Math 53 Midterm 2 Solutions

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Problem 1. Find a particular solution of the differential equation

$$2 \cdot y'' - y' = t.$$

Solution. The method of undetermined coefficients suggested that the solution has the form $y(t) = At^2 + Bt + C$. Substituting this in shows that $A = -1/2$, $B = -2$, and $C = 0$. So one solution is

$$\boxed{y(t) = -(1/2)t^2 - 2t}.$$

$\square$

Problem 2. Find the inverse Laplace transform of the function

$$F(s) = \frac{s^2 - 5s + 4}{s(s^2 + 1)}.$$

Solution. Write

$$\frac{s^2 - 5s + 4}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{As^2 + A + Bs^2 + Cs}{s(s^2 + 1)}.$$

Then $A = 4$, $B = -3$, and $C = -5$. Therefore

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{s^2 - 5s + 4}{s(s^2 + 1)}\right) &= \mathcal{L}^{-1}\left(\frac{4}{s}\right) + \mathcal{L}^{-1}\left(\frac{-3s - 5}{s^2 + 1}\right) \\ &= \mathcal{L}^{-1}\left(\frac{4}{s}\right) + \mathcal{L}^{-1}\left(\frac{-3s}{s^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{-5}{s^2 + 1}\right) \\ &= \boxed{4 - 3\cos t - 5\sin t}. \end{aligned}$$

$\square$

Problem 3. Find the Laplace transform of the function

$$g(t) = \begin{cases} 2t & : & t < 3 \\ 1 & : & t \geq 3. \end{cases}$$

Solution. Rewrite

$$g(t) = 2t(H(t) - H(t-3)) + H(t-3) = 2tH(t) - 2(t-3)H(t-3) - 5H(t-3).$$

Then

$$\mathcal{L}(g) = \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} - \frac{5e^{-3s}}{s} = \boxed{\frac{2 - 2e^{-3s} - 5se^{-3s}}{s^2}}.$$

□

Problem 4. Find the solution to the initial value problem

$$\begin{aligned} y'' + \pi^2 y &= \delta(t - 1/2) \\ y(0) &= 0 \\ y'(0) &= 0, \end{aligned}$$

where $\delta(t)$ is the delta function.

Solution. Let $Y(s)$ be the Laplace transform of $y(t)$. Taking the Laplace transform of both sides of the equation gives

$$Y(s) = \frac{e^{-s/2}}{s^2 + \pi^2}.$$

Taking the inverse Laplace transform gives

$$\boxed{y(t) = H(t - 1/2) \cdot \frac{1}{\pi} \cdot \sin \pi(t - 1/2)}.$$

□

Problem 5. Find the general solution to the system

$$\begin{aligned} x_1' &= 2x_1 - 3x_2 \\ x_2' &= x_1 - 2x_2. \end{aligned}$$

Solution. The system equals

$$\mathbf{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \cdot \mathbf{x}.$$

The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

So the eigenvalues are $\lambda = \pm 1$. Looking at

$$\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix},$$

we see that an eigenvector for the eigenvalue $\lambda = 1$ is $v_1 = (3,1)^{\mathrm{T}}$. Looking at

$$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix},$$

we see that an eigenvector for the eigenvalue $\lambda = -1$ is $v_2 = (1,1)^{\mathrm{T}}$. Therefore, the general solution to the given system is

$$\boxed{\mathbf{x} = C_1 e^t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$$

□

Problem 6. Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} \cdot \mathbf{x}.$$

Solution. The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) + 8 = \lambda^2 + 4 = (\lambda+2i)(\lambda-2i).$$

So the eigenvalues are $\lambda = \pm 2i$. Looking at

$$\begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix},$$

we see that an eigenvector for the eigenvalue $\lambda = 2i$ is $v_1 = (2, 1-i)^{\mathrm{T}}$. Looking at

$$\begin{pmatrix} 2+2i & -4 \\ 2 & -2+2i \end{pmatrix},$$

we see that an eigenvector for the eigenvalue $\lambda = -2i$ is $v_2 = (2, 1+i)^{\mathrm{T}}$. Therefore, the general solution to the given system is

$$\boxed{\mathbf{x} = C_1 e^{2it} \begin{pmatrix} 2 \\ 1-i \end{pmatrix} + C_2 e^{-2it} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}}.$$

Taking the real and imaginary parts of $e^{2it} v_1$ shows that the real-valued solutions have the form

$$\boxed{\mathbf{x} = C_1 \begin{pmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}}.$$

□

Problem 7. Solve the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ -x_1 - x_2 + x_3 + x_4 + x_5 &= 3 \\ -x_1 - x_2 - x_3 + x_4 - x_5 &= 1. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ -1 & -1 & 1 & 1 & 1 & 3 \\ -1 & -1 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 2 & 2 & 2 & 8 \\ 0 & 0 & 0 & 2 & 0 & 6 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right)$$
$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right).$$

Let $x_2 = s$ and $x_5 = t$. Then $x_1 = 1 - s$, $x_3 = 1 - t$ and $x_4 = 3$. So the general solution is

$$\boxed{\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}$$

for any real numbers s and t. □

Problem 8. Consider the following system with unknowns x and y:

$$\begin{aligned} x + y &= 2 \\ x + ay &= b, \end{aligned}$$

where a and b are constants. For what values of a and b does the system above have: (a) no solution, (b) a unique solution, (c) exactly two solutions, and (d) more than two solutions?

Solution. Looking at the associated augmented matrix, we have

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & a & b \end{array}\right)$$
$$\to \begin{pmatrix} 1 & 1 & 2 \\ 0 & a-1 & b-2 \end{pmatrix}.$$

This system has no solution if $a - 1 = 0$ and $b - 2 \neq 0$, that is, if $a = 1$ and $b \neq 2$. It has a unique solution if $a - 1 \neq 0$, that is, if $a \neq 1$. It has infinitely many solutions if $a - 1 = b - 2 = 0$, that is, if $a = 1$ and $b = 2$. It never has exactly two solutions.

Note that these cases correspond geometrically to the case where the two lines are not parallel ($a \neq 1$), parallel but not equal ($a = 1$ and $b \neq 2$), and equal ($a = 1$ and $b = 2$). $\square$