Nathan Brown
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Ordinary Differential Equations Ma53

Homework #3 - Solutions

5.1.6 Using integration by parts,

$$\begin{aligned}\int e^{-st}\sin(3t)\,dt &= -\frac{1}{s}e^{-st}\sin(3t)+\frac{3}{s}\int e^{-st}\cos(3t)\,dt\\ &= -\frac{1}{s}e^{-st}\sin(3t)-\frac{3}{s^2}e^{-st}\cos(3t)-\frac{9}{s^2}\int e^{-st}\sin(3t)\,dt.\end{aligned}$$

We can now solve for the integral as if it were a variable to obtain

$$\int e^{-st}\sin(3t)\,dt = -\frac{se^{-st}}{s^2+9}\sin(3t)-\frac{3}{s^2+9}e^{-st}\cos(3t).$$

Then the desired transform is

$$\begin{aligned}F(s) &= \lim_{T\to\infty} e^{-st}\left[-\frac{s}{s^2+9}\sin(3t)-\frac{3}{s^2+9}\cos(3t)\right]\Bigg|_0^T\\ &= \frac{3}{s^2+9},\end{aligned}$$

assuming $s>0$.

5.1.10 We can write

$$F(s)=\lim_{T\to\infty}\int_0^T e^{-(s+3)t}\sin(2t)\,dt.$$

Then a similar analysis as in question 6 gives

$$F(s)=\frac{2}{(s+3)^2+4},$$

provided $s>-3$.

5.2.6 Using linearity and the table on p204, we have

$$F(s)=\frac{6}{s^2+9}+\frac{3s}{s^2+25},$$

provided $s>0$.

5.2.10 Because $\frac{d}{dt}e^{2t} = 2e^{2t}$, we have

$$\mathcal{L}(y')(s) = \mathcal{L}(2e^{2t})(s) = \frac{2}{s-2}.$$

We can also check

$$s\mathcal{L}(e^{2t}) - y(0) = \frac{s}{s-2} - 1 = \frac{s-(s-2)}{s-2} = \frac{2}{s-2},$$

which agrees with our solution above.

5.3.4 We write

$$Y(s) = 5\frac{s}{s^2+9},$$

so that by linearity, $y(t) = 5\cos(3t)$.

5.3.6 Similarly,

$$Y(s) = \frac{1}{9} \cdot \frac{6}{s^4},$$

so that $y(t) = \frac{1}{9}t^3$.

5.3.16 We can write

$$Y(s) = \frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4},$$

in which case $y(t) = e^{-2t}(\cos(2t) - \sin(2t))$.

5.3.26 Using partial fractions,

$$\frac{4s+15}{2s^2+3s} = \frac{A}{s} + \frac{B}{2s+3}.$$

Upon cross-multiplication and comparing numerators, we have $4s + 15 = A(2s + 3) + Bs$. Plugging in $s = 0$ gives $A = 5$, with $s = -2/3$ giving $B = -6$. Now we can easily take the inverse transform of the difference, from which we see $y(t) = 5 - e^{-3t/2}$.

5.3.30 Similarly, we use partial fractions to write

$$\frac{7s^2+20s+53}{(s-1)((s+1)^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{(s+1)^2+4}.$$

This produces the equation $A((s+1)^2+4) + Bs(s-1) + C(s-1) = 7s^2+20s+53$. Plugging in $s = 1$ gives $A = 10$. With $s = 0$ we see $C = -3$. Finally, plugging in any other value for s shows $B = -3$. Taking the inverse transform of the above, we see

$$y(t) = 10e^t - 3e^{-t}\cos 2t.$$

5.4.2 Taking the transform of both sides, we have

$$\mathcal{L}(y' + 9y)(s) = sY(s) - y(0) + 9Y(s) = Y(s)(s+9) = \frac{1}{s+1} = \mathcal{L}(e^{-t})(s).$$

Hence,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+9)}\right] = \mathcal{L}^{-1}\left[\frac{1}{8(s+1)} - \frac{1}{8(s+9)}\right],$$

which is easily computed as $y(t) = \frac{1}{8}\left(e^{-t} - e^{-9t}\right)$.

