11. DEGREE, LINKING NUMBERS AND INDEX OF VECTOR FIELDS

Let \( f: N^n \to M^n \) be a smooth map between compact connected oriented manifolds of the same dimension \( n \). We have the commutative diagram

\[
\begin{array}{c}
H^n(M) \xrightarrow{H^n(f)} H^n(N) \\
\cong \downarrow \quad \cong \downarrow \\
\mathbb{R} \xrightarrow{\deg(f)} \mathbb{R}
\end{array}
\]

(1)

where the vertical isomorphisms are given by integration over \( M \) and \( N \) respectively; cf. Corollary 10.14. The lower horizontal arrow is multiplication by the real number \( \deg(f) \) that makes the diagram commutative. Thus for \( \omega \in \Omega^n(M) \),

\[
\int_N f^*(\omega) = \deg(f) \int_M \omega.
\]

(2)

This formulation can be generalized to the case where \( N \) is not connected:

**Proposition 11.1** Let \( f: N^n \to M^n \) be a smooth map between compact \( n \)-dimensional oriented manifolds with \( M \) connected. There exists a unique \( \deg(f) \in \mathbb{R} \) such that (2) holds for all \( \omega \in \Omega^n(M) \). We call \( \deg(f) \) the degree of \( f \).

**Proof.** We write \( N \) as a disjoint union of its connected components \( N_1, \ldots, N_k \) and denote the restriction of \( f \) to \( N_j \) by \( f_j \). We have already defined \( \deg(f_j) \); we set

\[
\deg(f) = \sum_{j=1}^k \deg(f_j).
\]

(3)

Thus for \( \omega \in \Omega^n(M) \),

\[
\int_N f^*(\omega) = \sum_{j=1}^k \int_{N_j} f_j^*(\omega) = \sum_{j=1}^k \deg(f_j) \int_M \omega = \deg(f) \int_M \omega.
\]

Corollary 11.2 \( \deg(f) \) depends only on the homotopy class of \( f: N \to M \).

**Proof.** By (3) we can restrict ourselves to the case where \( N \) is connected. The assertion then follows from diagram (1), since \( H^n(f) \) depends only on the homotopy class of \( f \). \( \square \)
Corollary 11.3 Suppose \( N^n \xrightarrow{f} M^n \xrightarrow{g} P^n \) are smooth maps between \( n \)-dimensional compact oriented manifolds and that \( M \) and \( P \) are connected. Then

\[
\text{deg}(gf) = \text{deg}(f)\text{deg}(g).
\]

Proof. For \( \omega \in \Omega^n(P) \),

\[
\text{deg}(gf) \int_P \omega = \int_N (gf)^*(\omega) = \int_N f^*(g^*(\omega)) = \text{deg}(f) \int_M g^*(\omega) = \text{deg}(f)\text{deg}(g) \int_P \omega.
\]

\[\square\]

Remark 11.4 If \( f: M^n \to M^n \) is a smooth map of a connected compact orientable manifold to itself then \( \text{deg}(f) \) can be defined by choosing an orientation of \( M \) and using it at both the domain and range. Change of orientation leaves \( \text{deg}(f) \) unaffected.

We will show that \( \text{deg}(f) \) takes only integer values. This follows from an important geometric interpretation of \( \text{deg}(f) \) which uses the concept of regular value. In general \( p \in M \) is said to be a regular value for the smooth map \( f: N^n \to M^m \) if

\[
D_q f: T_q N \to T_p M
\]

is surjective for all \( q \in f^{-1}(p) \). In particular, points in the complement of \( f(N^n) \) are regular values. Regular values are in rich supply:

Theorem 11.5 (Brown–Sard) For every smooth map \( f: N^n \to M^m \) the set of regular values is dense in \( M^m \).

When proving Theorem 11.5 one may replace \( M^m \) by an open subset \( W \subseteq M^m \) diffeomorphic to \( \mathbb{R}^n \), and replace \( N^p \) by \( f^{-1}(W) \). This reduces Theorem 11.5 to the special case where \( M^m = \mathbb{R}^m \).

In this case one shows, that almost all points in \( \mathbb{R}^m \) (in the Lebesgue sense) are regular values. By covering \( N^n \) with countably many coordinate patches and using the fact that the union of countably many Lebesgue null-sets is again a null-set, Theorem 11.5 therefore reduces to the following result:

Theorem 11.6 (Sard, 1942) Let \( f: U \to \mathbb{R}^m \) be a smooth map defined on an open set \( U \subseteq \mathbb{R}^n \) and let

\[
S = \{ x \in U \mid \text{rank} \, D_x f < m \}.
\]
Then \( f(S) \) is a Lebesgue null-set in \( \mathbb{R}^n \).

Note that \( x \in U \) belongs to \( S \) if and only if every \( m \times m \) submatrix of the Jacobi matrix of \( f \), evaluated at \( x \), has determinant zero. Therefore \( S \) is closed in \( U \) and we can write \( S \) as a union of at most countably many compact subsets \( K \subseteq S \). Theorem 11.6 thus follows if \( f(K) \) is a Lebesgue null-set for every compact subset \( K \) of \( S \). We shall only use and prove these theorems in the case \( m = n \), where they follow from

**Proposition 11.7** Let \( f: U \to \mathbb{R}^n \) be a \( C^1 \)-map defined on an open set \( U \subseteq \mathbb{R}^n \), and let \( K \subseteq U \) be a compact set such that \( \det (D_x f) = 0 \) for all \( x \in K \). Then \( f(K) \) is a Lebesgue null-set in \( \mathbb{R}^n \).

**Proof.** Choose a compact set \( L \subseteq U \) which contains \( K \) in the interior, \( K \subseteq L \). Let \( C > 0 \) be a constant such that

\[
\sup_{\xi \in L} \| \text{grad}_\xi f_j \| \leq C \quad (1 \leq j \leq n).
\]

Here \( f_j \) is the \( j \)-th coordinate function of \( f \), and \( \| \| \) denotes the Euclidean norm. Let

\[
T = \prod_{i=1}^{n} [t_i, t_i + a]
\]

be a cube such that \( K \subseteq T \), and let \( \varepsilon > 0 \). Since the functions \( \partial f_j / \partial x_i \) are uniformly continuous on \( L \), there exists a \( \delta > 0 \) such that

\[
\| x - y \| \leq \delta \Rightarrow \left| \frac{\partial f_j}{\partial x_i} (x) - \frac{\partial f_j}{\partial x_i} (y) \right| \leq \varepsilon, \quad (1 \leq i, j \leq n \text{ and } x, y \in L).
\]

We subdivide \( T \) into a union of \( N^n \) closed small cubes \( T_i \) with side length \( \frac{a}{N} \), and choose \( N \) so that

\[
\text{diam}(T_i) = \frac{a\sqrt{n}}{N} \leq \delta, \quad T_i \cap K \neq \emptyset \Rightarrow T_i \subseteq L.
\]

For a small cube \( T_i \) with \( T_i \cap K \neq \emptyset \) we pick \( x \in T_i \cap K \). If \( y \in T_i \) the mean value theorem yields points \( \xi_j \) on the line segment between \( x \) and \( y \) for which

\[
f_j(y) - f_j(x) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} (\xi_j) (y_i - x_i).
\]

Since \( \xi_j \in \hat{T}_i \subseteq L \), the Cauchy–Schwarz inequality and (4) give

\[
|f_j(y) - f_j(x)| \leq C \| y - x \|
\]
and by (6),

\[ \| f(y) - f(x) \| \leq C \sqrt{n} \text{diam}(T_i) = \frac{anC}{N}. \]

Formula (7) can be rewritten as

\[ f(y) = f(x) + D_x f(y - x) + z, \]

where \( z = (z_1, \ldots, z_n) \) is given by

\[ z_j = \sum_{i=1}^{n} \left( \frac{\partial f_j}{\partial x_i} (\xi_j) - \frac{\partial f_j}{\partial x_i} (x) \right) (y_i - x_i). \]

By (6), \( \| \xi_j - x \| \leq \delta \), so that (5) gives \( |z_j| \leq \epsilon n \frac{\delta}{N} \). Hence

\[ \| z \| \leq \epsilon a n \sqrt{n} \frac{\sqrt{n}}{N}. \]

Since the image of \( D_x f \) is a proper subspace of \( \mathbb{R}^n \), we may choose an affine hyperplane \( H \subseteq \mathbb{R}^n \) with

\[ f(x) + \text{Im}(D_x f) \subseteq H. \]

By (9) and (10) the distance from \( f(y) \) to \( H \) is less than \( \epsilon \frac{an \sqrt{n}}{N} \). Then (8) implies that \( f(T_i) \) is contained in the set \( D_i \) consisting of all points \( q \in \mathbb{R}^n \) whose orthogonal projection \( \text{pr}(q) \) on \( H \) lies in the closed ball in \( H \) with radius \( \frac{anC}{N} \) and centre \( f(x) \) and \( \| q - \text{pr}(q) \| \leq \epsilon \frac{an \sqrt{n}}{N} \). For the Lebesgue measure \( \mu_n \) on \( \mathbb{R}^n \) we have

\[ \mu_n(D_i) = 2 \epsilon \frac{an \sqrt{n}}{N} \left( \frac{anC}{N} \right)^{n-1} \text{Vol}(D^{n-1}) = \epsilon \frac{c}{N^n}, \]

where \( c = 2 a^n n^{n+\frac{1}{2}} C^{n-1} \text{Vol}(D^{n-1}) \). For every small cube \( T_i \) with \( T_i \cap K \neq \emptyset \)
we now have \( \mu_n(f(T_i)) \leq \epsilon \frac{c}{N^n} \). Since there are at most \( N^n \) such small cubes \( T_i \),
\( \mu_n(f(K)) \leq c \epsilon \). This holds for every \( \epsilon > 0 \) and proves the assertion. \( \square \)

