9.8

2. (a) If $\mathbf{x}_1(t) = (\sin 2t, 2\cos 2t)^T$, then $\mathbf{x}_1'(t) = (2\cos 2t, -4\sin 2t)^T$ and

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}_1(t) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} = \begin{pmatrix} 2\cos 2t \\ -4\sin 2t \end{pmatrix},$$

so $\mathbf{x}_1$ is a solution of

$$\mathbf{x}_1' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}.$$

Similarly, if $\mathbf{x}_2(t) = (\cos 2t, -2\sin 2t)^T$, then $\mathbf{x}_2'(t) = (-2\sin 2t, -4\cos 2t)^T$ and

$$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}_2(t) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} = \begin{pmatrix} -2\sin 2t \\ -4\cos 2t \end{pmatrix},$$

so $\mathbf{x}_2$ is also a solution of

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}.$$

To show independence, we need only show that the functions are independent at one value of t. However,

$$\mathbf{x}_1(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are clearly independent ($\mathbf{x}_2(0)$ is not a multiple of $\vec{x}_1(0)$).

(b) Because

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} + C_2 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix},$$

The first component of $\mathbf{x}(t)$ is $y(t) = C_1 \sin 2t + C_2 \cos 2t$. Thus,

$$y' = 2C_1 \cos 2t - 2C_2 \sin 2t, \quad \text{and}$$
$$y'' = -4C_1 \sin 2t - 4C_2 \cos 2t,$$

and

$$\begin{aligned} y'' + 4y &= (-4C_1 \sin 2t - 4C_2 \cos 2t) \\ &\quad + 4(C_1 \sin 2t + C_2 \cos 2t) \\ &= 0. \end{aligned}$$

6. If $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$, suppose that there exists constants C_1, C_2, and C_3 such that

$$C_1e^t + C_2te^t + C_3t^2e^t = 0$$

for all t. If $t = 1$, then

$$\begin{aligned} C_1e + C_2e + C_3e &= 0 \\ (C_1 + C_2 + C_3)e &= 0 \\ C_1 + C_2 + C_3 &= 0. \end{aligned}$$

If $t = -1$, then

$$\begin{aligned} C_1e^{-1} - C_2e^{-1} + C_3e^{-1} &= 0 \\ (C_1 - C_2 + C_3)e^{-1} &= 0 \\ C_1 - C_2 + C_3 &= 0 \end{aligned}$$

Finally, if $t = 0$, then $C_1 = 0$ and the last two equations become

$$\begin{aligned} C_2 + C_3 &= 0 \\ -C_2 + C_3 &= 0. \end{aligned}$$

Because the coefficient matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has determinant $D = 2$, the coefficient matrix is nonsingular and this last system has unique solution $C_2 = C_3 = 0$. Hence, $C_1 = C_2 = C_3 = 0$ and the solutions $y_1(t) = e^t4$, $y_2(t) = te^t$, and $y_3(t) = t^2e^t$ are linearly independent.

18. If $y''' + 2y'' - 5y' - 6y = 0$, then the characteristic equation is $p(\lambda) = \lambda^3 + 2\lambda^2 - 5\lambda - 6$. Note that -1 is a root of p, so

$$\begin{aligned} p(\lambda) &= (\lambda + 1)(\lambda^2 + \lambda - 6) \\ &= (\lambda + 1)(\lambda + 3)(\lambda - 2). \end{aligned}$$

Thus, the characteristic polynomial has roots -1, -3, and 2, leading to the general solution

$$y(t) = C_1e^{-t} + C_2e^{-3t} + C_3e^{2t}.$$

12. If $y_1 = \cos 3t$, then

$$y_1' = -3\sin 3t, \quad y_1'' = -9\cos 3t, \quad y_1''' = 27\sin 3t, \quad \text{and} \quad y_1^{(4)} = 81\cos 3t.$$

Thus,

$$y_1^{(4)} + 13y_1'' + 36y_1 81\cos 3t + 13(-9\cos 3t) + 36\cos 3t = 0$$

and y_1 is a solution of $y^{(4)} + 13y'' + 36y = 0$. In a similar manner, $y_2 = \sin 3t$, $y_3 = \cos 2t$, and $y_4 = \sin 2t$ are also solutions. Finally, the Wronskian is

$$W(t) = \det \begin{pmatrix} y_1 & y_1 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{pmatrix} = \det \begin{pmatrix} \cos 3t & \sin 3t & \cos 2t & \sin 2t \\ -3\sin 3t & 3\cos 3t & -2\sin 2t & 2\cos 2t \\ -9\cos 3t & -9\sin 3t & -4\cos 3t & -4\sin 2t \\ 27\sin 3t & -27\cos 3t & 8\sin 2t & -8\cos 2t \end{pmatrix}.$$

Using a computer, $W(t) = 150$, so the solutions y_1, y_2, y_3, and y_4 are linearly independent.

