#1 (a) (identity) If $e_i$ is the constant path at $x_i$, it has been shown in class that $e_i \cdot y \cdot e_j \sim y$ and $y \cdot e_j \sim e_j \cdot y$ $\forall y : (x_0, x_1, x_2) \to (x, x_1, x_2)$. 

(associativity) It was also shown in class that by reparametrizing $[0, 1]$ we get homotopies $(Y \cdot \beta) \cdot \alpha \sim Y \cdot (\beta \cdot \alpha)$ rel endpoints, for all composable paths $Y : \beta, \alpha : [0, 1] \to X$. This implies associativity.

(inverses). It was also shown in class that given any path $Y : (x_0, 0, 1) \to (x, x_1, x_2)$, $Y \cdot Y_1 \sim e_i$ and $Y \cdot Y_2 \sim e_j$ rel endpoints. Hence inverses exist.

(b) The empty word is the identity and $\omega_{-1} ... \omega_{-k}$ is the inverse of any word $\omega_1 ... \omega_k$. To check associativity, first observe that every word class $\omega_1 ... \omega_k$ has a unique representative in which consecutive letters belong to different groupoids. Existence follows from a simple argument inducting on the length of the word. For uniqueness, suppose that $\omega_1 = \omega_2$ and $\omega_1, \omega_2, \omega_3$ are reduced representatives of the same word. Then $\omega_1, \omega_2, \omega_3$ must be in the same class as the empty word. Since $\omega_1, \omega_2$ and $\omega_1, \omega_2, \omega_3$ are in different groupoids, necessarily $\omega_1$ and $\omega_3$ must be in the same $G_i$, and moreover we must have $\omega_1 = \omega_3$. Hence we can again induct on the length of the word to establish uniqueness.

The operation $G_1 \times G_2 \times G_1 \times G_2 \to G_1 \times G_2$ can now be defined at the level of reduced words: $\alpha \cdot \beta$ is the unique reduced representative of reduced words $\alpha, \beta$. Note that without this we would need to prove that the operation is well defined on equivalence classes (which we could have done). Associativity is now immediate since $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are both equal to the unique reduced representative of the concatenation $\alpha \cdot \beta \cdot \gamma$.

(c) Let $\pi_1(U_1 \times \mathcal{A})^+ \cong \pi_1(U_2 \times \mathcal{A}) \to \pi_1(X \times \mathcal{A})$ be the map defined by concatenation of composable paths.

(1) $f_*$ is surjective.

Given $[f] \in \pi_1(X \times \mathcal{A})$, we can find to $0 < t_1 < \ldots < t_N = 1$ such that $f([t_1, t_{n}])$ is contained in $U_1$ or $U_2$, by compactness of $[0, 1]$. Then we can write $[f]$ as the product $[f] = [f_{1}]_1 \cdot [f_{1} + f_{2} + \ldots + f_{N}]_2 = [g_1 \cdot f_1 + \ldots + f_N]_2$, where $g_1$ is a path in $U_1$ from $x_1$ or $x_2$ to $f(t_1)$.

(2) $f_*$ is injective.

The argument given in the text verifies the injectivity argument, where the paths connecting the basepoint to each vertex of a rectangle $R_k$ are now replaced with paths connecting either $x_1$ or $x_2$ to each vertex of $R_k$. 

(d) $\pi_1(U_i, p) = \mathbb{Z}^3$, $\pi_1(U_i, q) = \mathbb{Z}^5$, $\pi_1(U_i, r) = \mathbb{Z}^5$, $\pi_1(U_i, s) = \mathbb{Z}^5$. Therefore $\pi_1(S_1, p) \cong (\pi_1(U_1, q) \times \mathbb{Z}^3) \times \mathbb{Z}$, $\pi_1(U_2, q) = \mathbb{Z}^5$, with $(ab)^n$ the empty word and $(ab)^m$. Hence we obtain $\pi_1(S_1, p) \cong \mathbb{Z}$ as desired.

#3. Let $p \in X^0$ be a point belonging to the closure of infinitely many 1-cells $e_1, e_2, e_3, \ldots$. Suppose that $X = X^0 \cup X^1$ is first countable, let $U_1 \cup U_2 \cup \cdots$ be a countable basis at $p$ and choose $x_k \in e_k \cap U_k$, $k = 1, 2, 3, \ldots$. Since $\{x_k : k \in \mathbb{N}\}$ intersects each cell $e_k$ in a closed subset $\{x_k : k \in \mathbb{N}\}$, the subset $\{x_k : k \in \mathbb{N}\} \subseteq X$ is closed. However, $x_k \to p$ and $p \notin \{x_k : k \in \mathbb{N}\}$. This is a contradiction, hence $X$ is not first countable, hence is not metrizable. In the general case observe that any CW complex where a point lies in the closure of infinitely many cells can be subdivided into a finer CW complex where a point $p \in X^0$ is in the closure of infinitely many 1-cells. Since subspaces of metrizable spaces are metrizable, we are done.

