1. (20 pts) (a). Find the solution of the following initial value problem.

$$y'' + y' - 6y = 0 \quad y(0) = 2, \quad y'(0) = 9.$$

Char poly: $\lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$

$\lambda = 2, -3$

general sol'n: $y(t) = Ae^{2t} + Be^{-3t}$

$\left.\begin{aligned} 2 &= y(0) = A + B \\ 9 &= y'(0) = 2A - 3B \end{aligned}\right\} \Rightarrow A = 3,\ B = -1$

Solution: $\boxed{y(t) = 3e^{2t} - e^{-3t}}$

(b). Find the general solution to the following differential equation:

$$y'' - 4y' + 4y = 0.$$

Char poly: $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

$\lambda = 2$, repeated root

general sol'n: $\boxed{y(t) = Ae^{2t} + Bte^{2t}}$

2. (20 pts) (a). *Newton's law of cooling* asserts that the rate at which an object cools is proportional to the difference between the object's temperature (T) and the temperature of the surrounding medium (A). It therefore satisfies the ODE

$$T'(t) = -k(T(t) - A).$$

Solve for $T(t)$ in terms of A, $T_0 =$ the temperature of the body at time 0, and $k =$ the proportionality constant.

Separate variables, we get

$$\frac{dT}{T-A} = -k\,dt$$

Integrate both sides, $\int_{T_0}^{T} \frac{ds}{s-A} = -k\int_0^t du$

$$\ln\frac{|T-A|}{|T_0-A|} = \ln|T-A| - \ln|T_0-A| = -kt$$

Solve for T by exponentiating, and since $T-A$ and T_0-A have the same sign,

$$T(t) = A + (T_0-A)\,e^{-kt}.$$

(b) A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Calculate the victim's time of death. (*Note.* The "normal" temperature of a living human being is approximately 37°C).

Let $t=0$ be midnight. Let t_1 be the victim's time of death. Then $t_1 < 0$. (The unit for t is hour)

$T_0 = 31$. $A = 21$. we have

$$T(t) = 21 + (31-21)\,e^{-kt}.$$

In particular, $T(1) = 29 = 21 + (31-21)e^{-k} \Rightarrow e^{-k} = \frac{8}{10}$.

It implies that $T(t) = 21 + 10\cdot\left(\frac{8}{10}\right)^t$

If $T(t_1) = 37$, then $37 = 21 + 10\cdot\left(\frac{8}{10}\right)^{t_1} \Rightarrow \left(\frac{8}{10}\right)^{t_1} = \frac{16}{10} = \frac{8}{5}$.

$$t_1 = \ln\tfrac{8}{5}/\ln\tfrac{8}{10} = -\ln\tfrac{8}{5}/\ln\tfrac{5}{4} \quad (\approx -2)$$

The victim's time of death is $\ln\frac{8}{5}/\ln\frac{5}{4}$ hours before midnight. ($\approx$10pm) (c). Consider the equation

$$y' = f(at + by + c)$$

where a, b, and c are constants. Show that the substitution $x = at + by + c$ changes the equation to the separable equation $x' = a + bf(x)$. Use this method to find the general solution of the equation $y' = (y + t)^2$.

Since $x = at + by + c$

$$\frac{dx}{dt} = a + b\frac{dy}{dt}.$$

The equation $\frac{dy}{dt} = f(at + by + c)$ becomes

$$\frac{dx}{dt} = a + b\frac{dy}{dt} = a + bf(at + by + c) = a + bf(x).$$

Next. Let $f(at + by + c) = (y + t)^2$.

Let $x = y + t$.

Then $y' = (y+t)^2$ is equivalent to

$$x' = y' + 1 = (y+t)^2 + 1 = x^2 + 1.$$

This is a separable equation.

$$\frac{dx}{1+x^2} = dt$$

$$\int \frac{dx}{1+x^2} = \int dt \Rightarrow \arctan x = t + C$$

$x = \tan(t + C)$.

Since $y = x - t$, $y = \tan(t + c) - t$. is the general solution of $y' = (y+t)^2$.