5.4.6 The same approach as above gives

$$Y(s) = \frac{2}{s^3(s+8)} - \frac{1}{s+8}.$$

We need to take the partial fraction decomposition of the first term. We write

$$\frac{2}{s^3(s+8)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+8},$$

which produces the equation $2 = As^2(s+8) + Bs(s+8) + C(s+8) + Ds^3$. We take $s = 0$ to see $C = 1/4$. Taking $s = -8$, we have $D = -1/256$. We are left with two unknowns A and B for which the substitution method will not work. Instead, we can find a system of equations by taking $s = 1$ and $s = -1$, for example. In this case, we have

$$\begin{aligned} 2 &= 9A + 9B + 9C + D \\ 2 &= 7A - 7B + 7C - D. \end{aligned}$$

Because we know the values of C and D, this is two equations with two unknowns, which is easily solvable. We find $A = 1/256$ and $B = -1/32$. Finally, we can write

$$Y(s) = -\frac{257}{256(s+8)} + \frac{1}{256s} - \frac{1}{32s^2} + \frac{1}{4s^3},$$

which we can easily transform to find

$$y(t) = \frac{1}{256} - \frac{1}{32}t + \frac{1}{8}t^2 - \frac{257}{256}e^{-8t}.$$

Phew.

5.4.10 Going through the same procedure above, we have

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}.$$

Again, we need to take the partial fraction decomposition of the latter term. Write

$$\frac{2}{(s-4)(s+2)^3} = \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3},$$

which again gives $2 = A(s+2)^3 + (s-4)(B(s+2)^2 + C(s+2) + D)$. Taking $s = 4$ gives $A = 1/108$. With $s = -2$, we see $D = -1/3$. We again substitute $s = 1$ and $s = -1$ to obtain a system of equations, from which we see $B = -1/108$ and $C = -1/18$. Finally we have

$$Y(s) = \frac{109}{108(s-4)} - \frac{1}{108(s+2)} - \frac{1}{18(s+2)^2} - \frac{1}{3(s+2)^3}.$$

We now apply the eighth listed transform on p204 to find

$$y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}.$$

5.4.20 We take the transform of the left hand side:

$$\mathcal{L}[y'' - 2y' - 3y](s) = s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) - 3Y(s) = Y(s)(s^2 - 2s - 3) - s + 3.$$

Setting this equal to $\frac{1}{s-4} = \mathcal{L}[e^{4t}](s)$ and solving for $Y(s)$, we have

$$Y(s) = \frac{1}{s+1} + \frac{1}{(s-4)(s+1)(s-3)}.$$

Writing the latter term in partial fraction form

$$\frac{1}{(s-4)(s+1)(s-3)} = \frac{A}{s-4} + \frac{B}{s+1} + \frac{C}{s-3},$$

we obtain the equation $1 = A(s+1)(s-3) + B(s-4)(s-3) + C(s-4)(s+1)$. With $s = 4$, $s = -1$, and $s = 3$, we find $A = 1/5$, $B = 1/20$, and $C = -1/4$. Hence

$$Y(s) = \frac{21}{20(s+1)} - \frac{1}{4(s-3)} + \frac{1}{5(s-4)}.$$

A straightforward inverse transform yields $y(t) = \frac{21}{20}e^{-t} - \frac{1}{4}e^{3t} + \frac{1}{5}e^{4t}$.

