(1) Prove that a smooth function \( f : M \to \mathbb{R} \) is Morse if and only if its differential \( df : M \to T^*M \) is transverse to the zero section. Here \( M \) is a smooth, closed manifold and \( T^*M \) is its cotangent bundle.

**Solution.** The set of points \( p \in M \) where \( df(p) \) lies in the zero section is exactly the set of critical points. Both conditions imply that this set is isolated. So without loss of generality we can assume that this is the case.

Then it is enough to show that the statement of the exercise holds locally around each critical point \( p \in M \) of \( f \). Let \( (\varphi, U) \) be a chart of \( M \) around \( p \), with \( \varphi(p) = 0 \) and let \( V := \varphi(U) \subseteq \mathbb{R}^n \). Then \( \varphi \) naturally induces a chart \( (T^*\varphi, \pi^{-1}(U)) \) for \( T^*M \), where \( \pi : T^*M \to M \) is the projection and \( T^*\varphi : T^*U \to T^*V = V \times \mathbb{R}^n \) is a morphism of bundles lying over \( \varphi \). Set \( g := f\varphi^{-1} \). We have

\[
T^*\varphi \circ df \circ \varphi^{-1} = d(f\varphi^{-1}) = dg,
\]

as one can check directly from the definitions. Moreover, \( p \) is non-degenerate as a critical point of \( f \) if and only if 0 is an non-degenerate critical point of \( g \); and \( df \) is transverse to the zero-section of \( T^*M \) at \( p \), if and only if \( dg \) is transverse to the zero-section of \( T^*V \) at 0. It is therefore enough to check that the statement of the exercise holds for \( g \) locally around 0.

Since \( g \) is defined on an open subset of \( \mathbb{R}^n \), we can write down \( dg \) explicitly as

\[
dg(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n})^t.
\]

We find that its differential at 0 is

\[
D_0(dg) = \begin{pmatrix} \text{Id}_n \\ \text{Hess}_0(g) \end{pmatrix}.
\]

In the other hand, the differential of the zero section is

\[
\begin{pmatrix} \text{Id}_n \\ 0 \end{pmatrix}.
\]

The map \( dg \) is transverse to the zero section if and only if the columns of these two matrices span \( \mathbb{R}^{2n} \). We see that this is the case if and only if \( \text{Hess}_0(g) \) is non-degenerate.

(2) Let \( M \subset \mathbb{R}^L \) be a closed, smooth submanifold. For each \( v \in S^{L-1} \) let \( f_v : M \to \mathbb{R} \) be the map \( f_v(x) = \langle v, x \rangle \). (This is essentially orthogonal projection into the line through \( v \).) Show that the set of \( v \in S^{L-1} \) such that \( f_v \) is a Morse function is open and dense.

**Solution.** Consider the normal sphere bundle \( N^\circ_M \) to \( M \), which we define as the intersection \( N_M \cap \mathbb{R}^L \times S^{L-1} \) in \( \mathbb{R}^L \times \mathbb{R}^L \). One easily checks that this intersection is transverse, and thus \( N^\circ_M \) is a \( L-1 \)
dimensional manifold. Set-theoretically it is given by \( \{(p, v) \in M \times S^{L-1} | v \text{ is normal to } M \text{ at } p \} \).

The projection \( \pi : \mathcal{N}_M^\circ \to S^{L-1} \) is a smooth map. We claim that any \( v \in S^{L-1} \) is a regular value of \( \pi \) if and only if \( f_v \) is a Morse function. The set of regular values is dense by Sard’s theorem. Since \( \mathcal{N}_M^\circ \) is compact, and the set \( \Sigma \) of singular points closed, it follows that the set \( \pi(\Sigma) \) of singular values for \( \pi \) is closed. Thus the set of regular values of \( \pi \) is also open.

It remains to prove the claim. It follows from the following two statements which can be checked locally:

1. For any \( p \in M \) and \( v \in S^{L-1} \), the point \( p \) is a critical point of \( f_v \) if and only if \( (p,v) \in \mathcal{N}_M^\circ \).

2. If \( f_v \) has a critical point at \( p \), then it is a degenerate critical point if and only if \( (p,v) \) is a singular point of \( \pi \).

The first claim follows, since \( D_q f_v = (v, -) \) for all \( q \in bR^n \). On the other hand, \( D_p (f_v|M) = D_p f_v | T_p M \). This shows that \( p \) is a critical point of \( f_v \) if and only if the scalar product with \( v \) vanishes on \( T_p M \). This is the case if and only if \( (p,v) \in \mathcal{N}_M^\circ \).

For the second claim, assume that \( p \) is a critical point of \( f_v \). Choose a local parametrization \( \varphi = (\varphi_1, \ldots, \varphi_L) : \mathbb{R}^n \to \mathbb{R}^L \) of \( M \) around \( p \) with \( \varphi(0) = p \). Let also \( F = (F_1, \ldots, F_{L-n}) : B_r(p) \to \mathbb{R}^{L-n} \) be a submersion with \( M = F^{-1}(\{0\}) \). At every point \( q \in M \), the vectors \( D_q F_1, \ldots, D_q F_{L-n} \) (regarded as column vectors) give a basis for the normal space at that point. By the first claim, \( v \) is normal to \( M \) at \( p \), so in particular, \( v = \sum_{i=1}^{L-n} \lambda_i D_p F_i \) for some \( \lambda_1, \ldots, \lambda_{L-n} \in \mathbb{R} \).

