Math 215B suggested exercises, after Homework 6

1. The cohomology ring of a genus $g$ surface. Solve §3.2 (page 228), problem 1.

2. Restrictions on induced maps coming from cup product. Solve §3.2 (page 229), problem 11.

3. Non-equivalent spaces with isomorphic cohomology rings. Solve §3.2 (page 229), problem 12.

4. Orientability is unaffected by removing points. Solve §3.3 (page 257), problem 2.

5. The degree of maps between manifolds. For a map $f : M \to N$ between connected closed orientable $n$-manifolds with fundamental classes $[M]$ and $[N]$, the degree of $f$ is defined to be the integer $d$ such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes.
   a. A map to $S^n$ of degree 1. Solve §3.3 (page 258), problem 7.
   b. The degree of a covering map. Solve §3.3 (page 258), problem 9.
   c. The effect of degree 1 maps on $\pi_1$. Solve §3.3 (page 258), problem 10.

6. Homology as a module over cohomology. Show that $(\alpha \cap \phi) \cap \psi = \alpha \cap (\phi \cup \psi)$ for all $\alpha \in C_k(X; R)$, $\phi \in C^l(X; R)$, and $\psi \in C^m(X; R)$. Deduce that cap product makes $H_*(X; R)$ a right $H^*(X; R)$ module.

7. The homology groups of 3-manifolds. Solve the first part of §3.3 (page 259), problem 24. Namely: let $M$ be a closed, connected 3-manifold, and write $H_1(M; R)$ as $\mathbb{Z}^r \oplus T$, the direct sum of a free abelian group of rank $r$ and a finite group $T$. Show that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^r$ if $M$ is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ if $M$ is non-orientable. In particular, $r \geq 1$ when $M$ is nonorientable.