#4. Let $X$ be a CW complex and $\bar{X} = X \cup fD^n$, $f : S^{n-1} \to X$, where $n \geq 3$. Let $U = X \cup f \{ x \in D^n \mid 1/2 < r < 1/3 \}$, $V = X \setminus \{ x \in D^n \mid 1/3 < r < 2/3 \}$. Then we have, for $p \in U \cup V$, $\pi_1(U \cup V, p) \cong \pi_1(X, p)$ (since $U$ deformation retracts onto $X$); $\pi_1(V, p) = 1$ (since $V$ is contractible) and $\pi_1(U \cup V, p) = 1$ (since $S^{n-1}$ is simply connected for $n-1 \geq 2$). Von Kampen's Theorem then yields $\pi_1(\bar{X}, p) \cong \pi_1(X, p)$.

#5. $U \cong (S' \vee S') \times S'$, and $\pi_1((S' \vee S') \times S') \cong \pi_1(S' \vee S') \times \pi_1(S') = (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}$

#6. $T_f = \mathbb{D}_4$ obtained from $S' \vee S' \vee S'$ by attaching two disks $D_4$ and $D_2$ along the paths acf(a)^{-1}c^{-1}, bcf(b)^{-1}c^{-1}.$ Hence $\pi_1(T_f) \cong <a,b,c | acf(a)^{-1}c^{-1}, bcf(b)^{-1}c^{-1}>.$
For $S^1 \times S^1$ we have already a 2-disk in addition to two new ones $D_1 \& D_2$

$$T_f = \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array} \xrightarrow{(x,y)} \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array} \text{ so } \pi_1(T_f) \cong \langle a, b, c \mid aba^{-1}b^{-1}, abc(b^{-1}c)^{-1}, bcf(c^{-1})^{-1} \rangle$$

# 7 Since $X \cong S^2 \setminus S^1 \setminus S^1$ as seen before, $\pi_1(S^2) = 1, \pi_1(S^1) = \mathbb{Z}$, we have $\pi_1(X) \cong \mathbb{Z} \* \mathbb{Z}$. Since $Y$ is obtained from the usual Klein bottle by removing a disk from the 2-cell and gluing its boundary to $b$, we can give $Y$ the following cell decomposition:

$$Y \cong \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\end{array} \text{ whence } \pi_1(Y) \cong \langle a, b, c \mid aba^{-1}b^{-1}c b^{-1}c^{-1} a^{-1} \rangle$$

Finally, $Y$ deformation retracts to $\mathbb{R}^3 \setminus \mathbb{Z}$, so the have the same $\pi_1$.

# 8 By Corollary 0.20, we only need to show that if $(X, A)$ satisfies the homotopy extension property, then $(X \setminus X_0) \cup (A \setminus X_1)$ satisfies the homotopy extension property and $(X \setminus X_0) U (A \setminus X_1) \hookrightarrow X \setminus X_1$ is a homotopy equivalence.

For the first statement, we show that $X \setminus X_0 U (X \setminus X_0) U (A \setminus X_1)$ is a retract of $X \setminus X_1$, which is equivalent. But given a retract $\tilde{r} : X \setminus X \rightarrow (X \setminus X_0) U (A \setminus X_1)$, which exists since $(X, A)$ satisfies HEP, it suffices to change our viewpoint on the product retract $\tilde{r} \times \text{id} : X \setminus X_1 \rightarrow (X \setminus X_0) U (A \setminus X_1) \times X_1$

$$\tilde{r} : X \setminus I \rightarrow (X \setminus X_0) U (A \setminus X_1)$$

$$X \setminus X_0 \cup X \setminus X_0 \cup X \setminus X_1$$

Where we have used the homeo $I^2 \rightarrow I^2$ given by the picture

To show that $(X \setminus X_0) U (A \setminus X_1) \hookrightarrow X \setminus X_1$ is a homotopy equivalence, it is sufficient to observe that $X \setminus X_0$ is a deformation retract of both spaces.
#9 Consider \( \tilde{X} = X \times I / f \times 1 = \frac{(x,t) \in X \times I}{(x,1) \sim (y,1) \text{ if } x,y \in A \text{ and } f(x) = f(y)} \).

Observe that \( X / f \subset \tilde{X} \) as the points \((x,1), x \in X \). Similarly, observe that \( X \cup Mf \subset \tilde{X} \) as the points \((x,0), x \in X \) and \((x,t), x \in A \).

The deformation retraction of \( X \times I \) onto \( X \times 0 \cup A \times I \) induces a deformation retraction of \( \tilde{X} \) onto \( X \cup Mf \). On the other hand, \( \tilde{X} \) also deformation retracts onto \( X \times I / f \times 1 \) \( \approx \) \( X / f \). Hence \( X / f \) is homotopy equivalent to \( X \cup Mf \). But \( f \) is a homotopy equivalence, hence \( A \) is a deformation retract of \( Mf \), hence \( \tilde{X} \) is a deformation retract of \( X \cup Mf \), hence \( X \cup Mf \) is homotopy equivalent to \( X \), as desired.