**Lemma 11.8** Let \( p \in M^n \) be a regular value for the smooth map \( f: N^n \rightarrow M^n \), with \( N^n \) compact. Then \( f^{-1}(p) \) consists of finitely many points \( q_1, \ldots, q_k \). Moreover, there exist disjoint open neighborhoods \( V_i \) of \( q_i \) in \( N^n \), and an open neighborhood \( U \) of \( p \) in \( M^n \), such that

(i) \( f^{-1}(U) = \bigcup_{i=1}^{k} V_i \)

(ii) \( f_i \) maps \( V_i \) homeomorphically onto \( U \) for \( 1 \leq i \leq k \).
Proof. For each $q \in f^{-1}(p)$, $D_q f : T_q N \to T_q M$ is an isomorphism. From the inverse function theorem we know that $f$ is a local diffeomorphism around $q$. In particular $q$ is an isolated point in $f^{-1}(p)$. Compactness of $N$ implies that $f^{-1}(p)$ consists of finitely many points $q_1, \ldots, q_k$. We can choose mutually disjoint open neighborhoods $W_i$ of $q_i$ in $N$, such that $f$ maps $W_i$ diffeomorphically onto an open neighborhood $f(W_i)$ of $p$ in $M$. Let

$$U = \left( \bigcap_{i=1}^{k} f(W_i) \right) - f \left( N - \bigcup_{i=1}^{k} W_i \right).$$

Since $N - \bigcup_{i=1}^{k} W_i$ is closed in $N$ and therefore compact, $f(N - \bigcup_{i=1}^{k} W_i)$ is also compact. Hence $U$ is an open neighborhood of $p$ in $M$. We then set $V_i = W_i \cap f^{-1}(U)$.

Consider a smooth map $f : N^n \to M^n$ between compact $n$-dimensional oriented manifolds, with $M$ connected. For a regular value $p \in M$ and $q \in f^{-1}(p)$, define the local index

$$\text{Ind}(f; q) = \begin{cases} 1 & \text{if } D_q f : T_q N \to T_p M \text{ preserves orientation} \\ -1 & \text{otherwise.} \end{cases}$$

(11) $\text{Ind}(f; q) = \frac{1}{2}$ if $D_q f : T_q N \to T_p M$ preserves orientation 

$\text{Ind}(f; q) = -1$ otherwise.

Theorem 11.9 In the situation above, and for every regular value $p$,

$$\text{deg}(f) = \sum_{q \in f^{-1}(p)} \text{Ind}(f; q).$$

In particular $\text{deg}(f)$ is an integer.

Proof. Let $q_i$, $V_i$, and $U$ be as in Lemma 11.8. We may assume that $U$ and hence $V_i$ connected. The diffeomorphism $f|_{V_i} : V_i \to U$ is positively or negatively oriented, depending on whether $\text{Ind}(f; q_i)$ is 1 or $-1$. Let $\omega \in \Omega^n(M)$ be an $n$-form with

$$\text{supp}_M(\omega) \subseteq U, \quad \int_M \omega = 1.$$ 

Then $\text{supp}_N(f^*(\omega)) \subseteq f^{-1}(U) = V_1 \cup \ldots \cup V_k$, and we can write

$$f^*(\omega) = \sum_{i=1}^{k} \omega_i,$$

where $\omega_i \in \Omega^n(N)$ and $\text{supp}(\omega_i) \subseteq V_i$. Here $\omega_i|_{V_i} = (f|_{V_i})^*(\omega|_U)$. The formula is a consequence of the following calculation:

$$\text{deg}(f) = \text{deg}(f) \int_M \omega = \int_N f^*(\omega) = \sum_{i=1}^{k} \int_{V_i} \omega_i = \sum_{i=1}^{k} \int_{V_i} \int_U (f|_{V_i})^*(\omega|_U)$$

$$= \sum_{i=1}^{k} \text{Ind}(f; q_i) \int_U \omega|_U = \sum_{i=1}^{k} \text{Ind}(f; q_i).$$

\qed
In the special case where \( f^{-1}(p) = \emptyset \) the theorem shows that \( \deg(f) = 0 \) (in the
proof above we get \( f^* (\omega) = 0 \)). Thus we have

**Corollary 11.10** If \( \deg(f) \neq 0 \), then \( f \) is surjective. \( \Box \)

**Proposition 11.11** Let \( F: P^{n+1} \to M^n \) be a smooth map between oriented smooth
manifolds, with \( M^n \) compact and connected. Let \( X \subseteq P \) be a compact domain with
smooth boundary \( N^n = \partial X \), and suppose \( N \) is the disjoint union of submanifolds
\( N_1^n, \ldots, N_k^n \). If \( f_i = F|_{N_i} \), then

\[
\sum_{i=1}^{k} \deg(f_i) = 0.
\]

**Proof.** Let \( f = F|_{N} \) so that

\[
\deg(f) = \sum_{i=1}^{k} \deg(f_i).
\]

On the other hand, if \( \omega \in \Omega^n(M) \) has \( \int_{M} \omega = 1 \), then

\[
\deg(f) = \int_{N} f^*(\omega) = \int_{X} dF^*(\omega) = \int_{X} F^*(d\omega) = 0
\]

where the second equation is from Theorem 10.8. \( \Box \)

We shall give two applications of degree. We first consider linking numbers, and
then treat indices of vector fields.

**Definition 11.12** Let \( J^d \) and \( K^l \) be two disjoint compact oriented connected
smooth submanifolds of \( \mathbb{R}^{n+1} \), whose dimensions \( d \geq 1, l \geq 1 \) satisfy \( d + l = n \).
Their **linking number** is the integer

\[
\operatorname{lk}(J, K) = \deg(\Psi_{J,K})
\]

where

\[
\Psi = \Psi_{J,K} : J \times K \to S^n; \quad \Psi(x, y) = \frac{y - x}{\|y - x\|}.
\]

Here \( J \times K \) is equipped with the product orientation (cf. Remark 9.20) and \( S^n \) is
oriented as the boundary of \( D^{n+1} \) with the standard orientation of \( \mathbb{R}^{n+1} \). We note
that \( \operatorname{lk}(J, K) \) changes sign when the orientation of either \( J \) or \( K \) is reversed.
Proposition 11.13

(i) \( \text{lk}(K^d, J^d) = (-1)^{(d+1)(l+1)} \text{lk}(J^d, K^d) \)

(ii) If \( J \) and \( K \) can be separated by a hyperplane \( H \subset \mathbb{R}^{n+1} \) then \( \text{lk}(J, K) = 0 \).

(iii) Let \( g_t \) and \( h_t \) be homotopies of the inclusions \( g_0 : J \to \mathbb{R}^{n+1} \) and \( h_0 : K \to \mathbb{R}^{n+1} \) to smooth embeddings \( g_1 \) and \( h_1 \), such that \( g_t(J) \cap h_t(K) = \emptyset \) for all \( t \in [0, 1] \). Then \( \text{lk}(J, K) = \text{lk}(g_1(J), h_1(K)) \).

(iv) Let \( \Phi : P^{d+1} \to \mathbb{R}^{n+1} - J \) be a smooth map with \( P \) oriented. Given a compact domain \( R \subset P \) with smooth boundary \( \partial R \), let \( Q_1, \ldots, Q_k \) be the connected components of \( \partial R \). Suppose each \( \Phi|_{Q_j} \) is a smooth embedding. If \( K_i = \Phi(Q_i) \), then

\[
\sum_{i=1}^{k} \text{lk}(J; K_i) = 0.
\]

Proof. We look at the commutative diagram

\[
\begin{array}{ccc}
J \times K & \xrightarrow{\Psi_{J,K}} & S^n \\
\downarrow T & & \downarrow A \\
K \times J & \xrightarrow{\Psi_{K,J}} & S^n
\end{array}
\]

where \( T \) interchanges factors and \( A \) is the antipodal map \( Av = -v \). Then (i) follows from Corollary 11.3 upon using that

\[
\text{deg}(T) = (-1)^d, \quad \text{deg}(A) = (-1)^{n+1} = (-1)^{d+l+1}.
\]

In the situation of (ii) the image of \( \Psi \) will not contain vectors parallel to \( H \), and the assertion follows from Corollary 11.10.

Assertion (iii) is a consequence of the homotopy property, Corollary 11.2. Indeed, a homotopy \( J \times K \times [0, 1] \to S^n \) is given by

\[
(h_t(y) - g_t(x)) / \|h_t(y) - g_t(x)\|.
\]

Finally (iv) follows from Proposition 11.11 applied to the map \( F : J \times P \to S^n \) with

\[
F(x, y) = (\Phi(y) - x) / \|\Phi(y) - x\|,
\]

and to the domain \( X = J \times R \) with boundary components \( J \times Q_i \). Indeed, \( f_i = F|_{J \times Q_i} \) has degree \( \text{deg}(f_i) = \text{lk}(J, K_i) \).