9.7

3. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -6 & -15 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

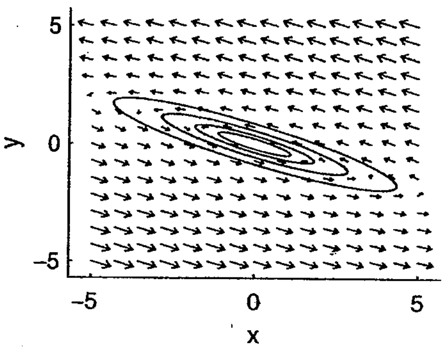
the coefficient matrix

$$A = \begin{pmatrix} -6 & -15 \\ 3 & 6 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^2 + 9,$$

producing eigenvalues $\lambda = \pm 3i$. Therefore, the equilibrium point at the origin is a stable center.



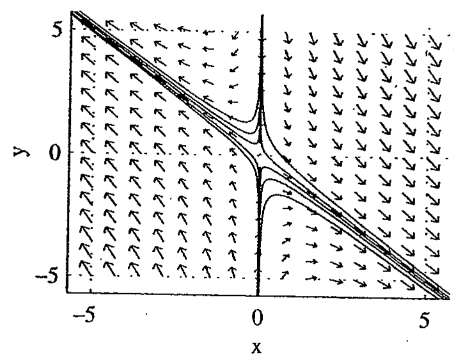
4. In matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The coefficient matrix

$$A = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$$

is lower triangular, so the eigenvalues lie on the diagonal, $\lambda_1 = 2$ and $\lambda_2 = -1$. Because there is at least one positive eigenvalue, the equilibrium point at the origin is unstable. Indeed, with one positive and one negative eigenvalue, the origin is a saddle.



9. Consider the system

$$\mathbf{y}' = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix} \mathbf{y}.$$

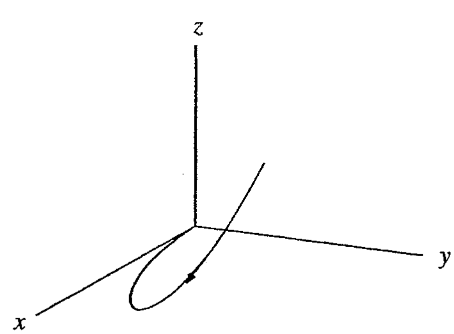
Using a computer, matrix

$$A = \begin{pmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6,$$

and eigenvalues $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$. Because the real parts of all eigenvalues are negative, the equilibrium point at the origin is asymptotically stable. One such solution, with initial condition $(1, 1, 1)^T$, is shown in the following figure.



9.9

3. If

$$A = \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

then the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - \lambda = \lambda(\lambda - 1),$$

generating eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$. The associated eigenvectors are

$$A - 0I = \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and}$$

$$A - I = \begin{pmatrix} -4 & 6 \\ -2 & 3 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Thus, the homogeneous solution is $\mathbf{y}_h = C_1\mathbf{y}_1 + C_2\mathbf{y}_2$, where

$$\mathbf{y}_1(t) = e^{0t}\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and}$$

$$\mathbf{y}_2(t) = e^t\mathbf{v}_2 = \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{pmatrix} 2 & 3e^t \\ 1 & 2e^t \end{pmatrix}.$$

The inverse of $Y(t)$ is calculated

$$Y^{-1}(t) = \frac{1}{e^t}\begin{pmatrix} 2e^t & -3e^t \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -e^{-t} & 2e^{-t} \end{pmatrix}.$$

Hence,

$$Y^{-1}(t)\mathbf{f}(t) = \begin{pmatrix} 2 & -3 \\ -e^{-t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 5e^{-t} \end{pmatrix},$$

and

$$\int Y^{-1}(t)\mathbf{f}(t)\,dt = \int \begin{pmatrix} -6 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} -6t \\ -5e^{-t} \end{pmatrix}.$$

