Math 215B

Take-home Midterm Exam February 6, 2020

Instructions. You are welcome to use the results from the books or class. If you have any questions about the exam, you may e-mail me or ask me in person. Discussion of the problems with anyone else is not permitted. Please send your solutions in a pdf file to me via email (rlc@stanford.edu) by 5:00 pm, Tuesday, February 11.

Good luck!

Name ______________________________

1.	(12)	____________
2.	(30)	____________
3.	(24)	____________
4.	(12 plus 10 extra credit)	____________
5.	(22)	____________
Total	(100 plus 10 extra credit)	____________

1. Let $\pi : \xi \to X$ be a k-dimensional vector bundle over a space X of the homotopy type of a CW-complex. Let $\pi_q : Fr_q(\xi) \to X$ be the associated *q-frame bundle*. This bundle consists of those q-tuples of vectors $(v_1, \cdots v_q) \in \xi^{\times q}$ for which

   - $\pi(v_1) = \cdots = \pi(v_q)$ (in other words, the vectors $v_1, \cdots v_k$ all live in the same fiber of $\pi : \xi \to X$).
   - The vectors $v_1, \cdots, v_q$ are linearly independent.
     Then $\pi_q(v_1 \cdots, v_q) = \pi(v_1) = \cdots = \pi(v_q) \in X$.

   Show that there is a continuous section $\sigma : X \to Fr_q(\xi)$ (i.e a map σ such that $\pi_q \circ \sigma = id_X$) if and only if there is a vector bundle isomorphism

$$\begin{array}{ccc} \xi & \xrightarrow[\cong]{\Psi_\sigma} & \zeta \times \mathbb{R}^q \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{p}} \\ X & \xrightarrow[=]{} & X \end{array}$$

   where $p : \zeta \to X$ is a $(k-q)$-dimensional vector bundle and $\tilde{p}$ is the composition

$$\tilde{p} : \zeta \times \mathbb{R}^q \xrightarrow{project} \zeta \xrightarrow{p} X.$$

   Notice in particular that if $q = k$, the one has a section of $Fr_k(\xi) \to X$ if and only if $\xi \to X$ is isomorphic to the trivial k-dimensional bundle.

2. For any space X let $Vect^d(X)$ denote the set of isomorphism classes of d-dimensional vector bundles over X.

   (a) Compute $Vect^d(S^1)$. Justify your answer.

   (b) Compute the fundamental group of the Grassmannian, $\pi_1(Gr_d(\mathbb{R}^\infty))$.

   (c) Let X be a simply-connected space. Prove that any one-dimensional vector bundle over X is trivial.

3. Let T be a closed, connected, orientable surface (two-dimensional manifold).

   (a) Show that there are infinitely many nonisomorphic complex line bundles over T.

(b) Let $x_0 \in T$. Show that the restriction of any complex line bundle $p : E \to T$ to the "punctured surface", $T - x_0$, is trivial.

4. A *Lie group* is a C^∞ differentiable manifold together with a group structure so that both the multiplication map $G \times G \to G$ and the inversion map $G \to G$ given by $g \to g^{-1}$ are C^∞ maps.

   (a) Show that a compact Lie group G has a trivial tangent bundle.

   Hint: Show that the tangent bundle TG is isomorphic to the trivial bundle $G \times T_{id}G$ where $T_{id}G$ is the tangent space at the identity element of G. Make use of the left and right translation maps $L_g : G \to G$ given by $L_g(h) = gh$ and $R_g : G \to G$ given by $R_g(h) = hg$. These are defined for any $g \in G$ and are differentiable maps.

   (b) *Extra Credit:* Show that for G a compact Lie group, and $K < H < G$ compact sub-Lie groups (i.e subgroups that are also submanifolds), then the projection map
   $$p : G/K \to G/H$$
   is a locally trivial fiber bundle with fiber H/K.

5. Let M^n be a closed differentiable manifold, and let $e_0 : M^n \looparrowright \mathbb{R}^N$ and $e_1 \looparrowright \mathbb{R}^N$ be two immersions of M^n. We say that e_0 and e_1 are *isotopic* if there is a one-parameter family of immersions connecting e_0 and e_1. That is, e_0 and e_1 are isotopic if there is a continuous map $H : M^n \times [0,1] \to \mathbb{R}^N$ so that

   (a) $H(x,0) = e_0(x)$ and $H(x,1) = e_1(x)$ for all $x \in M^n$

   (b) The map $H_t : M^n \to \mathbb{R}^N$ defined by $H_t(x) = H(x,t)$ is a differentiable immersion for every $t \in [0,1]$.

   Smale's theorem about "turning a sphere inside out" says that any two immersions $S^2 \looparrowright \mathbb{R}^3$ are isotopic.

   (a) Show, however, that there are infinitely many distinct isotopy classes of immersions $S^1 \looparrowright \mathbb{R}^2$. You may use Smale's theorem saying that the space of immersions $M \looparrowright \mathbb{R}^N$ is weakly homotopy equivalent to the space of bundle monomorphiisms $TM \to T\mathbb{R}^N$.

(b) Describe a representative of each isotopy class you find.