Here is a picture to illustrate (iv):
If \( \text{lk}(J, K) \neq 0 \) then (ii) and (iii) of Proposition 11.13 imply that \( J \) and \( K \) cannot be deformed to manifolds separated by a hyperplane.

We shall now specialize to the classical case of knots in \( \mathbb{R}^3 \) where \( J \) and \( K \) are disjoint oriented submanifolds of \( \mathbb{R}^3 \) diffeomorphic to \( S^1 \). Let us choose smooth regular parametrizations

\[
\alpha : \mathbb{R} \to J, \quad \beta : \mathbb{R} \to K
\]

with periods \( a \) and \( b \), respectively, corresponding to a single traversing of \( J \) and \( K \), respectively, agreeing with the orientation. For \( p \in S^2 \), consider the set

\[
I(p) = \{ (q_1, q_2) \in J \times K \mid q_2 - q_1 = \lambda p, \lambda > 0 \}.
\]

Let \( v(q_1) \) and \( w(q_2) \) denote the positively oriented unit tangent vectors to \( J \) and \( K \) in \( q_1 \) and \( q_2 \), respectively.

**Theorem 11.14** With the notation above we have:

(i) (Gauss)

\[
\text{lk}(J, K) = \frac{1}{4\pi} \int_0^a \int_0^b \frac{\det(\alpha(u) - \beta(v), \alpha'(u), \beta'(v))}{\|\alpha(u) - \beta(v)\|^3} \, du \, dv.
\]

(ii) There exists a dense set of points \( p \in S^2 \) such that

\[
\det(\alpha(q_1 - q_2), v(q_1), w(q_2)) \neq 0 \text{ for } (q_1, q_2) \in I(p).
\]

(iii) For such points \( p \), \( \text{lk}(J, K) = \sum_{(q_1, q_2) \in I(p)} \delta(q_1, q_2) \), where \( \delta(q_1, q_2) \) is the sign of the determinant in (ii).
Proof. We apply formula (2) to the map $\Psi = \Psi_{J,K}$ and the volume form $\omega = \text{vol}_{S^2}$ (with integral $4\pi$) to get

\begin{equation}
\text{lk}(J, K) = \text{deg}(\Psi) = \frac{1}{4\pi} \int_{J \times K} \Psi^*(\text{vol}_{S^2}).
\end{equation}

We write $\Psi = r \circ f$ with

\begin{align*}
f: J \times K &\to \mathbb{R}^3 - \{0\}; \quad f(q_1, q_2) = q_2 - q_1 \\
r: \mathbb{R}^3 - \{0\} &\to S^2; \quad r(x) = x/\|x\|.
\end{align*}

For $x \in \mathbb{R}^3 - \{0\}$, $r^*(\text{vol}_{S^2})_x \in \text{Alt}^2(\mathbb{R}^3)$ is given by

$$r^*(\text{vol}_{S^2})_x(v, w) = \det(x, v, w)/\|x\|^3$$

(cf. Example 9.18). The tangent space $T_{(q_1, q_2)}(J \times K)$ has a basis $\{v(q_1), w(q_2)\}$ and

$$Df_{(q_1, q_2)}(v(q_1)) = -v(q_1), \quad Df_{(q_1, q_2)}(w(q_2)) = w(q_2).$$

Therefore

\begin{equation}
\Psi^*(\text{vol}_{S^2})_{(q_1, q_2)}(v(q_1)w(q_2)) = r^*(\text{vol}_{S^2})_{q_2 - q_1}(-v(q_1), w(q_2))
= \|q_1 - q_2\|^{-3} \det(q_1 - q_2, v(q_1), w(q_2)).
\end{equation}

The integral of (12) can be calculated by integrating $(\alpha \times \beta)^*\Psi^*(\text{vol}_{S^2})$ over the period rectangle $[0, a] \times [0, b]$. This yields Gauss’s integral.

For $p \in S^2$, $I(p)$ is exactly the pre-image under $\Psi$. Thus $p$ is a regular value of $\Psi$ if and only if the determinant in (13) is non-zero for all $(q_1, q_2) \in I(p)$, and the sign $\delta(q_1, q_2)$ is determined by whether $D_{(q_1, q_2)}\Psi$ preserves or reverses orientation. Assertions (ii) and (iii) now follow from Theorems 11.5 and 11.9. □

Remark 11.15 In Theorem 11.14.(ii), after a rotation of $\mathbb{R}^3$, the regular value $p$ can be assumed to be the north pole $(0, 0, 1)$. The projections of $J$ and $K$ on the $x_1, x_2$-plane may be drawn indicating over- and undercrossings and orientations, e.g.
There is one element in \( I(p) \) for every place where \( K \) crosses over (and not under) \( J \). The corresponding sign \( \delta \) is determined by the orientation of the curves and of the standard orientation of the plane as shown in the picture.

![Diagram showing linking numbers](image)

In Fig. 2, \( \text{lk}(J, K) = -1 \). In Fig. 1,

\[
\text{lk}(J, K_2) = \text{lk}(J, K_4) = 0, \quad \text{lk}(J, K_3) = 1, \quad \text{lk}(J, K_1) = -1.
\]

The sum of these linking numbers is 0. This is in accordance with (iv) of Lemma 11.13.

We now apply the concept of degree to study singularities of vector fields. Consider a vector field \( F \in C^\infty(U, \mathbb{R}^n) \) on the open set \( U \subseteq \mathbb{R}^n, \ n \geq 2 \), and let us assume that \( 0 \in U \) is an isolated zero for \( F \). A zero for \( F \) is also called a singularity for the vector field. We can choose a \( \rho > 0 \) with

\[ \rho D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq \rho \} \subseteq U \]

and such that 0 is the only zero for \( F \) in \( \rho D^n \). Define a smooth map \( F_\rho: S^{n-1} \to S^{n-1} \) by

\[ F_\rho(x) = \frac{F(\rho x)}{\|F(\rho x)\|} \]

The homotopy class of \( F_\rho \) is independent of the choice of \( \rho \), and by Corollary 11.2 and Theorem 11.9, \( \deg F_\rho \in \mathbb{Z} \) is independent of \( \rho \).

**Definition 11.16** The degree of \( F_\rho \) is called local index of \( F \) at 0, and is denoted \( i(F; 0) \).

**Lemma 11.17** Suppose \( F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) has the origin as its only zero. Then

\[ F: \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\} \]

induces multiplication by \( i(F; 0) \) on \( H^{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{R} \).
Proof. Let $i: S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be the inclusion map and $r: \mathbb{R}^{n-1} \setminus \{0\} \to S^{n-1}$ the retraction $r(x) = x/\|x\|$. We have $\iota(F; 0) = \deg F_1$, where $F_1 = r \circ F \circ i$. The lemma follows from the commutative diagram below, where $H^{n-1}(i)$ and $H^{n-1}(r)$ are inverse isomorphisms:

$$
\begin{array}{ccc}
H^{n-1}(\mathbb{R}^n - \{0\}) & \xrightarrow{H^{n-1}(F)} & H^{n-1}(\mathbb{R} - \{0\}) \\
\uparrow H^{n-1}(r) & & \uparrow H^{n-1}(r) \\
H^{n-1}(S^{n-1}) & \xrightarrow{H^{n-1}(F_1)} & H^{n-1}(S^{n-1})
\end{array}
$$

\[ \square \]

Given a diffeomorphism $\phi: U \to V$ to an open set $V \subseteq \mathbb{R}^n$ and a vector field on $U$, we can define the direct image $\phi_* F \in C^\infty(V, \mathbb{R}^n)$ by

$$
\phi_* F(q) = D_q \phi(F(p)), \quad p = \phi^{-1}(q).
$$

**Lemma 11.18** If $F \in C^\infty(U, \mathbb{R}^n)$ has 0 as an isolated singularity and $\phi: U \to V$ is a diffeomorphism to an open set $V \subseteq \mathbb{R}^n$ with $\phi(0) = 0$, then

$$
\iota(\phi_* F; 0) = \iota(F, 0).
$$

**Proof.** By shrinking $U$ and $V$ we can restrict ourselves to considering the case where 0 is the only zero for $F$ in $U$, and where there exists a diffeomorphism $\psi: V \to \mathbb{R}^n$. The assertion about $\phi$ will follow from the corresponding assertions about $\psi$ and $\psi \circ \phi$, since

$$
\psi_* (\phi_* F) = (\psi \circ \phi)_* F.
$$

Thus it suffices to treat the case where $\phi: U \to \mathbb{R}^n$ is a diffeomorphism and where $Y = \phi_* F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only singularity.

Let $U_0 \subseteq U$ be open and star-shaped around 0. We define a homotopy

$$
\Phi: U_0 \times [0, 1] \to \mathbb{R}^n; \quad \Phi_t(x) = \Phi(x, t) = \begin{cases} 
(D_0 \phi)x & \text{if } t = 0 \\
\phi(tx)/t & \text{if } t \neq 0.
\end{cases}
$$

For $x \in U_0$,

$$
\phi(x) = \int_0^1 \frac{d}{dt} \phi(tx) dt = \int_0^1 \left( \sum_{i=1}^n x_i \frac{\partial \phi}{\partial x_i}(tx) \right) dt = \sum_{i=1}^n x_i \phi_i(x),
$$

where $\phi_i \in C^\infty(U_0, \mathbb{R}^n)$ is given by

$$
\phi_i(x) = \int_0^1 \frac{\partial \phi}{\partial x_i}(tx) dt,
$$
It follows that
\[ \Phi(x, t) = \sum_{i=1}^{n} x_i \phi_i(t x), \]
and in particular that \( \Phi \) has a smooth extension to an open set \( W \) with \( U_0 \times [0, 1] \subseteq W \subseteq U_0 \times \mathbb{R} \).

