p.138-9 # 1, 2
p.155-6 # 5 p.138, Exercise 1:

Solution. This is a straightforward generalization of exercise 3 in homework 5.

p.138, Exercise 2:

Solution. Here $M$ is an $m$-dimensional manifold, $W$ an $m+n$-dimensional manifold and $N$ an $n$-dimensional submanifold of $W$. All three manifolds are assumed closed and oriented. We also have a map $f : M \to W$, which - after possibly replacing it by a homotopy - we can assume is a smooth map transversal to $N$.

We show vanishing of the intersection number $\#(f, N; W)$ in the following cases:

(i) The map $f : M \to W$ extends to an oriented compact manifold bounded by $M$.

(ii) $N$ bounds a closed oriented submanifold of $W$.

(iii) $f$ or $N$ is null homotopic in $W$ (and neither $N$ nor $M$ are 0-dimensional).

First we do case (i): Say $f$ is the restriction of a map $g : K \to W$. Replacing $g$ by a homotopic map, and $f$ by the corresponding boundary map, which is homotopic to $f$, we can assume that $g$ is transversal to $N$ in $W$. Thus $g^{-1}(N)$ is a neat 1-submanifold of $K$, thus a union of circles and closed intervals. Moreover $f^{-1}(N)$ is the boundary of $g^{-1}(N)$ i.e. the collection of endpoints of those components which are intervals. We claim that for any interval $I$ contained in $g^{-1}(N)$, the two endpoints count with different sign towards $\#(f, N; W)$ and thus their contribution cancels. To see this, choose a parametrization $\varphi : [0, 1] \to I$ and pick sections $s_1, \ldots, s_m : I \to T_K$ of the tangent bundle of $K$ along $I$ with the property that at every point $p \in I$, the vectors $t(p), s_1(p), \ldots, s_m(p)$ form a basis for the tangent space of $K$ at $p$, where $t(p) = \varphi'(\varphi^{-1}(p))$ is the unit tangent vector to $I$ corresponding to the chosen parametrization. In particular, $(s_i(p_k))_i$ is a basis for the tangent space of $M$ at $p_k := \varphi(k)$ for $k = 0, 1$. For every $p \in I$ the induced map $T_g : T_p K \to T_{g(p)} W$ maps $s_1, \ldots, s_m$ to a basis of $TW/TN|_{g(p)}$. This map is either orientation preserving for all $p$, or orientation reversing for all $p$, since $I$ is connected. If we can show that the bases $(s_i(p_k))_i$ for $k = 0, 1$ induce different orientations on $M$, then we are done. This follows, since $K$ is orientable, and the bases $(t(p_k), s_1(p_k), \ldots, s_m(p_k))$ induce compatible orientations for $k = 0, 1$. Since one of \{t(p_k)\}_k=0,1 is inward pointing and the other outward pointing we are done.

Case (ii) works similarly: Let $L$ be a closed oriented submanifold of $W$ bounded by $N$. We can choose $f$ to be transverse to $L$. Then $f^{-1}(L)$ is a 1-submanifold of $M$, thus a disjoint union of circles and intervals. The endpoints of the intervals form exactly the set $f^{-1}(N)$. For any
interval $I \subset f^{-1}(L)$ the two endpoints are counted with opposite sign: Indeed choose a local frame $t, s_1, \ldots, s_{m-1}$ along $I$ as before, which we can choose to be compatible with the orientation of $M$. The images of $s_1, \ldots, s_{m-1}$ under $Tf$ give compatible choices of orientations of $TW/TL$ along the image of $I$. However, $t$ induces incompatible orientations of $TL/TN$ at the images of the respective endpoints of $I$.

For case (iii), if $f$ is null-homotopic, and $m \neq 0$, then $n < n + m$, so up to homotopy we can choose $f$ to be a map to a point $p$ such that $p \notin N$. Then $f$ is transverse to $N$ and $f^{-1}(N) = \emptyset$, so the intersection number is zero. Now assume instead that $N$ is null-homotopic in $W$. We claim that the intersection number $\#(f, N; W)$ is equal to the intersection number $\pm \#(\Gamma_f, M \times N; M \times W)$, where $\Gamma_f$ denotes the graph of $f$, which in our case is a closed oriented submanifold of $M \times W$. Assuming this, since $N$ is null-homotopic in $W$, we find that

$$\#(\Gamma_f, M \times N; M \times W) = \pm \#(i_M \times N, \Gamma_f; W) = \pm \#(i_M \times c_q, \Gamma_f; W),$$

where $c_q : N \to W$ is the constant map to a point. Since we can choose $q$ to not lie in the image of $f$, this map is zero.

Finally we show that

$$\#(f, N; W) = \pm \#(\Gamma_f, M \times N; M \times W).$$

Clearly the set $\Gamma_f \cap M \times N$ (regarded as a subset of $M$) agrees with $f^{-1}(N)$. One needs to verify that the orientation at each point behave in a consistent way: Either each point counts with the same sign towards both intersection numbers, or each point contributes with opposite signs. This is shown by the following computation.

