Math 215B: Solutions 5

Due Thursday, February 22, 2018

(1) Let X be a connected space. Let $[S^1, X]$ denote the set of homotopy classes of maps from the circle S^1 to X. Note that no basepoint requirements are made of either the maps or the homotopies. Show that there is a bijection between the set of conjugacy classes of elements in the fundamental group, $\pi_1(X, x_0)$ and $[S^1, X]$. That is, show that there is a bijection

$$\Phi : \pi_1(X, x_0)/\sim \longrightarrow [S^1, X],$$

where two elements g_1 and g_2 of $\pi_1(X, x_0)$ are equivalent if and only if there is another element $h \in \pi_1(X, x_0)$ with $hg_1h^{-1} = g_2$.

Solution. It is assumed that X is path-connected.

By forgetting the basepoint of a loop, and noting that two based loops that are equivalent through base-point preserving homotopies give the same element in $[S^1, X]$, we see that there is a map $\Phi' : \pi_1(X, x_0) \to [S^1, X]$.

We need to show that this map is surjective, and that the fiber over any element of $[S^1, X]$ is exactly a conjugacy class.

Surjectivity: Let $h : S^1 \to X$ be a loop. Let $\varphi : [0, 1] \to S^1$ be a map that descends to a homeomorphism when we identify the endpoints of the interval. Let $t_0 = \varphi(0)$, and $q := h(t_0)$. Pick a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = q$. Then the composition $g := \gamma^{-1} * (h \circ \varphi) * \gamma$ is a loop based at x_0, thus $[g] \in \pi_1(X, x_0)$ (here $*$ denotes composition of paths). We can write down a homotopy between g and h that retracts γ and γ^{-1} onto q. This shows that $\Phi'([g]) = h$.

More precisely, take

$$H(t,s) = \begin{cases} \gamma(s+3t) & ; \text{if } t \le (1-s)/3 \\ h(\frac{3}{1+2s}t - \frac{1-s}{3}) & ; \text{if } (1-s)/3 \le t \le (2+s)/3 \\ \gamma(1 - 3(t - \frac{2+s}{3})) & ; \text{if } t \ge (2+s)/3. \end{cases}$$

Fibers: First we show that if g_1 and g_2 are based loops at x_0 with $[g_1]$ and $[g_2]$ conjugate in $\pi_1(X, x_0)$, then they map to the same element in $[S^1, X]$ under Φ'. Indeed, in this case there is some loop h, such that $h * g_1 * h^{-1}$ is homotopic to g_2. We claim that g_1 and $h * g_1 * h^{-1}$ are homotopic. Indeed, this follows from the same argument as above with h^{-1} instead of γ: The homotopy retracts h^{-1} onto its endpoint and h onto its starting point. Thus $\Phi'([g_1]) = \Phi'([g_2])$.

Now suppose g_1 and g_2 are arbitrary loops based at x_0, such that $\Phi'([g_1]) = \Phi'([g_2])$. Let $G(t, s)$ be a homotopy between g_1 and g_2, such that $G(\cdot, 0) = g_1$ and $G(\cdot, 1) = g_1$ and moreover $G(0, s) = G(1, s)$ (so G is a homotopy of maps from the circle, not just from the interval $[0, 1]$. Since g_1 and g_2 are based at x_0, we find that $G(0, 0) = G(0, 1)x_0$. Thus the paths $h : t \mapsto G(0, t)$ is also a loop at x_0. Now g_1 and $h^{-1} * g_2 * h$ are homotopic as loops at x_0: They are connected by the family that follows h up to times s, then follows the loop $G(\cdot, s)$ and then follows h backwards from times s to time 0. Explicitly:

$$\widetilde{G}(t,s) = \begin{cases} G(3t,0) & ; \text{if } t \leq s/3 \\ G(s, \frac{3t-s}{3-2s}) & ; \text{if } s/3 \leq t \leq 1-s/3 \\ G(3(1-t),1) & ; \text{if } t \geq (1-s)/3. \end{cases}$$

Thus, we have shown that $[g_1] = [h]^{-1}[g_2][h]$ in $\pi_1(X, x_0)$, i.e. $[g_1]$ and $[g_2]$ are conjugate.

(3). Write $\mathbb{CP}^n$ in projective coordinates. $\mathbb{CP}^2 = \{[z_0, z_1, z_2] \in \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times\}$. That is $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action, via scalar multiplication, of the nonzero complex numbers $C^\times$. There are two natural copies of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$ given by $\{[z_0, z_1, 0]\}$ and by $\{[0, z_1, z_0]\}$. If we call one of these N and the other one K,

(a) Show that the intersection product $[N] \cdot [K] = 1 \in H_0(\mathbb{CP}^2)$. Conclude that each of these classes represent a generator of $H_2(\mathbb{CP}^2)$.