For each \( t \in [0, 1] \), \( \Phi_t \) is a diffeomorphism from \( U_0 \) to an open subset of \( \mathbb{R}^n \). Consider the direct image under \( \Phi_t^{-1} \) of \( Y \) restricted to \( \Phi_t(U_0) \):
\[ X_t = (\Phi_t^{-1})_* Y \in C^\infty(U_0, \mathbb{R}^n); \quad X_t(x) = (D_x \Phi_t)^{-1} Y(\Phi_t(x)). \]
The function \( X_t(x) \) is smooth on \( W \). Now \( X_1 = f_1|_{U_0} \) and \( X_0 = (A^{-1})_* Y \), where \( A = D_0 \psi \).
Choose \( \rho > 0 \) such that \( \rho D^n \subseteq U_0 \). The homotopy \( S^{n-1} \times [0, 1] \to S^{n-1} \) given by
\[ X_t(\rho x) / \|X_t(\rho x)\|, \quad 0 \leq t \leq 1, \]
and Corollary 11.2 shows that
\[ \iota(F; 0) = \iota(X_1; 0) = \iota(X_0; 0) = \iota((A^{-1})_* Y; 0). \]
Since \( A: \mathbb{R}^n \to \mathbb{R}^n \) is linear we have \( (A^{-1})_* Y = A^{-1} \circ Y \circ A: \mathbb{R}^n \to \mathbb{R}^n \). This yields the commutative diagram
\[ \begin{array}{ccc}
\mathbb{R}^n - \{0\} & \xrightarrow{(A^{-1})_* Y} & \mathbb{R}^n - \{0\} \\
\uparrow A & & \uparrow A \\
\mathbb{R}^n - \{0\} & \xrightarrow{Y} & \mathbb{R}^n - \{0\}
\end{array} \]
Now use the functor \( H^{n-1} \) and apply Lemma 11.17 to both \( Y \) and \( (A^{-1})_* Y \) to get \( \iota((A^{-1})_* Y; 0) = \iota(Y; 0) \). Hence \( \iota(F; 0) = \iota(Y; 0) \).

**Definition 11.19** Let \( X \) be a smooth tangent vector field on the manifold \( M^n \), \( n \geq 2 \) with \( p_0 \in M \) as an isolated zero. The local index \( \iota(X; p_0) \in \mathbb{Z} \) of \( X \) is defined by
\[ \iota(X; p_0) = \iota(h_* X|_{U}; 0), \]
where \( (U, h) \) is an arbitrary smooth chart around \( p_0 \) with \( h(p_0) = 0 \).

We note that Lemma 11.18 shows that the local index does not depend on the choice of \( (U, h) \). One can picture vector fields in the plane by drawing their integral curves, e.g.
Let $X$ be a smooth tangent vector field on $M^n$ and let $p_0 \in M^n$ be a zero. Let

$$F = h_*(X|_U) \in C^\infty(h(U), \mathbb{R}^n)$$

for a chart $(U, h)$ with $h(p_0) = 0$. If $D_0F: \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then $p_0$ is said to be a non-degenerate singularity or zero. Note that by the inverse function theorem $F$ is a local diffeomorphism around 0, such that 0 is an isolated zero for $F$. Hence $p_0 \in M^n$ is also an isolated zero for $X$.

**Lemma 11.20** If $p_0$ is a non-degenerate singularity, then

$$\iota(X, p_0) = \text{sign}(\det D_0F) \in \{\pm 1\}.$$

**Proof.** By shrinking $U$ we may assume that $h$ maps $U$ diffeomorphically onto an open set $U_0 \subseteq \mathbb{R}^n$, which is star-shaped around 0, and that $F$ is a diffeomorphism from $U_0$ to an open set. As in the proof of Lemma 11.18 we can define a homotopy

$$G: U_0 \times [0, 1] \to \mathbb{R}^n, \quad G(x, t) = \begin{cases} D_0F & \text{if } t = 0 \\ F(tx)/t & \text{if } t \neq 0, \end{cases}$$

where $G$ can be extended smoothly to an open set $W$ in $U_0 \times \mathbb{R}$ that contains $U_0 \times [0, 1]$. Choose $\rho > 0$ so that $\rho D^n \subseteq U_0$. We get a homotopy $\tilde{G}: S^{n-1} \times [0, 1] \to S^{n-1}$,

$$\tilde{G}(x, t) = G(\rho x, t) / \|G(\rho x, t)\|,$$

between the map $F_\rho$ in Definition 11.16 and the analogous map $A_\rho$ with $A = D_0F$. It follows from Corollary 11.2 that

$$\iota(X; p_0) = \iota(F; 0) = \deg(F_\rho) = \deg A_\rho = \iota(A; 0).$$

The map $f_A: \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ induced by $A$ operates on $H^{n-1}(\mathbb{R}^n - \{0\})$ by multiplication by $\iota(X; p_0)$; cf. Lemma 11.17. The result now follows from Lemma 6.14. \qed
Definition 11.21 Let $X$ be a smooth vector field on $M^n$, with only isolated singularities. For a compact set $R \subseteq M$ we define the total index of $X$ over $R$ to be

$$\text{Index}(X; R) = \sum \iota(X; p),$$

where the summation runs over the finite number of zeros $p \in R$ for $X$. If $M$ is compact we write $\text{Index}(X)$ instead of $\text{Index}(X; M)$.

Theorem 11.22 Let $F \in C^\infty(U, \mathbb{R}^n)$ be a vector field on an open set $U \subseteq \mathbb{R}^n$, with only isolated zeros. Let $R \subseteq U$ be a compact domain with smooth boundary $\partial R$, and assume that $F(p) \neq 0$ for $p \in \partial R$. Then

$$\text{Index}(F; R) = \deg f,$$

where $f: \partial R \to S^{n-1}$ is the map $f(x) = F(x) / \|F(x)\|$. 

Proof. Let $p_1, \ldots, p_k$ be the zeros in $R$ for $F$, and choose disjoint closed balls $D_j \subseteq R - \partial R$, with centers $p_j$. Define

$$f_j: \partial D_j \to S^{n-1}, \quad f_j(x) = F(x) / \|F(x)\|.$$

We apply Proposition 11.11 with $X = R - \bigcup_j \partial D_j$. The boundary $\partial X$ is the disjoint union of $\partial R$ and the $(n-1)$-spheres $\partial D_1, \ldots, \partial D_k$. Here $\partial D_j$, considered as boundary component of $X$, has the opposite orientation to the one induced from $D_j$. Thus

$$\deg(f) + \sum_{j=1}^k -\deg(f_j) = 0.$$

Finally $\deg(f_j) = \iota(F; p_j)$ by the definition of local index and Corollary 11.3. \hfill $\square$

Corollary 11.23 In the situation of Theorem 11.22, $\text{Index}(F; R)$ depends only on the restriction of $F$ to $\partial R$. \hfill $\square$

Corollary 11.24 In the situation of Theorem 11.22, suppose for every $p \in \partial R$ that the vector $F(p)$ points outward. Let $g: \partial R \to S^{n-1}$ be the Gauss map which to $p \in \partial R$ associates the outward pointing unit normal vector to $\partial R$. Then

$$\text{Index}(F; R) = \deg g.$$

Proof. By Corollary 11.2 it suffices to show that $f$ and $g$ are homotopic. Since $f(p)$ and $g(p)$ belong to the same open half-space of $\mathbb{R}^n$, the desired homotopy can be defined by

$$(1 - t)f(p) + tg(p) \over \|(1 - t)f(p) + tg(p)\| \quad (0 \leq t \leq 1).$$

\hfill $\square$
Lemma 11.25 Suppose \( F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) has the origin as its only zero. Then there exists an \( \tilde{F} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \), with only non-degenerate zeros, that coincides with \( F \) outside a compact set.

Proof. We choose a function \( \phi \in C^\infty(\mathbb{R}^n, [0, 1]) \) with

\[
\phi(x) = \begin{cases} 
1 & \text{if } \|x\| \leq 1 \\
0 & \text{if } \|x\| \geq 2.
\end{cases}
\]

We want to define \( \tilde{F}(x) = F(x) - \phi(x)w \) for a suitable \( w \in \mathbb{R}^n \). For \( \|x\| > 2 \) we have \( \tilde{F}(x) = F(x) \). Set

\[
c = \inf_{1 \leq \|x\| \leq 2} \|F(x)\| > 0
\]

and choose \( w \) with \( \|w\| < c \). For \( 1 \leq \|x\| \leq 2, \|\tilde{F}(x)\| \geq c - \|w\| > 0 \). Thus all zeros of \( \tilde{F} \) belong to the open unit ball \( \tilde{D}^n \). Since \( \tilde{F} \) coincides with \( F - w \) on \( \tilde{D}^n \),

\[
\tilde{F}^{-1}(0) = \tilde{D}^n \cap F^{-1}(w).
\]

We can pick \( w \) as a regular value of \( F \) with \( \|w\| < c \) by Sard's theorem. Then \( D_p\tilde{F} = D_pF \) will be invertible for all \( p \in \tilde{F}^{-1}(0) \), and \( \tilde{F} \) has the desired properties. \( \square \)

Note, by Corollary 11.23, that

\[
\iota(F; 0) = \sum_{p \in \tilde{F}^{-1}(p)} \iota(\tilde{F}, p)
\]

Here is a picture of \( F \) and \( \tilde{F} \) in a simple case:

![Figure 5](image)

The zero for \( F \) of index \(-2\) has been replaced by two non-degenerate zeros for \( \tilde{F} \), both of index \(-1\).
Corollary 11.26 Let $X$ be a smooth vector field on the compact manifold $M^n$ with isolated singularities. Then there exists a smooth vector field $\tilde{X}$ on $M$ having only non-degenerate zeros and with

$$\text{Index}(X) = \text{Index}(\tilde{X}).$$

Proof. We choose disjoint coordinate patches which are diffeomorphic to $\mathbb{R}^n$ around the finitely many zeros of $X$, and apply Lemma 11.25 on the interior of each of them to obtain $\tilde{X}$. The formula then follows from (14).