3. (15 pts) FACT: The function $y = (x^2+1)e^{-x^2}$ is a solution to the ordinary differential equation

$$(x^2+1)y' + 2x^3y = 0.$$

(a). Using the above fact, find the general solution $y(x)$ to the ordinary differential equation

$$(x^2+1)y' + 2x^3y = 2(x^3+x)e^{-x^2}.$$

Let $y_h = (x^2+1)e^{-x^2}$ and $y = vy_h$. Then plugging in, $(x^2+1)v'y_h + (x^2+1)vy_h' + 2x^3vy_h = 2(x^3+x)e^{-x^2}$

$$\iff (x^2+1)v'y_h + v\underbrace{\left((x^2+1)y_h' + 2x^3y_h\right)}_{=0} = 2(x^3+x)e^{-x^2}$$

$$\iff (x^2+1)v'y_h = 2(x^3+x)e^{-x^2} \iff v' = \frac{2(x^3+x)e^{-x^2}}{(x^2+1)y_h}$$

$$\iff v' = \frac{2x}{x^2+1} \implies v(x) = \ln(x^2+1) + C$$

so $\boxed{y(x) = (x^2+1)e^{-x^2}(\ln(x^2+1) + c)}$

(b). Find the solution $y(x)$ to the equation from part (a) satisfying the initial value

$$y(0) = 1.$$

$$1 = y(0) = (0+1)e^0(\ln(1) + c)$$
$$= c$$

$$\boxed{y(x) = (x^2+1)e^{-x^2}(\ln(x^2+1) + c)}$$

4. (15 pts) Suppose that the functions u and v are solutions to the linear, homogeneous equation

$$y'' + p(y)y' + q(t)y = 0$$

in the interval (α, β). Prove that the Wronskian of u and v is either identically equal to zero on (α, β), or it is never equal to zero there.

Proposition 1.26 pp. 142-143.

5. (20 pts) (a). Show that the following equation is exact and solve it.

$$(2x + y)dx + (x - 6y)dy = 0$$

Solution:

$$\frac{\partial}{\partial y}(2x + y) = 1 = \frac{\partial}{\partial x}(x - 6y)$$

implies that the equation is exact. The solution is

$$x^2 + xy - 3y^2 = C$$

(b). Suppose that $(x+y)dx+2xdy=0$ has an integrating factor that is a function of x alone (i.e $\mu=\mu(x)$). Find the integrating factor and use it to solve the differential equation.

Solution:

Suppose $\mu(x)$ is an integrating factor. We need

$$\mu(x)(x+y)\mathrm{d}x+\mu(x)(2x)\mathrm{d}y=0$$

to be exact. So we have

$$\frac{\partial}{\partial y}(\mu(x)(x+y))=\frac{\partial}{\partial x}(\mu(x)(2x))$$

i.e.

$$\mu(x)=\mu'(x)\cdot 2x+\mu(x)\cdot 2$$

We have solution for this ODE,

$$\mu(x)=x^{-1/2}=\frac{1}{\sqrt{x}}$$

Now the equation becomes

$$(\sqrt{x}+y/\sqrt{x})\mathrm{d}x+2\sqrt{x}\mathrm{d}y=0$$

Therefore we can get the solution

$$2\sqrt{x}y+\frac{2}{3}x^{\frac{3}{2}}=C$$

6. (10 pts) The undamped system

$$\frac{2}{5}x'' + kx = 0 \quad x(0) = 2, \quad x'(0) = v_0$$

is observed to have period $\pi/2$ and amplitude 2. Find k and v_0.

Solution:

$\omega_0 = \frac{2\pi}{\pi/2} = 4$, so $\frac{k}{2/5} = \omega_0^2 = 16$ and therefore $k = \frac{32}{5}$. The equation becomes $x'' + 16x = 0$, with general solution $x(t) = c_1 \cos 4t + c_2 \sin 4t$. But we know that $x(0) = 2 =$ amplitude, hence $x(t) = 2\cos 4t$ and therefore $v_0 = x'(0) = 0$.