#1
If \( \pi_1(X) \) is finite, \( f_*(\pi_1(X)) \leq \pi_1(S^1) \cong \mathbb{Z} \) is also finite, hence trivial. Therefore, \( f: X \to S^4 \) can be lifted to the universal cover \( p: \mathbb{R} \to S^4 \), in other words, we can factor \( f \) as \( X \xrightarrow{\phi} \mathbb{R}^2 \to S^4 \). Since \( \mathbb{R} \) is contractible, \( f \) is null homotopic.

#2
Consider the cover of \( S^4 \vee S^4 \) given by the 2-oriented graph \( X \) shown below:

Let \( \sigma: X \to X \) be the homeomorphism obtained by rotating \( X \) about the centre \( O \) by a \( \pi/2 \) clockwise angle.

Let \( \tau: X \to X \) be the homeomorphism obtained by composing the reflection about the line \( L \) with the reflection about the circle \( C \).

Both \( \sigma \) and \( \tau \) are deck transformations. Moreover, any vertex of \( X \) can be taken to any other vertex of \( X \) by repeatedly applying \( \sigma \) or \( \tau \) to \( X \). Hence \( p: X \to S^4 \vee S^4 \) is normal, hence \( N := p_*(\pi_1(X, x_0)) \leq \pi_1(S^4 \vee S^4, x) \) is a normal subgroup. Van Kampen's theorem shows that \( \pi_1(X, x_0) \) is free on the generators \( \{ a^2, ab^2a^{-1}, (ab)a^2(ab)^{-1}, (aba)b^2(ab)^{-1}, (ab)a^2(ab)^{-2}, (ababa)b^2(ababa)^{-1}, (ab)a^2(ab)^{-1}, (ab)^4 \} \) as shown below:

Therefore, \( a^2, b^2 \) and \( (ab)^4 \) are in \( N \), so that \( N \) contains the normal subgroup of \( \pi_1(S^4 \vee S^4, x) \) generated by \( a^2, b^2 \) and \( (ab)^4 \). On the other hand, conjugates of these three words have been shown to generate \( N \), so we also have the other inclusion. It follows that \( N = p_*(\pi_1(X, x_0)) \) is precisely the normal subgroup of \( \pi_1(S^4 \vee S^4, x) \cong \mathbb{F}_2 \) generated by \( a^2, b^2 \) and \( (ab)^4 \), as desired.

#3
Observe that any word \( w \in \mathbb{F}_2 = \langle a, b \rangle / \langle a^2, b^2 \rangle \) is either \( (ab)^n \), \( (ba)^n \) for some \( n \in \mathbb{Z} \) or conjugate to either \( a \) or \( b \). It follows that up to conjugacy the only subgroups of \( \mathbb{F}_2 = \pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) \) are:

\( \{ 1 \}, \langle a^2 \rangle, \langle b \rangle, \langle a, b(ab)^n \rangle, \langle b, a(ba)^n \rangle, \langle (ab)^n \rangle, \langle a/b \rangle. \)
where we have also used that \( b \cdot (ab)^n \cdot b^{-1} = (ba)^n \), so \( <(ab)^n> \cong <(ba)^n> \). Since unbased connected covers of \( \mathbb{RP}^2 \vee \mathbb{RP}^2 \) are in 1-1 correspondence with the conjugacy classes of subgroups of \( \pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) \), it will suffice to produce connected covers \( p: X \to \mathbb{RP}^2 \vee \mathbb{RP}^2 \) with \( p_*(\pi_1(X)) \leq \pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) \) equal to each of these groups. Consider the basic covers:

If we let \( \tilde{\iota}_1 \) and \( \tilde{\pi}_1 \) be the compositions of \( \iota \) and \( \pi \) with the inclusion of \( \mathbb{RP}^2 \) into the first copy of the wedge \( \mathbb{RP}^2 \vee \mathbb{RP}^2 \) and \( \tilde{\iota}_2 \) \( \tilde{\pi}_2 \) be the compositions of \( \iota \) and \( \pi \) with the inclusion of \( \mathbb{RP}^2 \) into the second copy, we have explicit covers given by:

\[
\langle a, b \rangle \leftrightarrow \tilde{\iota}_1 \vee \tilde{\iota}_2 \quad \text{(here and below wedge of maps will be used),}
\]

\[
\langle a \rangle \leftrightarrow \tilde{\pi}_1 \tilde{\iota}_2 \tilde{\pi}_1 \tilde{\iota}_2 \quad \text{weave of 2n spheres}
\]

\[
\langle ab \rangle \leftrightarrow \tilde{\pi}_1 \tilde{\iota}_2 \tilde{\pi}_1 \tilde{\iota}_2 \quad \text{weave of 2n spheres}
\]

\[
\langle a, b(ab) \rangle \leftrightarrow \tilde{\pi}_1 \tilde{\iota}_2 \tilde{\pi}_1 \tilde{\iota}_2 \quad \text{weave of 2n spheres}
\]

\[
\langle a, b(ab)^n \rangle \leftrightarrow \tilde{\pi}_1 \tilde{\iota}_2 \tilde{\pi}_1 \tilde{\iota}_2 \quad \text{weave of 2n spheres}
\]

Since \( \langle a, b(ab)^{2n} \rangle \) is conjugate to \( \langle b, a(ba)^{2n} \rangle \), we are done.