Theorem 11.27 Let $M^n \subseteq \mathbb{R}^{n+k}$ be a compact smooth submanifold and let $N_\epsilon$ be a tubular neighborhood of radius $\epsilon > 0$ around $M$. Denote by $g: \partial N_\epsilon \to S^{n+k-1}$ the outward pointing Gauss map. If $X$ is an arbitrary smooth vector field on $M^n$ with isolated singularities, then

$$\text{Index}(X) = \deg g.$$

Proof. By Corollary 11.26 one may assume that $X$ only has non-degenerate zeros. From the construction of the tubular neighborhood we have a smooth projection $\pi: N \to M$ from an open tubular neighborhood $N$ with $N_\epsilon \subseteq N \subseteq \mathbb{R}^{n+k}$, and can define a smooth vector field $F$ on $N$ by

$$F(q) = X(\pi(q)) + (q - \pi(q)).$$

Since the two summands are orthogonal, $F(q) = 0$ if and only if $q \in M$ and $X(q) = 0$. For $q \in \partial N_\epsilon$, $q - \pi(q)$ is a vector normal to $T_q\partial N_\epsilon$ pointing outwards. Hence $X(\pi(q)) \in T_q\partial N_\epsilon$, and $F(q)$ points outwards. By Corollary 11.24

$$\text{Index}(F; N_\epsilon) = \deg g,$$

and it suffices to show that $\iota(X; p) = \iota(F; p)$ for an arbitrary zero of $X$. In local coordinates around $p$ in $M$, with $p$ corresponding to $0 \in \mathbb{R}^n$, $X$ can be written in the form

$$X = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i},$$

where $f_i(0) = 0$, and by Lemma 11.20 $\iota(X; p)$ is the sign of

$$\det \left( \frac{\partial f_i}{\partial x_j}(0) \right).$$

By differentiating (16) and substituting 0 one gets

$$\frac{\partial X}{\partial x_j}(0) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(0) \frac{\partial}{\partial x_i}.$$

It follows from (15) that $D_pF: \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is the identity on $T_pM^\perp$, and by (18) $D_pF$ maps $T_pM$ into itself by the linear map with matrix $(\partial f_i/\partial x_j(0))$ (with respect to the basis $(\partial/\partial x_i)$). It follows that $p$ is a non-degenerate zero for $F$ and that $\det D_pF$ has the same sign as the Jacobian in (17). \qed
12. THE POINCARÉ–HOPF THEOREM

In the following, $M^n \subseteq \mathbb{R}^{n+k}$ will denote a fixed smooth submanifold. If the cohomology of $M^n$ is finite-dimensional (e.g. when $M^n$ is compact), the $i$-th Betti number is given by

\[(1) \quad b_i(M) = \dim_{\mathbb{R}} H^i(M^n).\]

The Euler characteristic of $M^n$ is defined to be

\[(2) \quad \chi(M^n) = \sum_{i=0}^{n} (-1)^i b_i(M).\]

This chapter’s main result is:

**Theorem 12.1** (Poincaré–Hopf) Let $X$ be a smooth vector field on a compact manifold $M$. If $X$ has only isolated zeros then

\[\text{Index}(X) = \chi(M).\]

By the final result of Chapter 11 it is sufficient to show the formula for just one such vector field $X$ on $M$. We shall do so by making use of a Morse function on $M^n$.

Given $f \in C^\infty(M, \mathbb{R})$, a point $p \in M$ is a critical point for $f$ if $d_pf = 0$.

**Proposition 12.2** Suppose that $p \in M$ is a critical point for $f \in C^\infty(M, \mathbb{R})$.

(i) There exists a quadratic form $d^2_pf$ on $T_pM$ characterized by the equation

\[d^2_pf(\alpha'(0)) = (f \circ \alpha)''(0)\]

where $\alpha: (-\delta, \delta) \rightarrow M$ is any smooth curve with $\alpha(0) = p$.

(ii) Let $h: U \rightarrow \mathbb{R}^n$ be a chart around $p$ and let $q = h(p)$. Then the composition

\[\mathbb{R}^n \xrightarrow{Dh^{-1}} T_pM \xrightarrow{d^2_pf} \mathbb{R}\]

is the quadratic form associated to the symmetric matrix

\[\left( \frac{\partial^2 f \circ h^{-1}}{\partial x_i \partial x_j} (q) \right).\]
12. THE PÖINCARÉ-HOPF THEOREM

Proof. Set $h \circ \alpha(t) = \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ and $\phi = f \circ h^{-1}$. A direct calculation yields

$$(f \circ \alpha)'(t) = (\phi \circ \gamma)'(t) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\gamma(t))\gamma_i'(t).$$

Since $p$ is critical, $(\partial \phi/\partial x_i)(\gamma(0)) = 0$. By differentiating once again and substituting $t = 0$, we get

$$(f \circ \alpha)''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\bar{q}) \gamma_i''(0) \gamma_j'(0).$$

This is the value at $\gamma'(0) = D_p h(\alpha'(0)) \in \mathbb{R}^n$ of the quadratic form from (ii). Both (i) and (ii) follow. \qed

Consider another chart $\tilde{h}: \tilde{U} \to \mathbb{R}^n$ around $p$ with $\tilde{q} = \tilde{h}(p)$ and let $F = \tilde{h} \circ h^{-1}$ defined in a neighborhood of $q$. The last formula in the proof above can be compared with

$$(f \circ \alpha)''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j}(\tilde{q}) \tilde{\gamma}_i''(0) \tilde{\gamma}_j'(0),$$

where $\tilde{\phi} = f \circ \tilde{h}^{-1}$ and $\tilde{\gamma} = \tilde{h} \circ \alpha$. Let $J$ denote the Jacobi matrix associated with $F$ in $q$. Then $\tilde{\gamma}'(0) = J\gamma'(0)$ for the column vectors $\tilde{\gamma}'(0)$ and $\gamma'(0)$. By substituting this and comparing, one obtains the matrix identity

$$(3) \quad \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\bar{q}) \right) = J^t \left( \frac{\partial^2 \tilde{\phi}}{\partial x_i \partial y_j}(\tilde{q}) \right) J.$$

Definition 12.3 A critical point $p \in M$ of $f \in C^\infty(M, \mathbb{R})$ is said to be non-degenerate, if the matrix in Proposition 12.2(ii) is invertible. We call $f$ a Morse function, if all critical points of $f$ are non-degenerate. The index of a non-degenerate critical point $p$ is the maximal dimension of a subspace $V \subseteq T_p M$ for which the restriction of $\partial^2_p f$ to $V$ is negative definite.

For smooth submanifolds $M^n \subseteq \mathbb{R}^{n+k}$ one can get Morse functions by:

Theorem 12.4 For almost all $p_0 \in \mathbb{R}^{n+k}$ the function $f: M \to \mathbb{R}$ defined by

$$f(p) = \frac{1}{2}\|p - p_0\|^2$$

is a Morse function.
Proof. Let \( g: \mathbb{R}^n \rightarrow M \) be a local parametrization and \( Y_j: \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}, j = 1, \ldots, k \) smooth maps, such that \( Y_1(x), \ldots, Y_k(x) \) is a basis of \( T_{g(x)}M^\perp \) for all \( x \in \mathbb{R}^n \). By Lemma 9.21 we know that \( M \) can be covered by at most countably many coordinate patches \( g(U) \) of this type. Therefore it suffices to prove the assertion for \( g(U) \) instead of \( M \).

We define \( \Phi: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k} \) by

\[
\Phi(x, t) = g(x) + \sum_{j=1}^{k} t_j Y_j(x) \quad (x \in \mathbb{R}^n, t \in \mathbb{R}^k).
\]

By Sard's theorem it suffices to prove that if \( p_0 \) is a regular value of \( \Phi \), then \( f \) becomes a Morse function on \( g(U) \). Set \( k = f \circ g: \mathbb{R}^n \rightarrow \mathbb{R} \); we can show instead that \( k \) becomes a Morse function on \( \mathbb{R}^n \). We have

\[
k(x) = \frac{1}{2} \langle g(x) - p_0, g(x) - p_0 \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product on \( \mathbb{R}^{n+k} \).

