Math 215B: Solutions 5
Due Thursday, February 22, 2018

(1) Let $X$ be a connected space. Let $[S^1, X]$ denote the set of homotopy classes of maps from the circle $S^1$ to $X$. Note that no basepoint requirements are made of either the maps or the homotopies. Show that there is a bijection between the set of conjugacy classes of elements in the fundamental group, $\pi_1(X, x_0)$ and $[S^1, X]$. That is, show that there is a bijection

$\Phi : \pi_1(X, x_0)/\sim \rightarrow [S^1, X],$

where two elements $g_1$ and $g_2$ of $\pi_1(X, x_0)$ are equivalent if and only if there is another element $h \in \pi_1(X, x_0)$ with $hg_1h^{-1} = g_2$.

Solution. It is assumed that $X$ is path-connected.

By forgetting the basepoint of a loop, and noting that two based loops that are equivalent through base-point preserving homotopies give the same element in $[S^1, X]$, we see that there is a map $\Phi' : \pi_1(X, x_0) \rightarrow [S^1, X]$.

We need to show that this map is surjective, and that the fiber over any element of $[S^1, X]$ is exactly a conjugacy class.

Surjectivity: Let $h : S^1 \rightarrow X$ be a loop. Let $\varphi : [0, 1] \rightarrow S^1$ be a map that descends to a homeomorphism when we identify the endpoints of the interval. Let $t_0 = \varphi(0)$, and $q = h(t_0)$. Pick a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = q$. Then the composition $g := \gamma^{-1} \ast (h \circ \varphi) \ast \gamma$ is a loop based at $x_0$, thus $[g] \in \pi_1(X, x_0)$ (here $\ast$ denotes composition of paths). We can write down a homotopy between $g$ and $h$ that retracts $\gamma$ and $\gamma^{-1}$ onto $q$. This shows that $\Phi'([g]) = h$.

More precisely, take

$H(t, s) = \begin{cases} 
\gamma(s + 3t) ; & \text{if } t \leq (1 - s)/3 \\
h(\frac{3}{1 + 2s} t - \frac{1 - s}{3}) ; & \text{if } (1 - s)/3 \leq t \leq (2 + s)/3 \\
\gamma(1 - 3(t - \frac{2 + s}{3})) ; & \text{if } t \geq (2 + s)/3.
\end{cases}$

Fibers: First we show that if $g_1$ and $g_2$ are based loops at $x_0$ with $[g_1]$ and $[g_2]$ conjugate in $\pi_1(X, x_0)$, then they map to the same element in $[S^1, X]$ under $\Phi'$. Indeed, in this case there is some loop $h$, such that $h \ast g_1 \ast h^{-1}$ is homotopic to $g_2$. We claim that $g_1$ and $h \ast g_1 \ast h^{-1}$ are homotopic. Indeed, this follows from the same argument as above with $h^{-1}$ instead of $\gamma$: The homotopy retracts $h^{-1}$ onto its endpoint and $h$ onto its starting point. Thus $\Phi'([g_1]) = \Phi'([g_2])$.

Now suppose $g_1$ and $g_2$ are arbitrary loops based at $x_0$, such that $\Phi'([g_1]) = \Phi'([g_2])$. Let $G(t, s)$ be a homotopy between $g_1$ and $g_2$, such that $G(\cdot, 0) = g_1$ and $G(\cdot, 1) = g_1$ and moreover $G(0, s) = G(1, s)$ (so $G$ is a homotopy of maps from the circle, not just from the interval $[0, 1]$). Since $g_1$ and $g_2$ are based at $x_0$, we find that $G(0, 0) = G(0, 1)x_0$. Thus the paths $h : t \mapsto G(0, t)$ is also a loop at $x_0$. Now $g_1$ and $h^{-1} \ast g_2 \ast h$ are homotopic as loops at $x_0$: They are connected by the family
that follows $h$ up to times $s$, then follows the loop $G(\cdot, s)$ and then follows $h$ backwards from times $s$ to time 0. Explicitly:

$$
\tilde{G}(t, s) = \begin{cases} 
G(3t, 0) & ; \text{if } t \leq s/3 \\
G(s, \frac{3t-s}{3-2s}) & ; \text{if } s/3 \leq t \leq 1 - s/3 \\
G(3(1-t), 1) & ; \text{if } t \geq (1-s)/3.
\end{cases}
$$

Thus, we have shown that $[g_1] = [h]^{-1}g_2[h]$ in $\pi_1(X, x_0)$, i.e. $[g_1]$ and $[g_2]$ are conjugate.

(3). Write $\mathbb{C}P^n$ in projective coordinates. $\mathbb{C}P^2 = \{[z_0, z_1, z_2] \in \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^x\}$. That is $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action, via scalar multiplication, of the nonzero complex numbers $\mathbb{C}^x$.

There are two natural copies of $\mathbb{C}P^1$ inside $\mathbb{C}P^2$ given by $\{[z_0, z_1, 0]\}$ and $\{[0, z_1, 0]\}$. If we call one of these $N$ and the other one $K$.

