Section 2.4

4. We have $a(t) = -2t$, so $u(t) = e^{t^2}$. Multiplying by u, the equation becomes

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}.$$

We verify that the left-hand side is the derivative of $e^{t^2}y$, so when we integrate we get

$$e^{t^2}y(t) = \frac{5}{2}e^{t^2} + C.$$

Solving for y we get the general solution

$$y(t) = \frac{5}{2} + Ce^{-t^2}.$$

6. If we write the equations as $x' = (4/t)x + t^3$, we see that $a(t) = 4/t$. Thus the integrating factor is

$$u(t) = e^{-\int(4/t)\,dt} = e^{-4\ln t} = t^{-4}.$$

Multiplying by u, the equation becomes

$$t^{-4}x' - 4t^{-5}x = t^{-1}.$$

After verifying that the left-hand side is the derivative of $t^{-4}x$, we can integrate and get

$$t^{-4}x(t) = \ln t + C.$$

Hence the general solution is

$$x(t) = t^4\ln t + Ct^4.$$

18. Solve $xy' + 2y = \sin x$ for y'.

$$y' = -\frac{2}{x}y + \frac{\sin x}{x}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -2/x$ and $f(x) = (\sin x)/x$. It is important to note that neither a nor f is continuous at $x = 0$, a fact that will heavily influence our interval of existence.

An integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$x^2y' + 2xy = x\sin x$$
$$(x^2y)' = x\sin x$$

Integration by parts yields

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + C.$$

Consequently,

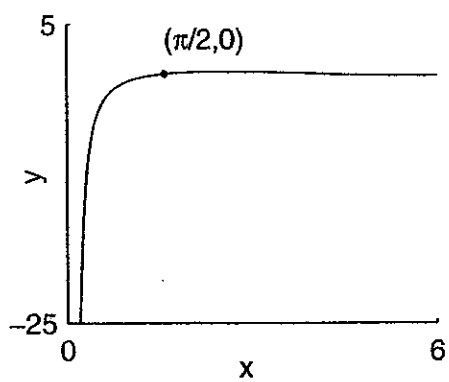
$$x^2y = -x\cos x + \sin x + C,$$
$$y = -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x + \frac{C}{x^2}.$$

The initial condition provides

$$0 = y(\pi/2) = \frac{4}{\pi^2} + \frac{4C}{\pi^2}.$$

Consequently, $C = -1$ and $y = -(1/x)\cos x + (1/x^2)\sin x - 1/x^2$.

We cannot extend any interval to include $x = 0$, as our solution is undefined there. The initial condition $y(\pi/2) = 0$ forces the solution through a point with $x = \pi/2$, a fact which causes us to select $(0, +\infty)$ as the interval of existence. The solution curve is shown in the following figure. Note how it drops to negative infinity as x approaches zero from the right.



Section 2.5

4. Let $x(t)$ represent the amount of salt in the solution at time t. Let r represent the rate (gal/min) that water enters (and leaves) the tank. Consequently, the rate at which salt enters the tank is 0 gal/min, but the

$$\text{rate out} = r\ \text{gal/min} \times \frac{x(t)}{500}\ \text{lb/gal} = \frac{r}{500}x(t)\ \text{lb/min}.$$

Thus,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out},$$
$$\frac{dx}{dt} = -\frac{r}{500}x.$$

Let $c(t)$ represent the concentration at time t. Thus, $c(t) = x(t)/500$, or $500c(t) = x(t)$ and $500c'(t) = x'(t)$. Substitute these into the rate equation to produce

$$500c' = -\frac{r}{500}(500c),$$
$$c' = -\frac{r}{500}c.$$

This equation is separable, with solution $c = Ae^{-(r/500)t}$. Use the initial concentration, $c(0) = .05$ lb/gal, to produce

$$c = 0.05e^{-(r/500)t}.$$

The concentration must reach 1% in one hour (60 min), so $c(60) = 0.01$ and

$$0.01 = 0.05e^{-(r/500)(60)},$$
$$\frac{1}{5} = e^{-(3/25)r},$$
$$r = \frac{25}{3}\ln 5,$$
$$r \approx 13.4\ \text{gal/min}.$$

Section 2.2

4. Separate the variables and integrate.

$$\frac{dy}{dx} = (1+y^2)e^x$$
$$\frac{1}{1+y^2}\,dy = e^x dx$$
$$\tan^{-1} y = e^x + C$$
$$y(x) = \tan(e^x + C)$$

10.

$$x\frac{dy}{dx} = y(1+2x^2)$$
$$\frac{dy}{y} = \frac{1+2x^2}{x}\,dx = \left[\frac{1}{x} + 2x\right] dx$$
$$\ln|y| = \ln|x| + x^2 + C$$
$$|y(x)| = e^{\ln|x| + x^2 + C} = e^C|x|e^{x^2}$$
$$y(x) = Axe^{x^2}$$

18.

$$\frac{dy}{dx} = \frac{x}{1+2y}$$
$$(1+2y)\,dy = x\,dx$$
$$y + y^2 = x^2/2 + C$$

34. Let $y(t)$ be the temperature of the beer at time t minutes after being placed into the room. From Newton's law of cooling, we obtain

$$y'(t) = k(70 - y(t)) \quad y(0) = 40$$

Note k is positive since $70 > y(t)$ and $y'(t) > 0$ (the beer is warming up). This equation separates as

$$\frac{dy}{70-y} = k\,dt$$

which has solution $y = 70 - Ce^{-kt}$. From the initial condition, $y(0) = 40$, $C = 30$. Using $y(10) = 48$, we obtain $48 = 70 - 30e^{-10k}$ or $k = (-1/10)\ln(11/15)$ or $k = .0310$. When $t = 25$, we obtain $y(25) = 70 - 30e^{-.598} \approx 56.18°$.