Since \( \langle \partial g/\partial x_i(x), Y_\nu(x) \rangle = 0 \), it follows by differentiation with respect to \( x_j \) that

\[
\langle \frac{\partial^2 g}{\partial x_i \partial x_j}, Y_\nu \rangle = -\langle \frac{\partial g}{\partial x_i}, \frac{\partial Y_\nu}{\partial x_j} \rangle.
\]

From (5) we have

\[
\frac{\partial k}{\partial x_i} = \left\langle g(x) - p_0, \frac{\partial g}{\partial x_i} \right\rangle,
\]

and therefore

\[
\frac{\partial^2 k}{\partial x_i \partial x_j} = \left\langle \frac{\partial g}{\partial x_i}, \frac{\partial g}{\partial x_j} \right\rangle + \left\langle g(x) - p_0, \frac{\partial^2 g}{\partial x_i \partial x_j} \right\rangle.
\]

Furthermore, by (4),

\[
\frac{\partial \Phi}{\partial x_j} = \frac{\partial g}{\partial x_j} + \sum_{\nu=1}^{k} t_\nu \frac{\partial Y_\nu}{\partial x_j}, \quad \frac{\partial \Phi}{\partial t_\nu} = Y_\nu.
\]

Assume that \( p_0 \) is a regular value of \( \Phi \) and let \( x \) be a critical point of \( k \). It follows from (7) that \( g(x) - p_0 \in T_{g(x)}M^\perp \). Hence there exists a unique \( t \in \mathbb{R}^k \) with

\[
g(x) - p_0 = -\sum_{\nu=1}^{k} t_\nu Y_\nu(x),
\]
and \((x, t) \in \Phi^{-1}(p_0)\). The \(n + k\) vectors in (9) are linearly independent at the point \((x, t)\). At this point, the equations (8), (10), (6) and (9) give

\[
\frac{\partial^2 k}{\partial x_i \partial x_j} = \left\langle \frac{\partial g}{\partial x_j}, \frac{\partial g}{\partial x_i} \right\rangle = \left\langle \sum_{\nu=1}^{k} t_\nu Y_\nu, \frac{\partial^2 g}{\partial x_i \partial x_j} \right\rangle \\
= \left\langle \frac{\partial g}{\partial x_j}, \frac{\partial g}{\partial x_i} \right\rangle + \left\langle \sum_{\nu=1}^{k} t_\nu \frac{\partial Y_\nu}{\partial x_j}, \frac{\partial g}{\partial x_i} \right\rangle = \left\langle \frac{\partial \Phi}{\partial x_j}, \frac{\partial g}{\partial x_i} \right\rangle.
\]

Let \(A\) denote the invertible \((n + k) \times (n + k)\) matrix with the vectors from (9) as rows. Then \(AD_x g\) takes the following form:

\[
\begin{pmatrix}
\frac{\partial^2 k}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 k}{\partial x_1 \partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 k}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 k}{\partial x_n \partial x_n} \\
0 & \cdots & 0
\end{pmatrix}
\]

Since \(D_x g\) has rank \(n\), so does \(AD_x g\). Hence the \(n \times n\) matrix

\[
\begin{pmatrix}
\frac{\partial^2 k}{\partial x_i \partial x_j}(x)
\end{pmatrix}
\]

is invertible. This shows that \(x\) is a non-degenerate critical point. \(\square\)

**Example 12.5** Let \(f: \mathbb{R}^n \to \mathbb{R}\) be the function

\[
f(x) = c - x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots x_n^2,
\]

where \(c \in \mathbb{R}\), \(\lambda \in \mathbb{Z}\) and \(0 \leq \lambda \leq n\). Since

\[
\text{grad}_x(f) = 2(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n),
\]

0 is the only critical point of \(f\). We find that

\[
\begin{pmatrix}
\frac{\partial^2 f}{\partial x_i \partial x_j}(0)
\end{pmatrix} = \text{diag}(-2, \ldots, -2, 2, \ldots, 2)
\]

with exactly \(\lambda\) diagonal entries equal to \(-2\). Thus the origin is non-degenerate of index \(\lambda\). We note that the vector field \(\text{grad}(f)\) has the origin as its only zero and that it is non-degenerate of index \((-1)^\lambda\).
Theorem 12.6 Let \( p \in M^n \) be a non-degenerate critical point for \( f \in C^\infty(M, \mathbb{R}) \). There exists a \( C^\infty \)-chart \( h: U \rightarrow h(U) \subseteq \mathbb{R}^n \) with \( p \in U \) and \( h(p) = 0 \) such that

\[
f \circ h^{-1}(x) = f(p) + \sum_{i=1}^{n} \delta_i x_i^2, \quad x \in h(U),
\]

where \( \delta_i = \pm 1 \) (1 \( \leq i \leq n \)) (By an additional permutation of coordinates we can put \( f \) into the standard form given in Example 12.5.)

Proof. After replacing \( f \) with \( f - f(p) \) we may assume that \( f(p) = 0 \). Since the problem is local and diffeomorphism invariant, we may also assume that \( f \in C^\infty(W, \mathbb{R}) \), where \( W \) is an open convex neighborhood of 0 in \( \mathbb{R}^n \) and that 0 is the considered non-degenerate critical point with \( f(0) = 0 \).

We write \( f \) in the form

\[
f(x) = \sum_{i=1}^{n} x_i g_i(x), \quad g_i(x) = \int_{0}^{1} \frac{\partial f(tx)}{\partial x_i} \, dt.
\]

Since \( g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0 \), we may repeat to get

\[
g_i(x) = \sum_{j=1}^{n} x_j g_{ij}(x), \quad g_{ij}(x) = \int_{0}^{1} \frac{\partial g_i(sx)}{\partial x_j} \, ds.
\]

On \( W \) we now have that

\[
f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j g_{ij}(x),
\]

where \( g_{ij} \in C^\infty(W, \mathbb{R}) \). If we introduce \( h_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) \) then \((h_{ij})\) becomes a symmetric \( n \times n \) matrix of smooth functions on \( W \), and

\[
f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j h_{ij}(x). \quad (11)
\]

By differentiating (11) twice and substituting 0, we get

\[
\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = 2h_{ij}(0).
\]

In particular the matrix \((h_{ij}(0))\) is invertible.

Let us return to the original \( f \in C^\infty(M, \mathbb{R}) \). By induction on \( k \), we attempt to show that the \( C^\infty \)-chart \( h \) from the theorem can be choosen such that \( f \circ h^{-1} \) is given by (11) with

\[
(h_{ij}) = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix},
\]
with $D$ a $(k-1) \times (k-1)$ matrix of the form diag$(\pm 1, \ldots, \pm 1)$, and $E$ some symmetric $(n-k+1) \times (n-k+1)$ matrix of smooth functions. So suppose inductively that

$$
(12) \quad f(x) = \sum_{i=1}^{k-1} \delta_i x_i^2 + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} x_i x_j h_{ij}(x), \quad \delta_i = \pm 1
$$

for $x$ in a neighborhood $W$ of the origin. We know that the minor $E$ is invertible at $0$. To start off we can perform a linear change of variables in $x_k, \ldots, x_n$, so that our new variables satisfy (12) with $h_{kk}(0) \neq 0$. By continuity we may assume that $h_{kk}(x)$ has constant sign $\delta_k = \pm 1$ on the entire $W$. Set

$$
q = \sqrt{|h_{kk}|} \in C^\infty(W, \mathbb{R}),
$$

and introduce new variables:

$$
y_k = q(x) \left( x_k + \sum_{i=k+1}^{n} x_i \frac{h_{ik}(x)}{h_{kk}(x)} \right),
$$

$$
y_j = x_j \quad \text{for } j \neq k, 1 \leq j \leq n.
$$

The Jacobi determinant for $y$ as function of $x$ is easily seen to be $\partial y_k/\partial x_k(0) = q(0) \neq 0$. The change of variables thus defines a local diffeomorphism $\Psi$ around $0$. In a neighborhood around $0$ we have for $y = \Psi(x)$:

$$
\begin{align*}
f \circ \Psi^{-1}(y) &= f(x) \\
&= \sum_{i=1}^{k-1} \delta_i x_i^2 + x_k^2 h_{kk}(x) + 2x_k \sum_{j=k+1}^{n} x_j h_{jk}(x) + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} x_i x_j h_{ij}(x) \\
&= \sum_{i=1}^{k-1} \delta_i x_i^2 + h_{kk}(x) \left( x_k + \sum_{j=k+1}^{n} x_j \frac{h_{jk}(x)}{h_{kk}(x)} \right)^2 \\
&\quad - h_{kk}(x) \left( \sum_{j=k+1}^{n} x_j \frac{h_{jk}(x)}{h_{kk}(x)} \right)^2 + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} x_i x_j h_{ij}(x) \\
&= \sum_{i=1}^{k} \delta_i y_i^2 + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} x_i x_j h_{ij}(x) \\
&= \sum_{i=1}^{k} \delta_i y_i^2 + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} y_i y_j h_{ij}(y),
\end{align*}
$$

where $h_{ij} \in C^\infty(W, \mathbb{R})$. This completes the induction step. \qed

We point out that with the assumptions of Theorem 12.6 $p$ is the only critical point in $U$. If $M$ is compact and $f \in C^\infty(M, \mathbb{R})$ is a Morse function then $f$ has only finitely many critical points. Among them there will always be at least one local minimum (index $\lambda = 0$) and at least one local maximum (index $\lambda = n$).
Definition 12.7 Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function. A smooth tangent vector field $X$ on $M$ is said to be gradient-like for $f$, if the following conditions are satisfied:

(i) For every non-critical point $p \in M$, $d_p f (X(p)) > 0$.

