Extra Problem

Let us take an indirect approach by first constructing a double cover \( \tilde{M} \) over \( M \) based on \( n \)-forms and proving it is the same as the orientation double cover \( M \). It will then be easy to argue that giving a nowhere vanishing \( n \)-form is equivalent to picking an orientation (if any).

Consider

\[
\tilde{M} := \{(p, \omega_p) : p \in M, 0 \neq \omega_p \in \Lambda^n T_p M \} / \sim
\]

where the only equivalence relation is \( (p, \omega_p) \sim (p, c \omega_p) \) for any \( c > 0 \). Projection to the first coordinate gives a map \( \pi : \tilde{M} \to M \). For any open set \( U \subset M \) and nowhere vanishing \( n \)-form \( \omega \) on \( U \), we define the set of equivalence classes \( U_\omega := \{[p, \omega_p] : p \in U \} \subset \tilde{M} \) to be open; it is straightforward to check that such \( U_\omega \) form the basis of a topology on \( \tilde{M} \) and \( \pi \) is a double covering. If we now fix a generator \( a_0 \in H_n(\mathbb{R}, \mathbb{R} - \{0\}) \) and let \( \omega_0 = dx_1 \wedge \cdots \wedge dx_n \in \Lambda^n (\mathbb{R}^n) \), we can define a map \( \tilde{M} \to \tilde{M} \) as follows: given \( [p, \omega_p] \in \tilde{M} \), let \( \alpha : (U, \{p\}) \to (\mathbb{R}, \mathbb{R} - \{0\}) \) be a coordinate chart such that \( \alpha^* \omega_0|_p \) is a positive multiple of \( \omega_p \); one checks that \( \alpha^*_{-1}(a_0) \in H_n(U, U - \{p\}) \cong H_n(M, M - \{p\}) \in \tilde{M} \) is well-defined. Similarly, there is a map \( \tilde{M} \to \tilde{M} \); given \( a \in H_n(M, M - \{p\}) \), let \( \alpha : (U, \{p\}) \to (\mathbb{R}, \mathbb{R} - \{0\}) \) be a coordinate chart such that \( \alpha^*_{-1}(a_0) = a \); one again checks \( [p, \alpha^* \omega_0|_p] \in \tilde{M} \) is well-defined. The two maps are clearly inverse to each other and thus \( \tilde{M} \cong \tilde{M} \) as double covers over \( M \).

We now turn to the proof. If \( \omega \in \Omega^m(M) \) is nowhere vanishing, \( \{[p, \omega_p] : p \in M \} \) defines a global section of \( \tilde{M} \) and hence a global section of \( \tilde{M} \), i.e. an orientation on \( M \). Conversely, a global section of \( \tilde{M} \), or \( \tilde{M} \), defines local nonvanishing \( n \)-forms on \( M \) which can be glued into a global one via a partition of unity.

M&T 9.3

It only remains to show that \( \varphi \) maps \( M \) homeomorphically to its image (with the subspace topology). Since \( \varphi \) injects, \( \varphi^{-1} \) exists. For any closed subset \( C \subset M \), \( \varphi(C) \) is closed in \( N \), hence in \( \varphi(M) \); therefore \( \varphi^{-1} \) is continuous. This completes the proof. [Note: \( \varphi^{-1} \) is automatically smooth.]

M&T 9.18

It is easy to see \( A \) is a diffeomorphism of order 2, and thus \( A^* \) is an isomorphism of order 2 on any \( \Omega^r(M) \); we have a decomposition \( \Omega^r(M) = \Omega^r(M)_+ \oplus \Omega^r(M)_- \) into the eigenspaces \( A^* = \pm 1 \). [Details: every \( \omega \) can be written as \( \frac{1}{2}(\omega + A^* \omega) + \frac{1}{2}(\omega - A^* \omega) \in \Omega^r(M)_+ \oplus \Omega^r(M)_- \).