We want to reformulate the condition that \( (p,v) \) is a singular point of \( \pi \) in terms of \( F \). Note that since \( \pi \) is the restriction of the projection \( \bar{\pi} : \mathcal{N}_M^\circ \to \mathbb{R}^L \) to \( \pi^{-1}(S^{L-1}) \to S^{L-1} \), and \( S^{L-1} \) is transversal to \( \bar{\pi} \). Thus the singular values of \( \pi \) are exactly the singular values of \( \bar{\pi} \) which lie in \( \mathcal{N}_M^\circ \).

A natural parametrization for the normal bundle \( \mathcal{N}_M \) over \( U = B_r(p) \) is given by

\[
\Psi : U \times \mathbb{R}^{L-n} \to \mathbb{R}^L \times \mathbb{R}^L, (q, \alpha_1, \ldots, \alpha_{L-n}) \mapsto (q, \sum_{i=1}^{L-n} \alpha_i D_q F_i).
\]

We consider the map \( \bar{\pi} \circ \Psi : U \times \mathbb{R}^{L-n} \to \mathbb{R}^L \). Let \( w_1, \ldots, w_n \) be a basis for the tangent space of \( M \) at \( p \). In these coordinates we find

\[
D(\bar{\pi} \circ \Psi) = \begin{pmatrix} \partial_{w_1} (\sum_i \alpha_i D F_i) \mid q & \cdots & \partial_{w_n} (\sum_i \alpha_i D F_i) \mid q & D_q F_1 & \cdots & D_q F_{L-n} \end{pmatrix}
\]

Since the last \( L-n \) columns span exactly the normal space at a point of \( M \), the map is non-degenerate at \( (q, \alpha_i) \) if the projections of the vectors \( \partial_{w_i} (\sum_i \alpha_i D F_i) \mid q \) to \( T_q M \) form a basis. Thus if the matrix with entries

\[
\left( \langle \partial_{w_j} (\sum_i \alpha_i D F_i) \mid q, w_k \rangle \right)_{j,k}
\]

is nonsingular. At the point \( \Psi^{-1}(p,v) \) and with the choice \( w_j = \partial_j \varphi \mid 0 \) this specializes to the matrix

\[
\left( \langle \partial_{\partial_j \varphi} (\sum_i \lambda_i D F_i) \mid p, \partial_k \varphi \mid 0 \rangle \right)_{j,k} = \left( \langle \partial_j (\sum_i \lambda_i D F_i \circ \varphi) \mid 0, \partial_k \varphi \mid 0 \rangle \right)_{j,k}
\]
Now we compare this to the behaviour of $f_v$. The Hessian in the coordinates given by $\varphi$ is

$$\text{Hess } f_v \circ \varphi = (\partial_j \partial_k (v, \varphi))_{j,k} = \langle (v, \partial_j \partial_k \varphi) \rangle_{j,k}. \quad (2)$$

By definition, $p$ is nondegenerate if and only if this matrix is nonsingular at 0.

It remains to compare the two matrices (1) and (2) that we got. We claim that they only differ by a sign. That is

$$\langle \partial_j (\sum_i \lambda_i DF_i \circ \varphi) | 0, \partial_k \varphi | 0 \rangle + \langle v, \partial_j \partial_k \varphi | 0 \rangle = 0$$

for all $j, k$. To see this, note that

$$\langle \sum_i \lambda_i DF_i \circ \varphi, \partial_k \varphi \rangle = 0$$

in a neighborhood of 0 in the domain, since we are taking the scalar product of a normal vector to $M$ with a tangent vector. Thus if we take $\partial_j$ and evaluate at 0, we get zero. On the other hand, if we compute the derivative via the product rule, we get the left hand side of the desired equality. This finishes the proof

(3) Let $M \subset \mathbb{R}^L$ be a closed, smooth submanifold. Show that the set of points $u \in \mathbb{R}^L$ such that the map $x \mapsto |x - u|^2$ is a Morse function on $M$, is open and dense.

Solution. This can be done similarly to (2). See Ch. I, §6 in Milnor’s “Morse theory” for the computation. They show that the map $E : N_M \to \mathbb{R}^L$ given by $(p, v) \mapsto p + v$ has $u \in \mathbb{R}^L$ as a regular value if and only if the distance function associated to $u$ is Morse. It follows again from Sard’s theorem that this set has full measure. Openness requires a little more care, since $N_M$ is not compact. But as one easily checks, it is still proper (the preimage of a compact set is compact). This condition ensures that $E$ maps closed sets to closed sets. Thus the set of singular values of $E$ is closed, and the set of regular values therefore open.

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