(ii) If $p \in M^n$ is a critical point of $f$ then there exists a $C^\infty$-chart $h : U \to h(U) \subseteq \mathbb{R}^n$ with $p \in U$ and $h(p) = 0$ such that

$$f \circ h^{-1}(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2, \quad x \in h(U),$$

and $h_* X|_U = \text{grad} (f \circ h^{-1})$.

A smooth parametrized curve $\alpha : I \to M$ is an integral curve for $X$, if

$$\alpha'(t) = X(\alpha(t)) \quad \text{for } t \in I.$$ 

Hence one gets $(f \circ \alpha)'(t) = d_{\alpha(t)} f(X(\alpha(t)))$. If $c(I)$ does not contain any critical points, then $f \circ c : I \to \mathbb{R}$ is a monotone increasing function by condition (i).

Lemma 12.8 Every Morse function on $M$ admits a gradient-like vector field.

Proof. We can find a $C^\infty$-atlas $(U_\alpha, h_\alpha)_{\alpha \in A}$ for $M$ which satisfies the following two conditions:

(i) Every critical point of $f$ belongs to just one of the coordinate patches $U_\alpha$.

(ii) For any $\alpha \in A$ either $f$ has no critical point in $U_\alpha$ or $f$ has precisely one critical point $p$ in $U_\alpha$, $h_\alpha(p) = 0$, and $f \circ h_\alpha^{-1}$ has the form listed in Example 12.5.

Let $X_\alpha$ be a tangent vector field on $U_\alpha$ determined by $X_\alpha = (h_\alpha^{-1})_* (\text{grad} (f \circ h_\alpha^{-1}))$. Choose a smooth partition of unity $(\rho_\alpha)_{\alpha \in A}$ subordinate to $(U_\alpha)_{\alpha \in A}$, and define a smooth tangent vector field on $M$ by

$$X = \sum_{\alpha \in A} \rho_\alpha X_\alpha,$$

where $\rho_\alpha X_\alpha$ is taken to be $0$ outside $U_\alpha$. If $p \in M$ is not a critical point for $f$ then, for every $\alpha \in A$ with $p \in U_\alpha$ and $q = h_\alpha(p)$, we have

$$d_p f(X_\alpha(p)) = d_q (f \circ h_\alpha^{-1})(\text{grad}_q (f \circ h_\alpha^{-1})) > 0.$$

Indeed, there is at least one $\alpha$ with $\rho_\alpha(p) > 0$ and

$$d_p f(X(p)) = \sum_{\alpha} \rho_\alpha(p) d_p f(X_\alpha(p)).$$

We see that $X$ satisfies condition (i) in Definition 12.7.
If $p$ is a critical point of $f$ then there exists a unique $\alpha \in A$ with $p \in U_\alpha$. It follows from (a) that $X$ coincides with $X_\alpha$ on a neighborhood of $p$, and condition (b) above shows that assertion (ii) in Definition 12.7 is satisfied.

The next lemma relates the index of a Morse function to the local index of vector fields as defined in Chapter 11.

**Lemma 12.9** Let $f$ be a Morse function on $M$ and $X$ a smooth tangent vector field such that $d_pf(X(p)) > 0$ for every $p \in M$ that is not a critical point for $f$. Let $p_0 \in M$ be a critical point for $f$ of index $\lambda$. If $X(p_0) = 0$, then

$$\iota(X; p_0) = (-1)^\lambda.$$

**Proof.** We choose a gradient-like vector field $\tilde{X}$. By Definition 12.7(ii) and Example 12.5,

$$\iota(\tilde{X}; p_0) = (-1)^\lambda.$$

Let $U$ be an open neighborhood of $p_0$ that is diffeomorphic to $\mathbb{R}^n$ and chosen so small that $p_0$ is the only critical point in $U$. The inequalities

$$d_pf(X(p)) > 0, \quad d_pf(\tilde{X}(p)) > 0,$$

valid for $p \in U - \{p_0\}$, show that $X(p)$ and $\tilde{X}(p)$ belong to the same open half-space in $T_pM$. Thus

$$(1 - t)X(p) + t\tilde{X}(p) \quad (0 \leq t \leq 1)$$

defines a homotopy between $X$ and $\tilde{X}$ considered as maps from $U - \{p_0\}$ to $\mathbb{R}^n - \{0\}$, and $\iota(X; p_0) = \iota(\tilde{X}; p_0)$. \hfill $\square$

**Remark 12.10** Given a Riemannian metric on $M$ and $f \in C^\infty(M, \mathbb{R})$, one can define the gradient vector field $\text{grad} f$ by the equation

$$\langle \text{grad}_f(p), X_p \rangle = d_pf(X_p)$$

for all $X_p \in T_pM$. Then Lemma 12.9 holds for $\text{grad}(f)$.

**Theorem 12.11** Let $M^n$ be a compact differentiable manifold and $X$ a smooth tangent vector field on $M^n$ with isolated singularities. Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function and $c_\lambda$ the number of critical points of index $\lambda$ for $f$. Then we have that

$$\text{Index}(X) = \sum_{\lambda=0}^{n} (-1)^\lambda c_\lambda.$$
Proof. It is a consequence of Theorem 11.27 that any two tangent vector fields with isolated singularities have the same index. Thus we may assume that $X$ is gradient-like for $f$. The zeros for $X$ are exactly the critical points of $f$, and the claimed formula follows from Lemma 12.9. □

It is a consequence of the above theorem that the sum

$$
\sum_{\lambda=0}^{n} (-1)^\lambda c_\lambda
$$

is independent of the choice of Morse function $f \in C^\infty(M, \mathbb{R})$. Given Theorem 12.11, the Poincaré-Hopf theorem is the statement that the sum (13) is equal to the Euler characteristic; cf. (2).

We will give a proof of this based on the two lemmas below, whose proofs in turn involve methods from dynamical systems and ordinary differential equations, and will be postponed to Appendix C.

Let us fix a compact manifold $M^n$ and a Morse function $f$ on $M$. For $a \in \mathbb{R}$ we set

$$
M(a) = \{ p \in M \mid f(p) < a \}.
$$

Recall that a number $a \in \mathbb{R}$ is a critical value if $f^{-1}(a)$ contains at least one critical point.

Lemma 12.12 If there are no critical values in the interval $[a_1, a_2]$, then $M(a_1)$ and $M(a_2)$ are diffeomorphic.

Lemma 12.13 Suppose that $a$ is a critical value and that $p_1, \ldots, p_r$ are the critical points in $f^{-1}(a)$. Let $p_i$ have index $\lambda_i$. There exists an $\epsilon > 0$, and disjoint open neighbourhoods $U_i$ of $p_i$, such that

(i) $p_1, \ldots, p_r$ are the only critical points in $f^{-1}([a - \epsilon, a + \epsilon])$.
(ii) $U_i$ is diffeomorphic to an open contractible subset of $\mathbb{R}^n$.
(iii) $U_i \cap M(a - \epsilon)$ is diffeomorphic to $S^{\lambda_i-1} \times V_i$, where $V_i$ is an open contractible subset of $\mathbb{R}^{n-\lambda_i+1}$ (in particular $U_i \cap M(a - \epsilon) = \emptyset$ if $\lambda_i = 0$).
(iv) $M(a + \epsilon)$ is diffeomorphic to $U_1 \cup \ldots \cup U_r \cup M(a - \epsilon)$. □

Proposition 12.14 In the situation of Lemma 12.13 suppose that $M(a - \epsilon)$ has finite-dimensional cohomology. Then the same will be true for $M(a + \epsilon)$, and

$$
\chi(M(a + \epsilon)) = \chi(M(a - \epsilon)) + \sum_{i=1}^{r} (-1)^{\lambda_i}.
$$
Proof. For $U = U_1 \cup \ldots \cup U_r$, Lemma 12.13.(ii) and Corollary 6.10 imply that

$$H^p(U) \simeq \begin{cases} 0 & \text{if } p \neq 0 \\ \mathbb{R}^r & \text{if } p = 0. \end{cases}$$

This gives $\chi(U) = r$. Condition (iii) of Lemma 12.13 shows that $U_i \cap M(a - \epsilon)$ is homotopy equivalent to $S^{\lambda_i - 1}$, and Example 9.29 gives

$$\chi(U_i \cap M(a - \epsilon)) = 1 + (-1)^{\lambda_i - 1}.$$

Since $U \cap M(a - \epsilon)$ is a disjoint union of the sets $U_i \cap M(a - \epsilon)$, it has a finite-dimensional de Rham cohomology, and

$$\chi(U \cap M(a - \epsilon)) = \sum_{i=1}^r (1 + (-1)^{\lambda_i - 1}) = r - \sum_{i=1}^r (-1)^{\lambda_i} = \chi(U) - \sum_{i=1}^r (-1)^{\lambda_i}.$$

The claimed formula now follows from Lemma 12.13.(iv) and the lemma below, applied to $U$ and $V = M(a - \epsilon)$.

**Lemma 12.15** Let $U$ and $V$ be open subsets of a smooth manifold. If $U$, $V$ and $U \cap V$ have finite dimensional de Rham cohomology, the same is true for $U \cup V$, and

$$\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Proof. We use the long exact Mayer–Vietoris sequence

$$\cdots \rightarrow H^{p-1}(U \cap V) \rightarrow H^p(U \cup V) \rightarrow H^p(U) \oplus H^p(V) \rightarrow H^p(U \cap V) \rightarrow \cdots$$

First we conclude that $\dim H^p(U \cup V) < \infty$. Second, the alternating sum of the dimensions of the vector spaces in an exact sequence is equal to zero; cf. Exercise 4.4.

