M&T 8.2

(φ smooth ⇒ i ◦ φ is smooth) Because i is smooth, by definition of an embedded submanifold. (See M&T, p.60)

(i ◦ φ smooth ⇒ φ is smooth) Let p ∈ N and q = φ(p). By the inverse function theorem and the definition of an embedded submanifold, there exists a diffeomorphism $\alpha : U \rightarrow U'$ of open subsets in $\mathbb{R}^k$, such that $q \in U$, and $\alpha(U \cap M) = U' \cap (\mathbb{R}^m \times 0)$, and we may use $\alpha|_{U \cap M}$ as a coordinate chart of $M$. [Details: let $W \subset \mathbb{R}^k$, $W' \subset \mathbb{R}^m$ be open sets and $\gamma : W' \rightarrow \mathbb{R}^k$ a smooth map such that $q \in W \cap M = \gamma(W')$ and $D\gamma$ is injective at every point in $W'$. We may then extend $\gamma$ to a map $\gamma : W' \times (-\epsilon, \epsilon)^{k-m} \rightarrow \mathbb{R}^k$ with $D\gamma$ nonsingular at $\gamma^{-1}(q)$. The IFT then gives us the open sets $U, U'$ above, with $\alpha = \gamma^{-1}|_{U'}$.]

Since $i \circ \varphi$ is smooth, by definition there is a coordinate chart $\beta : V \rightarrow V' \subset \mathbb{R}^n$ of $N$ near $p$ such that $(i \circ \varphi) \circ \beta^{-1} = \varphi \circ \beta^{-1} : V' \rightarrow \mathbb{R}^k$ is smooth. Of course, it in fact maps into $U \cap M$ and $(\alpha|_{U \cap M}) \circ \varphi \circ \beta^{-1}$ as a map from $V'$ to $U' \cap (\mathbb{R}^m \times 0)$ is also smooth, which by definition means $\varphi$ is smooth at $p$. This completes the proof.

M&T 8.5

Straightforward. For the last part, the smooth structure on $T^2$ induces a (the) smooth structure on $K^2$ through the covering map $\pi : T^2 \rightarrow K^2$: if $\alpha : U \rightarrow \mathbb{R}^2$ is a coordinate chart on $T^2$ and $U$ is homeomorphic to $\pi(U)$, then let $\alpha \circ \pi^{-1} : \pi(U) \rightarrow \mathbb{R}^2$ be a coordinate chart. This is well-defined because the deck transformation $A : T^2 \rightarrow T^2$ is smooth.

M&T 8.6

[Some may think that this question is tautological because the stereographic projection clearly defines a coordinate chart on $S^n$ and is by definition a diffeomorphism. In fact, the starting point of the question is that $S^n$ is endowed with the smooth structure from its standard embedding in $\mathbb{R}^{n+1}$ (see M&T, p.60), and then we show the stereographic projection is a diffeomorphism with respect to this smooth structure. This is not totally trivial because for example it is known there are smooth manifolds homeomorphic but not diffeomorphic to $S^7$.]

The maps between $S^n - \{p\}$ and $\mathbb{R}^n$ are:

$$f : (S^n - \{p\}) \rightarrow \mathbb{R}^n, \quad (x_1, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}\right)$$

$$g : \mathbb{R}^n \rightarrow (S^n - \{p\}), \quad (y_1, \ldots, y_n) \mapsto \frac{1}{r^2 + 1}(2y_1, \ldots, 2y_n, r^2 - 1)$$

where $r^2 = y_1^2 + \ldots + y_n^2$. One easily checks that $f$ and $g$ are inverse to each other. $f$ is the composition of the inclusion $(S^n - \{p\}) \hookrightarrow (\mathbb{R}^{n+1} - \{x_{n+1} = 0\})$ and a map from $\mathbb{R}^{n+1} - \{x_{n+1} = 0\}$ to $\mathbb{R}^n$ which is a natural extension of $f$; the inclusion is smooth by definition of embedded submanifold and the second map is clearly smooth, so $f$ is smooth. On the other hand, the smoothness of $g$ follows from 8.2 above.
M&T 9.1

For any smooth path \( \alpha : (-\epsilon, \epsilon) \to M \) with \( \alpha(0) = p_0 \), we have

\[
0 = \left. \frac{d}{dt} \langle \alpha(t) - p, \alpha(t) - p \rangle \right|_{t=0} = 2\langle \alpha'(0), p_0 - p \rangle,
\]

and hence \( p_0 - p \perp \alpha'(0) \). Since every element in \( T_m M \) arises as \( \alpha'(0) \) for some \( \alpha \), we have \( p_0 - p \perp T_{p_0} M \).

**Extra Problem 1**

If \( f : M \to N \) is not surjective, say \( p \notin \text{im} f \), that \( f \) can be factored as follows:

\[
f : M \to N - \{p\} \hookrightarrow N.
\]

Applying \( H_n \) we get

\[
f_* : H_n(M) \to H_n(N - \{p\}) \to H_n(N).
\]

Since \( N - \{p\} \) is not compact, \( H_n(N - \{p\}) = 0 \) and thus \( f_* \) is the zero map, i.e. \( \text{deg} f = 0 \).

**Extra Problem 2**

It suffices to show that the local topological degree and the local analytic degree coincide. Expressed in terms of local coordinates compatible with the orientations (i.e. \( \alpha : \mathbb{R}^n \to M \) such that \( \alpha_* \) maps a fixed prescribed generator in \( H_* (\mathbb{R}^n, \mathbb{R}^n - \{0\}) \) to the generator of \( H_* (M, M - \{\alpha(0)\}) \) coming from the orientation, and likewise for \( N \)), it in turn suffices to consider a diffeomorphism \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) and show that \( f_* = \text{id} \) (resp. \( -\text{id} \)) \iff \det Df(0) > 0 \) (resp. \( < 0 \)).

Next, we reduce it to the case that \( f \) is a linear isomorphism. Let \( f(x) = Df(0)x + f_1(x) \).
Since \( \lim_{x \to 0} \frac{|f_1(x)|}{|x|} = 0 \), \( \exists r > 0 \) such that \( |f_1(x)| < |Df(0)| \cdot |x|, \forall |x| < 2r \). Let \( \rho : [0, \infty) \to [0, 1] \) be a smooth function with \( \rho(t) = 1 \) for \( t < r \) and \( \rho(t) = 0 \) for \( t > 2r \). Then, it is easy to check that for every \( 0 \leq s \leq 1 \),

\[
x \mapsto Df(0)x + [1 - s\rho(|x|)]f_1(x)
\]
maps \( \mathbb{R}^n - \{0\} \) into itself, thus defining a homotopy of maps from \( (\mathbb{R}^n, \mathbb{R}^n - \{0\}) \) to itself between \( f \) (\( s = 0 \)) and a map \( g \) (\( s = 1 \)) satisfying \( g(x) = Df(0)x \) for \( |x| < r \). By homotopy and excision, the local degree of \( f \) at \( x = 0 \) equals that of the linear map \( x \mapsto Df(0)x \).

Finally, it remains to show that if \( A \) is an \( n \times n \) nonsingular matrix, the local topological degree of \( x \mapsto Ax \) is 1 (resp. \(-1\)) if \( \det A > 0 \) (resp. \(< 0\)). Instead of writing down all the details, let me just say that one way to do it is to use the fact that every such \( A \) is a product of the so called 'elementary matrices' and thus it suffices to prove the statement above for these special cases, which is easy.