5.4.27 a. The corresponding characteristic polynomial is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$. This polynomial clearly has roots $\lambda = 3$ and $\lambda = 1$. So the general solution to the differential equation is

$$y(t) = C_1e^{3t} + C_2e^t.$$

Taking $t = 0$ and using the initial conditions, we obtain the system

$$\begin{aligned} 1 &= C_1 + C_2 \\ -1 &= 3C_1 + C_2, \end{aligned}$$

so that $C_1 = -1$ and $C_2 = 2$, yielding $y(t) = 2e^t - e^{3t}$.

b. Taking the transform,

$$\mathcal{L}[y'' - 4y' + 3y](s) = s^2Y(s) - s + 1 - 4sY(s) + 4 + 3Y(s) = Y(s)(s^2 - 4s + 3) + 5 = 0.$$

Then

$$Y(s) = -\frac{5}{(s-3)(s-1)} = \frac{2}{s-1} - \frac{1}{s-3},$$

where we have of course used partial fractions. The inverse transform gives $y(t) = 2e^t - e^{3t}$, as in part a.

5.5.2 Using the notation of proposition 5.6, we have in this case $f(t-1) = e^{2(t-1)}$. We need $f(t)$, so we can take

$$f(t) = f((t+1)-1) = e^{2((t+1)-1)} = e^{2t}.$$

Then by the aforementioned proposition,

$$\mathcal{L}[H(t-1)e^{2(t-1)}](s) = e^{-s}\mathcal{L}[e^{2t}](s) = \frac{e^{-s}}{s-2}.$$

5.5.6 Again, using the notation of proposition 5.6, we have here $f(t-2) = e^{-t}$, so that $f(t) = e^{-(t+2)} = e^{-t}/e^2$. Then

$$\mathcal{L}[H(t-2)e^{-t}](s) = e^{-2s}e^{-2}\mathcal{L}[e^{-t}](s) = \frac{e^{-2(s+1)}}{s+1}.$$

5.6.4 Since $\mathcal{L}[\delta(t)](s) = 1$, we obtain

$$s^2E(s) - sy(0) - y'(0) + 4E(s) = E(s)(s^2+4) = 1.$$

Then $E(s) = \frac{1}{s^2+4}$ with inverse $e(t) = \frac{1}{2}\sin 2t$. You could also multiply the solution by $H(t)$ (see below and p231). The book is terribly inconsistent on whether this is required or not.

5.6.8 a. Taking the transform of both sides, we have $s^2X(s) = 2$ or $X(s) = 2/s^2$. Now taking the inverse transform of $X(s)$ gives $x(t) = 2t$. But as discussed on p231, this will not satisfy the initial conditions; specifically, we do not have $x'(0) = 0$. To account for this, we take $x(t) = 0$ for $t < 0$; that is, we let $x(t) = 2tH(t)$. Notice that we still do not satisfy $x'(0) = 0$ since in fact the derivative will not exist there. Instead, we interpret the requirement $x'(0) = 0$ to mean $\lim_{t\to 0^-} x'(t) = 0$, which is met in this case.

b. We note that because $\delta_0^\epsilon = \frac{1}{\epsilon}(H(t) - H(t-\epsilon))$,

$$\mathcal{L}[\delta_0^\epsilon(t)](s) = \frac{1-e^{-s\epsilon}}{s\epsilon}.$$

So in this case, we solve as above to find

$$X(s) = 2\frac{1-e^{-s\epsilon}}{s^3\epsilon}.$$

Instead of taking the inverse to find $x_\epsilon(t)$, we instead invert the proposed solution. If we find $X_\epsilon(s) = X(s)$ above, then the two solutions must in fact be equal, at least for $t \geq 0$, by the uniqueness of transforms.