**Theorem 12.16** If $f$ is a Morse function on the compact manifold $M^n$, then

$$\chi(M^n) = \sum_{\lambda=0}^n (-1)^{\lambda} c_{\lambda},$$

where $c_{\lambda}$ denotes the number of critical points for $f$ of index $\lambda$.

Proof. Let $a_1 < a_2 < \ldots < a_{k-1} < a_k$ be the critical values. Choose real numbers $b_0 < a_1$, $b_j \in (a_j, a_{j+1})$ for $1 \leq j \leq k - 1$ and $b_k > a_k$. Lemma 12.12 shows that the dimensions of $H^d(M(b_j))$ are independent of the choice of $b_j$ from the relevant interval. If $M(b_{j-1})$ has finite-dimensional de Rham cohomology, the same will be true for $M(b_j)$ according to Proposition 12.14, and

$$\chi(M(b_j)) - \chi(M(b_{j-1})) = \sum_{p \in f^{-1}(a_j)} (-1)^{\lambda(p)}$$
Here the sum runs over the critical points \( p \in f^{-1}(a_j) \), and \( \lambda(p) \) denotes the index of \( p \). We can start from \( M(b_0) = \emptyset \). An induction argument shows that \( \dim H^d(M(b_j)) < \infty \) for all \( j \) and \( d \). The sum of the formulas of (14) for \( 1 \leq j \leq k \) gives

\[
\chi(M) = \chi(M(b_k)) = \sum_p (-1)^{\lambda(p)}
\]

where \( p \) runs over the critical points. \( \square \)

The Poincaré–Hopf theorem 12.1 follows by combining Theorems 12.11 and 12.16.

**Corollary 12.17** If \( M^n \) is compact and of odd dimension \( n \) then \( \chi(M^n) = 0 \).

**Proof.** Let \( f \) be a Morse function on \( M \). Then \( -f \) is also a Morse function, and \( -f \) has the same critical points as \( f \). If a critical point \( p \) has index \( \lambda \) with respect to \( f \), then \( p \) has index \( n - \lambda \) with respect to \( -f \). Theorem 12.16 applied to both \( f \) and \( -f \) gives

\[
\chi(M) = \sum_{\lambda=0}^{n} (-1)^{\lambda} c_\lambda = \sum_{\lambda=0}^{n} (-1)^{n-\lambda} c_\lambda.
\]

The two sums differ by the factor \( (-1)^n \), and the assertion follows. \( \square \)

**Example 12.18** (Gauss–Bonnet in \( \mathbb{R}^3 \)). We consider a compact regular surface \( S \subseteq \mathbb{R}^3 \), oriented by means of the Gauss map \( N : S \rightarrow S^2 \). The Gauss curvature of \( S \) at the point \( p \) is

\[
K(p) = \det (d_p N); \quad T_p S = \{ p \} ^{\perp} = T_p S^2.
\]

Sard’s theorem implies that we can find a pair of antipodal points in \( S^2 \) that are both regular values of \( N \). After a suitable rotation of the entire situation we can assume that \( p_{\pm} = (0, 0, \pm 1) \) are regular values of \( N \).

Let \( f \in C^\infty(S, \mathbb{R}) \) be the projection on the third coordinate axis of \( \mathbb{R}^3 \). The critical points \( p \in S \) of \( f \) are exactly the points for which \( T_p S \) is parallel with the \( x_1, x_2 \)-plane, i.e. \( N(p) = p_{\pm} \). At such a point \( p \), the differential of the Gauss map \( d_p N \) is an isomorphism. Hence \( K(p) \neq 0 \). A neighborhood of \( p \) in \( S \) can be parametrized by \( (u, v, f(u, v)) \), and in these local coordinates the Gauss curvature has the following expression:

\[
K = \left( 1 + \left( \frac{\partial f}{\partial u} \right)^2 + \left( \frac{\partial f}{\partial v} \right)^2 \right)^{-1} \det \begin{pmatrix}
\frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial u \partial v} \\
\frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2}
\end{pmatrix}
\]
(see e.g. do Carmo page 163). Since \( K(p) \neq 0 \), the determinant in the expression does not vanish at \( p \), so \( p \) is a non-degenerate critical point for \( f \). If \( K(p) > 0 \) the determinant is positive and \( p \) has index 0 or 2. If \( K(p) < 0 \) the determinant is negative and \( p \) has index 1. We apply Theorem 12.16 to get

\[
\chi(S) = \# \{ p \in S \mid N(p) = p_{\pm}, K(p) > 0 \} - \# \{ p \in S \mid N(p) = p_{\pm}, K(p) < 0 \}.
\]

Since \( p_{+} \) is a regular value for \( N \), we have by Theorem 11.9

\[
\operatorname{deg}(N) = \# \{ p \in N^{-1}(p_{+}) \mid K(p) > 0 \} - \# \{ p \in N^{-1}(p_{+}) \mid K(p) < 0 \}
\]

and analogously with \( p_{-} \) instead of \( p_{+} \). It follows that

\[
\chi(S) = 2 \operatorname{deg}(N).
\]

(15)

The map

\[
\text{Alt}^2(dN_p): \text{Alt}^2(T_{N(p)}S^2) \to \text{Alt}^2(T_pS)
\]

is multiplication by \( \det(d_pN) = K(p) \), so \( N^*(\text{vol}_{S^2}) = K(p)\text{vol}_S \). Hence

\[
\int_S K\text{vol}_S = \int_{S^2} N^*(\text{vol}_{S^2}) = (\operatorname{deg}N) \int_{S^2} \text{vol}_{S^2} = 4\pi \operatorname{deg}N.
\]

Combined with (15) this yields the Gauss–Bonnet formula

\[
\frac{1}{2\pi} \int_S K\text{vol}_S = \chi(S).
\]

(16)

Example 12.19 Consider the torus \( T \) in \( \mathbb{R}^3 \). The height function \( f: T \to \mathbb{R} \) is a Morse function with the four indicated critical points. The Gauss curvature of \( T \) is positive at \( p \) and \( s \), but negative at \( q \) and \( r \) (cf. Example 12.18). Hence \( f \) is a Morse function on \( T \). The index at \( p, q, r \) and \( s \) is 0, 1, 1 and 2, respectively. Theorem 12.16 gives \( \chi(T) = 0 \). Since we know that \( \dim H^0(T) = \dim H^2(T) = 1 \), we can calculate \( \dim H^1(T) = 2 \).
Example 12.20 (Morse function on $\mathbb{R}P^n$) Real functions on $\mathbb{R}P^n$ are equivalent to even functions $f: S^n \to \mathbb{R}$, i.e. $f(x) = f(-x)$ for all $x \in S^n$. Let us try

$$f(x) = \sum_{i=0}^{n} a_i x_i^2$$

for $x = (x_0, x_1, \ldots, x_n) \in S^n \subseteq \mathbb{R}^{n+1}$, and real numbers $a_i$. The differential of $f$ at $x$ is given by

$$d_x f(v_0, \ldots, v_n) = 2 \sum_{i=0}^{n} a_i x_i v_i,$$

where $v = (v_0, v_1, \ldots, v_n) \in T_x S^n$ so that

$$\sum_{i=0}^{n} x_i v_i = 0.$$

Thus $x$ is a critical point for $f$ if and only if the vectors $x$ and $(a_0 x_0, a_1 x_1, \ldots, a_n x_n)$ are linearly independent. If the coefficients $a_i$ are distinct, this occurs precisely for $x = \pm e_i = (0, \ldots, \pm 1, \ldots, 0)$, and $f$ has exactly $2n + 2$ critical points. The induced smooth map $\tilde{f}: \mathbb{R}P^n \to \mathbb{R}$ then has $n + 1$ critical points $[e_j]$. In a neighborhood of $\pm e_0 \in S^n$ we have the charts $h$ with

$$h^{-1}_\pm(u_1, \ldots, u_n) = \left( \pm \sqrt{1 - \sum_{i=1}^{n} u_i^2}, u_1, \ldots, u_n \right),$$

and in a neighborhood of $0 \in \mathbb{R}^n$,

$$f \circ h^{-1}_\pm(u_1, \ldots, u_n) = a_0 \left( 1 - \sum_{i=1}^{n} u_i^2 \right) + \sum_{i=1}^{n} a_i u_i^2 = a_0 + \sum_{i=1}^{n} (a_i - a_0) u_i^2.$$

The matrix of the second-order partial derivatives for $f \circ h^{-1}_\pm$ (at 0) is the diagonal matrix

$$\text{diag}(2(a_1 - a_0), 2(a_2 - a_0), \ldots, 2(a_n - a_0)).$$

Hence $\pm e_0$ are non-degenerate critical points for $f$; the index for each is equal to the number of indices $i$ with $1 \leq i \leq n$ and $a_i < a_0$. An analogous result holds for the other critical points $\pm e_j$. For simplicity, suppose that $a_0 < a_1 < a_2 < \ldots < a_n$. Then the two critical points $\pm e_j$ for $f: S^n \to \mathbb{R}$ have index $j$. The induced function $\tilde{f}: \mathbb{R}P^n \to \mathbb{R}$ is a Morse function with critical points $[e_j]$ of index $j$. We apply Theorem 12.16 to $\tilde{f}$. Since $c_\lambda = 1$ for $0 \leq \lambda \leq n$, we get

$$\chi(\mathbb{R}P^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This agrees with Example 9.31.