. (15 points) a. Consider the unforced oscillator, which is modelled by the initial value problem

$$y'' + 2cy' + \omega_0^2 y = 0 \quad y(0) = y_0, \quad y'(0) = v_0$$

where y_0 and v_0 are the initial displacement and velocity of the mass. Show that the Laplace transform of the solution can be written

$$Y(s) = \frac{y_0 s + v_0 + 2cy_0}{(s+c)^2 + (\omega_0^2 - c)^2}. \leftarrow (\omega_0^2 - c^2)$$

Using the Laplace transform,

$$0 = (s^2 Y(s) - sy(0) - y'(0)) + 2c(sY(s) - y(0)) + \omega_0^2 Y(s);$$

hence,

$$Y(s) = \frac{(s+2c)y(0) + y'(0)}{s^2 + 2cs + \omega_0^2}$$

$$= \frac{(s+2s)y_0 + V_0}{(s+c)^2 + (\omega_0^2 - c^2)}$$

b. For $Y(s)$ as in part a, consider the case when $\omega_0^2 - c^2 = 0$. This is the so-called *"critically damped"* case. Compute the inverse Laplace transform of $Y(s)$ to obtain the solution to the initial value problem in part a in this case.

Case $\omega_0 = c$:

$$Y(s) = \frac{y_0 s + V_0 + 2cy_0}{(s+c)^2}$$

$$= y_0 \frac{1}{s+c} + (V_0 + cy_0)\frac{1}{(s+c)^2}$$

Therefore, $y(t) = y_0 e^{-ct} + (V_0 + cy_0)te^{-ct}$.

3. (20 points) a. Without using partial fractions, find the inverse Laplace transform of the function

$$F(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+1}.$$

Hint. Use the convolution product.

Using the table of Laplace transforms, we get

$$\mathcal{L}(t) = \frac{1}{s^2}, \qquad \mathcal{L}(\sin t) = \frac{1}{s^2+1}.$$

Since $\mathcal{L}^{-1}(F_1(s)\cdot F_2(s)) = f_1(t) * f_2(t)$.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= t * \sin t = \int_0^t u \cdot \sin(t-u)\, du$$

$$= \int_0^t u \, d\cos(t-u)$$

$$= u\cos(t-u)\Big|_{u=0}^{u=t} - \int_0^t \cos(t-u)\, du$$

$$= (t-0) + \sin(t-u)\Big|_{u=0}^{u=t} = t - \sin t.$$

b. Consider the initial value problem

$$y'' + 2y' + 5y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Find the initial impulse response function $e(t)$ for this system.

The unit impulse response function $e(t)$ is the solution to the equation

$$e'' + 2e' + 5e = \delta(t), \qquad e(0) = 0, \quad e'(0) = 0.$$

Apply Laplace transform on both sides, we get

$$s^2E(s) + 2sE(s) + 5E(s) = 1.$$

$$E(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} = \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

From the table of Laplace transforms, we obtain

$$e(t) = \frac{1}{2}e^{-t}\sin 2t \, H(t).$$

c. Find the solution of the initial value problem in part b in terms of the forcing function $g(t)$. (Your answer may involve an integral.)

The solution is

$$y(t) = y_s(t) + y_i(t)$$
$$= e(t) * g(t) + a y_0 e'(t) + (a y_1 + b y_0) e(t).$$

Since $e(t) = \frac{1}{2} e^{-t} \sin 2t$,

$$e'(t) = e^{-t}\left(-\frac{1}{2}\sin 2t + \cos 2t\right)$$

$a = 1$, $b = 2$; $\quad y_0 = y(0) = 1$, $y_1 = y'(0) = -1$.

$$y(t) = \left(\frac{1}{2} e^{-t} \sin 2t\right) * g(t) + e^{-t}\left(-\frac{1}{2}\sin 2t + \cos 2t\right) + (-1+2)\cdot \frac{1}{2} e^{-t} \sin 2t$$
$$= \frac{1}{2}\int_0^t e^{-u} \sin 2u \cdot g(t-u)\,du + e^{-t}\cos 2t$$

4. (25 points) a. Prove that if λ is an eigenvalue of an $n \times n$ matrix A and $\mathbf{v}$ is a corresponding eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to the ODE $\mathbf{x}' = A\mathbf{x}$.

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c. Find the solution of the initial value problem in part b in terms of the forcing function $g(t)$. (Your answer may involve an integral.)

4. (25 points) a. Prove that if λ is an eigenvalue of an $n \times n$ matrix A and $\mathbf{v}$ is a corresponding eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to the ODE $\mathbf{x}' = A\mathbf{x}$.

Let $x(t) = e^{\lambda t} v$, where λ is an eigenvalue, and v is a corresponding eigenvector. To prove that $x' = Ax$, we compute both sides, and show they're equal.

Left side: $x(t) = e^{\lambda t} v \Rightarrow x'(t) = \lambda e^{\lambda t} v = \lambda x(t)$.