# 4 Let \( x \in X \) and choose a neighborhood \( U \) of \( x \) such that \( U \cap g(U) = \emptyset \) only for \( g = g_1 \ldots g_n \) \( g \in G \). Since \( G \) acts freely on \( X \), \( g_k x \neq x \) \( \forall k = 1 \ldots n \). Choose neighborhoods \( U_k \) and \( V_k \) of \( x \) and \( g_k x \) such that \( U_k \cap V_k = \emptyset \). Set \( W_k = U_k \cap g_k^{-1} V_k \) and \( W = \cap_{k=1}^n W_k \), a neighborhood of \( x \). Then \( W \cap g W = \emptyset \) for all \( g \in G \), so indeed \( G \times X \) is a covering space action.

# 5 For \( (x, y) \in \mathbb{R}^2 \) such that \( x \neq 0 \), \( U = (3x/4, 3x/2) \times \mathbb{R} \) satisfies \( Nu \cap Nu = \emptyset \) \( \forall n \neq 0 \). Similarly, for \( (x, y) \in \mathbb{R}^2 \) such that \( y \neq 0 \), \( U = \mathbb{R} \times (3y/4, 3y/2) \) satisfies \( Nu \cap Nu = \emptyset \) \( \forall n \neq 0 \). Observe that the orbits of \( (1,0) \) and \( (0,1) \) cannot be separated by open neighborhoods, so \( X/\mathbb{Z} \) is not Hausdorff.
Repeated application of Van Kampen shows that $\pi_3(X; \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$.

Alternatively, we know that $\pi_3(X; \mathbb{Z})/p_\ast (\pi_1(X)) \cong G$, so $\pi_3(X; \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}$. If we also knew that $\pi_3(X; \mathbb{Z})$ is abelian, this implies that $\pi_3(X; \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. To see this, let $x_0 = (1, 0)$, take paths $\alpha, \beta, \gamma$ as shown, note $(p \circ \alpha)(x_0)$ or $(p \circ \alpha)(x_0)$ each generate $\pi_3(X; \mathbb{Z})$, and:

$\Rightarrow [\alpha] \cdot [\beta] \cdot [\gamma] = [\alpha \beta \gamma] \Rightarrow \pi_3(X; \mathbb{Z})$ abelian

# 6 Let $X = V^n S^1$, $(x_{0}) \sim (x_{0})$ cover with $p_\ast (\pi_1(X; \mathbb{Z})) = N < \pi_1(X; \mathbb{Z}) \cong \text{free}$ $\forall N$ a nontrivial normal subgroup. Suppose that $N$ is finitely generated. Since $p_\ast$ is injective, this implies that $\pi_1(\tilde{X}, \tilde{x}_0)$ is free on finitely many generators. Choose loops $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ based at $\tilde{x}_0$ generating $\pi_1(\tilde{X}, \tilde{x}_0)$. Let $\gamma \in N$ be nontrivial and let $k$ be its length. Then every lift $\tilde{\gamma}$ of $\gamma$ has also length $k$. Choose a lift $\tilde{\gamma}$ whose starting point in $p^{-1}(\gamma_0)$ is so far away from $\tilde{x}_0$ that $\tilde{\gamma}$ is disjoint from the $\tilde{\gamma}_i$. This is possible since $\gamma$, $\ldots, \gamma_m$, $\tilde{x}$ have finite length, $\tilde{x}$ has degree $n$ at each vertex and $\tilde{x}$ has infinitely many vertices. Then we obtain a contradiction with the fact that $\gamma_1, \ldots, \gamma_m$ generate $\pi_1(\tilde{X}, \tilde{x}_0)$. Since $\gamma \neq \text{free}, [H \triangleleft G] \text{ index } = k \iff [H \triangleleft G] \text{ index } = k$, we are done.

# 7 $\exists$ subgroups of $G$ $\subseteq \exists$ connected $k$-sheeted $\subseteq \exists k$-sheeted covers $\subseteq \exists$ covers of $X_G$ $\subseteq \exists$ of $X_G$ $\triangleleft \text{Hom}(G, S_k)$

where $X_G$ is the CW-complex naturally associated to $G$ (with $\pi_1(X_G) \cong G$) and $S_k$ is the symmetric group on $k$ letters. If $G$ is finitely generated, then $\text{Hom}(G, S_k)$ is finite, so we are done.

# 8 Let $H \triangleleft G$, $X_G$ the CW-complex naturally associated to $G$ and $p: (\tilde{x}_G, \tilde{x}_0) \sim (X_G, x_0)$ the cover corresponding to $H$. The subgroups of $G$ conjugate to $H$ are precisely the subgroups $p_\ast (\pi_1(\tilde{x}_G, \tilde{x}_k)) \leq \pi_1(X_G, x_0) \cong G$, where $\tilde{x}_k \in p^{-1}(x_0)$ as in Thm 1.3 of Hatcher's book. If $H$ has finite index in $G$, $\# p^{-1}(x) = [H : G]$, so there are $\leq [H : G]$ subgroups conjugate to $H$. Let $g_1, Hg_1, \ldots, g_k, Hg_k$ be the complete list. $K = \cap g_3 H g_3^{-1}$ is normal and $G/K \to \overline{\pi}_1(\text{cosets of } g_3 H g_3^{-1})$ in $G$ is injective, hence $[K : G] < \infty$ also.