For $p \in f^{-1}(N)$ and $q = f(p)$, let $W := (w_1, \ldots, w_m)$ be an oriented basis for $T_p M$, and $V := (v_1, \ldots, v_n)$ an oriented basis for $T_q N$. Then $(V, (Tf)W) = v_1, \ldots, v_n, Tf w_1, \ldots, Tf w_m$ is a basis of $T_q W$. Let $\sigma \in \{\pm 1\}$ be its orientation, so that the point $p$ counts $\sigma$ towards the intersection number $\#(f, N; W)$ (here we made a choice in which order to append the bases, the opposite choice would change the sign for every point, so this doesn’t matter for our purposes).

A basis for $T_{(p, q)} M \times N$ is given by

$$V' := (\pi_M W, \pi_W V) = (\pi_M^* w_1, \ldots, \pi_M^* w_m, \pi_W^* v_1, \ldots, \pi_W^* v_n),$$

and a basis of $T_{(p, q)} \Gamma_f$ by

$$W' := (w'_1, \ldots, w'_m) := (\pi_M^* w_1 + \pi_W^* Tf w_1, \ldots, \pi_M^* w_m + \pi_W^* Tf w_m).$$

Then the orientation of the basis $(V', W')$ for $T_{(p, q)} M \times W$ determines the sign of the count towards $\#(\Gamma_f, M \times N; M \times W)$. The orientation of $(V', W')$ is equal to the one of

$$(V', \pi_W (Tf W)) = (\pi_M^* W, \pi_W V, \pi_W (Tf W))$$

since the base change matrix is unipotent. By the standard choice of orientation on the product, this basis has orientation $\sigma$. 

2
p.138, Exercise 2:

Solution. Let \( \pi : \mathbb{R}^2 \to S^1 \times S^1, (\alpha, \beta) \mapsto (\overline{\alpha}, \overline{\beta}) \mod 2\pi \) be the covering map of the torus. The map \( X \mapsto \tilde{X} := X \circ \pi \) gives a bijection between vector fields on \( T \) and \( 2\pi \mathbb{Z}^2 \)-periodic vector fields on \( \mathbb{R}^2 \). If \( X \) is any vector field on \( T \), and \( \gamma : J \to \mathbb{R}^2 \) is the maximal trajectory of \( \tilde{X} \) starting at \( p \in \mathbb{R}^2 \), then \( \pi \circ \gamma \) is the maximal trajectory of \( X \) starting at \( \pi(p) \).

(a) The vector field \( X_{\alpha,\beta} \) lifts to the vector field \( \tilde{X}_{\alpha,\beta} : p \to (\alpha, \beta) \) on \( \mathbb{R}^2 \). Let \( p \in \mathbb{R}^2 \) be arbitrary. Then the trajectory of \( \tilde{X}_{\alpha,\beta} \) through \( p \) is \( \gamma : t \mapsto p + t(\alpha, \beta) \). By the assumption on the pair \( (\alpha, \beta) \) there exists a smallest constant \( t_0 > 0 \), such that \( t_0(\alpha, \beta) \in 2\pi \mathbb{Z}^2 \). Thus the map \( \pi \circ \gamma : \mathbb{R} \to T \) is periodic with period \( t_0 \), and thus induces a map \( \gamma : S^1 \simeq \mathbb{R}/t_0\mathbb{Z} \to T \), which in fact parametrizes the trajectory of \( X_{\alpha,\beta} \) through \( \pi(p) \). Thus the trajectory is a circle. Since we can choose \( p \) freely, this holds for all trajectories of the vector field.

(b) In any case, the trajectory corresponding to \( X_{\alpha,\beta} \) through a point \( \pi(p) \) with \( p = (x_0, y_0) \) is \( \gamma : t \mapsto (x_0 + t\alpha, y_0 + t\beta) \mod 2\pi \). Suppose that \( \alpha \) and \( \beta \) are not linearly dependent over \( \mathbb{Q} \). We want to show that \( \gamma(\mathbb{R}) \) is dense in \( T \). By a suitable choice of coordinates on \( \mathbb{R}^2 \) and \( T \), we can assume that \( p = (0,0) \), in which case \( \gamma \) is a group homomorphism.

We claim that the vertical segment \( L := \{(\overline{x}, y) | y \in \mathbb{R}\} \) lies in the closure \( C := \overline{\gamma(\mathbb{R})} \). Since \( C \) is invariant under translations by elements of \( \gamma(\mathbb{R}) \), and since the elements of \( \gamma(\mathbb{R}) \) assume all possible \( x \)-coordinates, this shows that \( C \) is equal to \( T \). Consider the set of points \( P_k := \{\gamma(2\pi k/\alpha)\}_{k \in \mathbb{Z}} \). These all lie in \( L \) and since we assumed \( \alpha \) and \( \beta \) are \( \mathbb{Q} \)-linearly independent, are all distinct. They are also closed under addition, since \( \gamma \) is a group homomorphism. Let \( N > 0 \) be arbitrary. By the pigeonhole principle, there exists \( i \neq j \), such that \( P_i \) and \( P_j \) have distance at most \( 2\pi/N \). Thus \( P_{i-j} \) lies on \( L \) and has positive distance at most \( 2\pi/N \) from \((0,\overline{0})\). Thus every point on \( L \) has at distance at most \( 2\pi/N \) from one of the points \( \{P_k(\overline{i-j})\}_{k \in \mathbb{Z}} \subset \gamma(\mathbb{R}) \). Since \( N \) was arbitrary, this proves that \( L \subset C \).