Right side: $Ax = A \cdot e^{\lambda t} v = e^{\lambda t} A v$, since A ~~can~~ preserves scalar multiplication. Since v is an eigenvector, $Av = \lambda v$. So

$$Ax = e^{\lambda t} A v = e^{\lambda t} \cdot \lambda v = \lambda e^{\lambda t} v = \lambda x(t).$$

Thus the left side and the right side are equal.

b. Use the definition of Laplace transform to show that if $f(t) = e^{at}\sin(bt)$, then its Laplace transform is given by

$$F(s) = \frac{b}{(s-a)^2 + b^2} \qquad s > a$$

By definition, $F(s) = \int_0^\infty e^{at}\sin(bt)\,e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\sin(bt)\,dt$

We compute by integration by parts: $u = e^{(a-s)t}$ $\quad dv = \sin(bt)\,dt$

$du = (a-s)e^{(a-s)t}\,dt \quad v = -\frac{1}{b}\cos(bt)$

$$F(s) = -\frac{1}{b}e^{(a-s)t}\cos(bt)\Big|_0^\infty + \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos(bt)\,dt$$

$$= \frac{1}{b} + \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos(bt)\,dt \quad \text{since } \lim_{t\to\infty} e^{(a-s)t}\cos(bt) = 0 \text{ for } s > a.$$

Using integration by parts again, with $u = e^{(a-s)t}$ $\quad dv = \cos(bt)\,dt$

$du = (a-s)e^{(a-s)t}, \quad v = \frac{1}{b}\sin(bt)$

$$\int_0^\infty e^{(a-s)t}\cos(bt)\,dt = \frac{1}{b}e^{(a-s)t}\sin(bt)\Big|_0^\infty - \left(\frac{a-s}{b}\right)\int_0^\infty e^{(a-s)t}\sin(bt)\,dt$$

$$= \frac{1}{b}e^{(a-s)t}\sin(bt)\Big|_0^\infty - \left(\frac{a-s}{b}\right)F(s) = -\left(\frac{a-s}{b}\right)F(s)$$

since $\lim_{t\to\infty} e^{(a-s)t}\sin(bt) = 0$ for $s > a$.

Putting this together, we have

$$F(s) = \frac{1}{b} - \frac{(a-s)^2}{b^2}F(s)$$, or, solving for $F(s)$

$$F(s) = \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

5. (25 points) a. Find the general solution of $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -1 & 0 \\ 4 & -2 \end{pmatrix}.$$

A upper triangular, so $\lambda_1 = -1$, $\lambda_2 = -2$

$A - \lambda_1 I = \begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$A - \lambda_2 I = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_1 \neq \lambda_2$, so $\vec{v}_1$ and $\vec{v}_2$ are linearly independent

$$\vec{y}(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

b. The matrix $A = \begin{pmatrix} -2 & 2 \\ -10 & -10 \end{pmatrix}$ has complex eigenvalues. Find a fundamental set of **real** solutions of the system $\mathbf{y}' = A\mathbf{y}$.

$P(\lambda) = \det(A - \lambda I) = (-2-\lambda)(-10-\lambda) + 20 = \lambda^2 + 12\lambda + 40$

$$\lambda_{1,2} = \frac{-12 \pm i\sqrt{160 - 144}}{2} = -6 \pm 2i$$

Then $A - \lambda_1 I = \begin{pmatrix} 4-2i & 2 \\ -10 & -4-2i \end{pmatrix} \sim \begin{pmatrix} 2-i & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 2-i \end{pmatrix}$

Take $z(t) = e^{(-6+2i)t}\begin{pmatrix} -1 \\ 2-i \end{pmatrix}$. Then $Re(z(t))$ and $Im(z(t))$ are two independent real solutions to $\vec{y}' = A\vec{y}$.

$$e^{(-6+2i)t}\begin{pmatrix} -1 \\ 2-i \end{pmatrix} = e^{-6t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} -1 \\ 2-i \end{pmatrix} = \underbrace{e^{-6t}\begin{pmatrix} -\cos(2t) \\ 2\cos(2t)+\sin(2t) \end{pmatrix}}_{\vec{X}_1(t)} + i\underbrace{e^{-6t}\begin{pmatrix} -\sin(2t) \\ -\cos(2t)+2\sin(2t) \end{pmatrix}}_{\vec{X}_2(t)}$$

c. Find the general solution of the system

$$x' = -3x$$
$$y' = -5x + 6y - 4z$$
$$z' = -5x + 2y$$

Re-write as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$p(\lambda) = (-3-\lambda)\left[(6-\lambda)(-\lambda) + 8\right] = -(3+\lambda)(\lambda^2 - 6\lambda + 8)$$
$$= -(3+\lambda)(\lambda - 4)(\lambda - 2)$$

$$\lambda_1 = -3: \quad A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 9 & -4 \\ -5 & 2 & +3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ -5 & 9 & -4 \\ 0 & -7 & 7 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ -5 & 9 & -4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 \\ -5 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4: \quad A - \lambda_2 I = \begin{pmatrix} -7 & 0 & 0 \\ -5 & 2 & -4 \\ -5 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 2: \quad A - \lambda_3 I = \begin{pmatrix} -5 & 0 & 0 \\ -5 & 4 & -4 \\ -5 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and since $\lambda_1, \lambda_2, \lambda_3$ are distinct, $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are lin. independent, so that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$