(a) Show that the intersection product $N \cdot [K] = 1 \in H_0(\mathbb{C}P^2)$. Conclude that each of these classes represent a generator of $H_2(\mathbb{C}P^2)$.

(b) Compute the ring structure of the cohomology $H^*(\mathbb{C}P^2)$.

**Solution.**  (a) Set theoretically, the intersection is $\{[0, z_1, 0]\}$, which is a single point $P$ in projective space. This point lies in the open subset $U_z$ of $\mathbb{C}P^2$ consisting of points with nonzero $z_1$-coordinate. We can parametrize $U_z$ by $\mathbb{C}^2$ using the chart $\varphi(w_1, w_2) = [w_1, 1, w_2]$. In these coordinates, $N$ corresponds to $\{(w_1, 0)\}$ and $K$ to $\{(0, w_2)\}$. Clearly these submanifolds intersect transversely. Thus $N \cdot [K] = \pm [P]$. In order to determine the sign, we need to consider the orientations. Any complex vector space, when regarded as a real vector space is canonically oriented. The orientation is given by any basis of the form $(e_1, i e_1, \ldots, e_n)$, where $(e_1, \ldots, e_n)$ form a complex basis. Since the chart we picked is compatible with the holomorphic structure, and we are considering complex submanifolds, we see that an oriented basis $(e_1, i e_1)$ for the tangent space of $N$ at $0$ and $(e_2, i e_2)$ for $K$ at $0$, fit together to an oriented basis $(e_1, i e_1, e_2, i e_2)$ of $\mathbb{C}^2$ at $0$. Thus the right sign is ‘+’. It follows that each of $[K]$, $[N]$ represents a generator for $H_2(\mathbb{C}P^2)$: Since $H^2(\mathbb{C}P^2) \cong \mathbb{Z}$, if this were false we would have for example $[N] = k \beta$, where $\beta$ is the class of a generator and $k \geq 2$. Thus $[P] = [N] \cdot [K] = k(\beta \cdot [K]) \in kH_0(\mathbb{C}P^2)$, but since the class of a point generates $H_0(\mathbb{C}P^2) \cong \mathbb{Z}$, it cannot be a proper multiple of another class.

(b) We claim that there exists a canonical ring isomorphism $\mathbb{Z}[X]/(X^3) \xrightarrow{\sim} H^*(\mathbb{C}P^2)$, where $X$ has degree 2, which sends $X$ to the Poincaré dual of the class of $N$. Clearly such a map exists as a homomorphism of graded rings, since the cup product is graded and commutative in even degree. Now notice that $[N] = [K]$, since we can transform one into the other along the homotopy $N_t = \{[\cos t z_0, z_1, \sin t z_2]\}$ (this should be interpreted as a family of embeddings of $\mathbb{C}P^1$). So $D([N])^2 = D([P])$ is a generator of $H^4(\mathbb{C}P^2)$. Thus the map is surjective. It is bijective, since as a map of abelian groups, it is surjective between free groups of the same finite rank.
(4) Let $M^n$ be a closed oriented $n$-dimensional manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Let $\Delta : H_q(M \times M) \to H_{q-n}(M)$ be the shriek map in homology. Show that for any homology classes $\alpha$ and $\beta$ of $M$, then $\alpha \cdot \beta = \pm \Delta_!(\beta \times \alpha)$.

Solution. Since $M^n$ satisfies Poincaré duality, it is equivalent to show that $D_M(\alpha \cdot \beta) = \pm D_M(\Delta_!(\beta \times \alpha))$. First, we have by definition of intersection and cup products

$$D_M(\alpha \cdot \beta) = D_M(\alpha) \cup D_M(\beta) = \Delta^*(D_M(\alpha) \times D_M(\beta)).$$

The shriek map is defined as $\Delta_! = D_M \Delta^* D_M \times M$. Let $p_i$ for $i = 1, 2$ be the projection on first and second factor respectively. Then we find that

$$D_M(\Delta_!(\beta \times \alpha)) = \Delta^*(D_M \times M(\beta \times \alpha)).$$

Thus, we are reduced to showing that $D_M(\alpha) \times D_M(\beta)$ (cross product in cohomology) is equal to $D_M \times M(\beta \times \alpha)$ (cross product in homology).

Now we use that $M \times M$ is again a closed oriented manifold. In particular, since we know that $H^{2n}(M \times M) \cong \mathbb{Z}$ is torsion free, by the Künneth formula cross product induces an isomorphism $H^n(M) \otimes H^n(M) \cong H^{2n}(M \times M)$. Thus we must have that $[M \times M] = \pm [M] \times [M]$. By the compatibility of the cohomological and the homological cross product with capping, we find that

$$(D_M(\alpha) \times D_M(\beta)) \cap ([M \times M]) = \pm D_M(\alpha) \cap [M] \times D_M(\beta) \cap [M] = \pm \alpha \times \beta.$$

This exhibits $D_M(\alpha) \times D_M(\beta)$ as the Poincaré dual of $\alpha \times \beta$, finishing the proof.