Clearly, \( \Omega^r(M)_+ \cap \Omega^r(M)_- = \{0\} \) Since \( dA^* = A^* d, d \) preserves the eigenvalue of \( A^* \) and thus \( \Omega^r(M)_+, \Omega^r(M)_- \) are subcomplexes. One easily sees that their cohomologies are precisely \( H^*(\tilde{M})_+ \subset H^*(\tilde{M}) \), the eigenspaces of \( H^*(A) = \pm 1 \).
Since $A \circ \pi = \pi$, $\pi^*\omega$ maps $\Omega^r(M)$ into $\Omega^r(M_+)$ and is injective because if $0 \neq \omega_p \in \Lambda^r(T_pM)$, then clearly $\pi^*(\omega_p) \neq 0$ at any point in $\pi^{-1}(p)$. \footnote{In general, if $f$ is a surjective submersion, $f^*$ is injective on forms.} As to surjectivity, let $\omega' \in \Omega^r(M_+)$, and $s_1, s_2 : U \to \pi^{-1}(U)$ be the two sections of $\pi$ over a small open subset $U \subset M$. Since $s_1 = A \circ s_2$ and $A^*\omega' = \omega'$, we have $\omega_U := s_1^*(\omega'_{|U_1}) = s_2^*(\omega'_{|U_2})$. As $U$ varies, $\omega_U$ clearly paste together into an $\omega \in \Omega^r(M)$ satisfying $\pi^*\omega = \omega'$. This proves $\pi^* : \Omega^*(M) \to \Omega^*(M_+)$ is an isomorphism, which is of course a chain map. Taking cohomology, we have $H^*(M) \cong H^*(M_+)$. \footnote{continued from M&\textit{T} 9.1}\ [\textit{M&\textit{T} 10.4}]

Let $\tilde{M}$ be the universal covering. If $A^*\omega' = \omega'$, then clearly $\omega$ descends to an $n$-form $\omega$ in $M$. However, since $\omega'$ is nowhere vanishing, so is $\omega$, contradicting the nonorientability of $M$. Therefore, we have $H^n(M) \cong H^n(\tilde{M}) = \{0\}$. \footnote{\textit{M&\textit{T} 10.5}}

From M&\textit{T} 10.4, we have $H^2(K^2) = \{0\}$. Also, we clearly have $H^0(K^2) = \mathbb{R}$. From M&\textit{T} 9.18, we have $H^1(K^2) \cong H^1(T^2)|_{A^*-1}$, where $H^1(T^2) = \mathbb{R}dx \oplus \mathbb{R}dy$ and $A$ is the map sending (the equivalence class of) $(x,y)$ to $(x + \frac{1}{2}, -y)$. Since $A^*dx = dx$ and $A^*dy = -dy$, we have $H^1(K^2) \cong \mathbb{R}$.

Let $\pi : \mathbb{R}^n \to T^n$ be the universal covering. If $\omega \in \Omega^1(T^n)$ is closed, then $\pi^*\omega$ is also closed, thus exact; let $\pi^*\omega = df$. For any $p \in \mathbb{R}^n$,

$$f(p + e_j) - f(p) = \int_p^{p+e_j} df = \int_p^{p+e_j} \pi^*\omega = \int_{C_j} \omega.$$  

Therefore, if the last integral vanishes for all $j$, $f$ is 1-periodic in every coordinate and descends to a function $f'$ on $T^n$. In that case, $\omega = df'$ is exact. Conversely, if $\omega$ is exact, the above integrals clearly vanish.

We have shown that

$$[\omega] \mapsto \left(\int_{C_1} \omega, \ldots, \int_{C_n} \omega\right)$$

is a well-defined and injective linear map from $H^1(T^n)$ to $\mathbb{R}^n$. On the other hand, given any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $\omega = a_1dx_1 + \ldots + a_n dx_n$ is a well-defined closed form on $T^n$ with $\int_{C_j} \omega = a_j$. We thus have an isomorphism $H^1(M) \cong \mathbb{R}^n$. 

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