(b) Compute the ring structure of the cohomology $H^*(\mathbb{CP}^2)$.

Solution. (a) Set theoretically, the intersection is $\{[0, z_1, 0]\}$, which is a single point P in projective space. This point lies in the open subset U_{z_1} of $\mathbb{CP}^2$ consisting of points with nonzero z_1-coordinate. We can parametrize U_{z_1} by $\mathbb{C}^2$ using the chart $\varphi(w_1, w_2) = [w_1, 1, w_2]$. In these coordinates, N corresponds to $\{(w_1, 0)\}$ and K to $\{(0, w_2)\}$. Clearly these submanifolds intersect transversely. Thus $[N] \cdot [K] = \pm[P]$. In order to determine the sign, we need to consider the orientations. Any complex vector space, when regarded as a real vector space is canonically oriented. The orientation is given by any basis of the form $(e_1, ie_, \ldots, e_, ie_n)$, where $(e_1, \ldots, e_n)$ form a complex basis. Since the chart we picked is compatible with the holomorphic structure, and we are considering complex submanifolds, we see that an oriented basis (e_1, ie_1) for the tangent space of N at 0 and (e_2, ie_2) for K at 0, fit together to an oriented basis (e_1, ie_1, e_2, ie_2) of $\mathbb{C}^2$ at 0. Thus the right sign is '+'. It follows that each of $[K]$, $[N]$ represents a generator for $H_2(\mathbb{CP}^2)$: Since $H^2(\mathbb{CP}^2) \cong \mathbb{Z}$, if this were false we would have for example $[N] = k\beta$, where β is the class of a generator and $k \geq 2$. Thus $[P] = [N] \cdot [K] = k(\beta \cdot [K]) \in kH_0(\mathbb{CP}^2)$, but since the class of a point generates $H_0(\mathbb{CP}^2) \cong \mathbb{Z}$, it cannot be a proper multiple of another class.

(b) We claim that there exists a canonical ring isomorphism $\mathbb{Z}[X]/(X^3) \xrightarrow{\cong} H^*(\mathbb{CP}^2)$, where X has degree 2, which sends X to the Poincaré dual of the class of N. Clearly such a map exists as a homomorphism of graded rings, since the cup product is graded and commutative in even degree. Now notice that $[N] = [K]$, since we can transform one into the other along the homotopy $N_t = \{[\cos tz_0, z_1, \sin tz_2]\}$ (this should be interpreted as a family of embeddings of $\mathbb{CP}^1$). So $D([N])^2 = D([P])$ is a generator of $H^4(\mathbb{CP}^2)$. Thus the map is surjective. It is bijective, since as a map of abelian groups, it is surjective between free groups of the same finite rank.

(4) Let M^n be a closed oriented n-dimensional manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Let $\Delta_! : H_q(M\times M) \to H_{q-n}(M)$ be the shriek map in homology. Show that for any homology classes α and β of M, then $\alpha \cdot \beta = \pm\Delta_!(\beta \times \alpha)$.

Solution. Since M^n satisfies Poincaré duality, it is equivalent to show that $D_M(\alpha\cdot\beta) = \pm D_M(\Delta_!(\beta\times\alpha))$. First, we have by definition of intersection and cup products

$$D_M(\alpha \cdot \beta) = D_M(\alpha) \cup D_M(\beta) = \Delta^*(D_M(\alpha) \times D_M(\beta)).$$

The shriek map is defined as $\Delta_! = D_M\Delta^*D_{M\times M}$. Let p_i for $i = 1, 2$ be the projection on first and second factor respectively. Then we find that

$$D_M(\Delta_!(\beta \times \alpha)) = \Delta^*(D_{M\times M}(\beta \times \alpha)).$$

Thus, we are reduced to showing that $D_M(\alpha) \times D_M(\beta)$ (cross product in *cohomology*) is equal to $D_{M\times M}(\beta \times \alpha)$ (cross product in *homology*).

Now we use that $M \times M$ is again a closed oriented manifold. In particular, since we know that $H^{2n}(M \times M) \cong \mathbb{Z}$ is torsion free, by the Künneth formula cross product induces an isomorphism $H^n(M) \otimes H^n(M) \xrightarrow{\sim} H^{2n}(M \times M)$. Thus we must have that $[M \times M] = \pm[M] \times [M]$. By the compatibility of the cohomological and the homological cross product with capping, we find that

$$(D_M(\alpha) \times D_M(\beta)) \cap ([M \times M]) = \pm D_M(\alpha) \cap [M] \times D_M(\beta) \cap [M] = \pm\, \alpha \times \beta.$$

This exhibits $D_M(\alpha) \times D_M(\beta)$ as the Poincaré dual of $\alpha \times \beta$, finishing the proof.