Section 2.3

4. In the first 60s the rocket rises to an elevation of $(100-9.8)t^2/2 = 162,360$m and achieves a velocity of $v(60) = (100-9.8)*60 = 5412$m/s. After that the velocity is $5412 - 9.8t$. This is zero at the highest point, reached when $t_1 = 552.2$s. The altitude at that point is $162,360 + 5412\,t_1 - 9.8\,t_1^2/2 = 1.657 \times 10^6$m. From there to the ground it takes t_2s, where $4.9t_2^2 = 1.657 \times 10^6$, or $t_2 = 581.5$s. The total trip takes $60 + 552.2 + 581.5 = 1193.7$s.

11. Without air resistance, $v_0 = \sqrt{2 \times 13.5g} = 16.2665$m/s. With air resistance, v_0 is defined by

$$\int_{v_0}^{0} \frac{v\,dv}{v + mg/r} = -\frac{r}{m}\int_{1.5}^{15} dy.$$

Hence,

$$\begin{aligned}
&- v_0 + (mg/r)\ln(v_0 + mg/r) - (mg/r)\ln(mg/r) \\
&= -13.5\frac{r}{m} \quad \text{or} \\
&- v_0 + 49\ln(v_0 + 49) - 49\ln(49) = -2.7
\end{aligned}$$

This is an implicit equation for v_0. Solving on a calculator or a computer yields $v_0 = 18.1142$m/s.

Section 3.3

6. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned}
\left(e^{-rt}P\right)' &= -we^{-rt}, \\
e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\
P &= \frac{w}{r} + Ce^{rt}.
\end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of four years. Consequently, $P(4) = 0$, so

$$\begin{aligned}
0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(4)}, \\
\frac{w}{r}\left(e^{4r} - 1\right) &= P_0e^{4r}, \\
P_0 &= \frac{w}{r}\left(1 - e^{-4r}\right) \\
P_0 &= \frac{(225)(12)}{0.08}\left(1 - e^{-4(0.08)}\right), \\
P_0 &\approx \$9,242,47
\end{aligned}$$

7. Let $P(t)$ represent the loan balance after t years. Let r represent the annual rate, w the annual payment, and P_0 the amount of the loan. Then

$$P' = rP - w \quad P(0) = P_0.$$

The equation is linear with integrating factor e^{-rt}. Consequently,

$$\begin{aligned}
\left(e^{-rt}P\right)' &= -we^{-rt}, \\
e^{-rt}P &= \frac{w}{r}e^{-rt} + C, \\
P &= \frac{w}{r} + Ce^{rt}.
\end{aligned}$$

Use $P(0) = P_0$ to produce $C = P_0 - w/r$ and

$$P(t) = \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{rt}.$$

Now, the loan is exhausted at the end of 30 years. Consequently, $P(30) = 0$, so

$$\begin{aligned}
0 &= \frac{w}{r} + \left(P_0 - \frac{w}{r}\right)e^{r(30)}, \\
\frac{w}{r}\left(e^{30r} - 1\right) &= P_0e^{30r}, \\
w &= \frac{rP_0}{1 - e^{-30r}}, \\
w &= \frac{0.08(100000)}{1 - e^{-30(0.08)}} \\
w &\approx \$8,798.15
\end{aligned}$$

11. (a) Let $P(n)$ represent the balance at the end of n compounding periods, I the annual interest rate, m the number of compounding periods per year, and P_0 the initial investment. Thus,

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n), \quad P(0) = P_0.$$

(b) Compare

$$P(n+1) = \left(1 + \frac{I}{m}\right)P(n), \quad P(0) = P_0.$$

with

$$a(n+1) = ra(n), \quad a(0) = a_0,$$

and note that $r = 1 + I/m$ and $a_0 = P_0$. Consequently,

$$a(n) = a_0r^n$$

becomes

$$P(n) = P_0\left(1 + \frac{I}{m}\right)^n.$$

Section 3.4

12. (a) Let $P(t)$ represent the balance at the end of t years. Let r represent the annual rate and P_0 the initial investment. Thus,

$$P' = rP, \quad P(0) = P_0.$$

Consequently,

$$\begin{aligned} P(t) &= P_0 e^{rt}, \\ P(10) &= 2000e^{0.06(10)}, \\ P(10) &\approx \$3,644.24 \end{aligned}$$

(b) In semiannual case, $m = 2$. Furthermore, there are 20 compounding periods in 10 years, so

$$P(20) = 2000\left(1 + \frac{0.06}{2}\right)^{20} \approx \$3,612.22.$$

In the monthly case, $m = 12$. There are 120 compounding periods in 10 years, so

$$P(120) = 2000\left(1 + \frac{0.06}{12}\right)^{120} \approx \$3,638.79.$$

In the daily case, $m = 365$. There are 3650 compounding periods in 10 years, so

$$P(3650) = 2000\left(1 + \frac{0.06}{365}\right)^{3650} \approx \$3,644.06.$$

14. $I(t) = (E + (RI_0 - E)e^{-Rt/L})/R$