Thus,

$$\mathbf{y}_p = Y(t)\int Y^{-1}\mathbf{f}(t)\,dt = \begin{pmatrix} 2 & 3e^t \\ 1 & 2e^t \end{pmatrix}\begin{pmatrix} -6t \\ -5e^{-t} \end{pmatrix} = \begin{pmatrix} -12t - 15 \\ -6t - 10 \end{pmatrix}.$$

Finally, the general solution is

$$\mathbf{y}(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix} + \begin{pmatrix} -12t - 15 \\ -6t - 10 \end{pmatrix}.$$

#4 $A = \begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix}$ $\quad p(\lambda) = (-3-\lambda)(8-\lambda) + 30$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -6 & 10 \\ -3 & 5 \end{pmatrix} \sim \begin{pmatrix} -3 & 5 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\lambda = 2$: $A - 2I = \begin{pmatrix} -5 & 10 \\ -3 & 6 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$Y = \begin{pmatrix} 5e^{3t} & 2e^{2t} \\ 3e^{3t} & e^{2t} \end{pmatrix}$ $\quad \det(Y) = -e^{5t}$ $\quad Y^{-1} = -e^{-5t} \begin{pmatrix} e^{2t} & -2e^{2t} \\ -3e^{3t} & 5e^{3t} \end{pmatrix}$

$$= \begin{pmatrix} -e^{-3t} & 2e^{-3t} \\ 3e^{-2t} & -5e^{-2t} \end{pmatrix}$$

$$Y \int Y^{-1} \vec{f} \, dt = Y \int \begin{pmatrix} -e^{-3t} & 2e^{-3t} \\ 3e^{-2t} & -5e^{-2t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix} dt$$

$$= Y \int \begin{pmatrix} -e^{-4t} + 2e^{-t} \\ 3e^{-3t} - 5 \end{pmatrix} dt$$

$$= \begin{pmatrix} 5e^{3t} & 2e^{2t} \\ 3e^{3t} & e^{2t} \end{pmatrix} \begin{pmatrix} +\frac{1}{4}e^{-4t} - 2e^{-t} \\ -e^{-3t} - 5t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{4}e^{-t} - 10e^{2t} - 2e^{-t} - 10te^{2t} \\ \frac{3}{4}e^{-t} - 6e^{2t} - e^{-t} - 5te^{2t} \end{pmatrix}$$

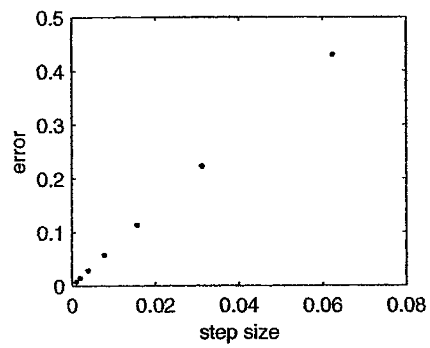
$$= \begin{pmatrix} -3/4\, e^{-t} - 10e^{2t}(1+t) \\ -1/4\, e^{-t} - e^{2t}(6+5t) \end{pmatrix}$$

$$= \vec{y}_p$$

$$\boxed{\vec{y}(t) = \vec{y}_p + C_1 e^{3t} \begin{pmatrix} 5 \\ 3 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

6.1

11. (a) Note that a plot of the error versus the step size signifies a linear relationship. Indeed, a line through the origin with the appropriate slope should pass through or close to each data point.



(b) We can estimate the proportionality constant by picking two points from the figure and calculating the slope of the line through the chosen points. Let's use the first and last points.

$$\lambda = \frac{0.0072076631 - 0.4303893417}{0.0009765625 - 0.0625000000} \approx 6.8784$$

We can use $E = \lambda h$ to calculate the step size.

$$E = \lambda h$$
$$h = \frac{E}{\lambda}$$
$$h = \frac{0.001}{6.8784}$$
$$h = 1.454 \times 10^{-4}$$

Since $h = (b - a)/N$,

$$N = \frac{b-a}{h}$$
$$N \frac{2-0}{1.454 \times 10^{-4}}$$
$$N = 13757.$$

It will take about 13,757 iterations to achieve the required accuracy.

(c) We ran our Euler routine on a 300 MHz PC using the step size from part (b). The run took approximately 56 seconds and reported an error from the true value of 0.00107417614304.