We can write $x_\epsilon(t) = \frac{1}{\epsilon}t^2(H(t)-H(t-\epsilon))+(2t-\epsilon)H(t-\epsilon)$, using the heavyside function. To transform $t^2H(t-\epsilon)$, we need by proposition 5.6 the transform of $(t+\epsilon)^2 = t^2+2\epsilon t+\epsilon^2$. For the transform of $(2t-\epsilon)H(t-\epsilon)$, we need the transform of $2(t+\epsilon)-\epsilon = 2t+\epsilon$. This gives

$$\begin{aligned}
\mathcal{L}[x_\epsilon(t)](s) &= \frac{1}{\epsilon}\left[\mathcal{L}[t^2](s) - \mathcal{L}[t^2+2\epsilon t+\epsilon^2](s)e^{-s\epsilon}\right] + \mathcal{L}[2t+\epsilon](s)e^{-s\epsilon} \\
&= \frac{1}{\epsilon}\left[\frac{2}{s^3} - e^{-s\epsilon}\left(\frac{2}{s^3}+\frac{2\epsilon}{s^2}+\frac{\epsilon^2}{s}\right)\right] + e^{-s\epsilon}\left[\frac{2}{s^2}+\frac{\epsilon}{s}\right] \\
&= \frac{2}{s^3\epsilon} - e^{-s\epsilon}\frac{2}{s^3\epsilon},
\end{aligned}$$

which agrees with our $X(s)$ from above. Hence $x_\epsilon(t)$ must solve the equation.

c. Note first that $\lim_{\epsilon\to 0} x_\epsilon(0) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\cdot 0^2 = 0$, which agrees with $x(0)$ from part a. If $t > 0$, then $\lim_{\epsilon\to 0} x_\epsilon(t) = \lim_{\epsilon\to 0} 2t-\epsilon = 2t$, which again agrees with part a. We conclude $\lim_{\epsilon\to 0} x_\epsilon(t) = 2tH(t)$, at least for $t \geq 0$.

5.6.9 a. This is done on p231.

b. Similar to above, we obtain

$$X(s) = \frac{1-e^{-s\epsilon}}{\epsilon s(s^2+2s+2)}.$$

We proceed by inverting $X(s)$. Let

$$F(s) = \frac{1}{s((s+1)^2+1)},$$

so that $X(s) = \frac{1}{\epsilon}(F(s) - e^{-s\epsilon}F(s))$. We write

$$F(s) = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1},$$

to arrive at the equation $1 = A((s+1)^2+1) + Bs^2 + Cs$. Letting $s = 0$ we obtain $A = 1/2$. Taking, for example, $s = 1$ and $s = -1$, produces the system

$$\begin{aligned}
1 &= 5A + B + C \\
1 &= A + B - C,
\end{aligned}$$

which upon using $A = 1/2$ gives $B = -1/2$ and $C = -1$, so that

$$F(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2+1}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right].$$

We can easily invert this now, so that $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}\left[1 - e^{-t}(\cos t + \sin t)\right]$. To find $\mathcal{L}^{-1}[e^{-s\epsilon}F(s)]$, we use proposition 5.12, which gives

$$\mathcal{L}^{-1}[e^{-s\epsilon}F(s)] = H(t-\epsilon)f(t-\epsilon) = H_\epsilon(t) \cdot \frac{1}{2}\left[1 - e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon))\right].$$

Putting all this together, we obtain

$$x_\epsilon(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\epsilon}\left[1 - e^{-t}(\cos t + \sin t) + H_\epsilon(t)\left[1 - e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon))\right]\right],$$

which is exactly what we wanted (upon writing as a piecewise function).

c. We take $t > 0$. Then

$$\lim_{\epsilon \to 0} x_\epsilon(t) = \lim_{\epsilon \to 0} \frac{-e^{-t}(\cos t + \sin t) + e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon))}{2\epsilon}.$$

We note l'Hôpital's rule applies since the numerator and denominator both have limit zero. The rule states that the above limit is equal to the limit of the quotient of the derivatives of the numerator and denominator (with respect to the variable ϵ). That is,

$$\lim_{\epsilon \to 0} x_\epsilon(t) = \lim_{\epsilon \to 0} \frac{e^{-(t-\epsilon)}(\cos(t-\epsilon) + \sin(t-\epsilon)) + e^{-(t-\epsilon)}(\sin(t-\epsilon) - \cos(t-\epsilon))}{2},$$

which is just $e^{-t}\sin t$, as claimed.