Math 215B HW 6 Solutions

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1 Orienting Tangent Bundles

Every chart (U_i, ϕ_i) on M induces a trivialization $TU_i \to U_i \times \mathbb{R}^n$ of the tangent bundle $f(x, v) = (x, d\phi_i(v))$. The clutching function for the tangent bundle $d\phi_i \circ d\phi_j^{-1} = d(\phi_i \circ \phi_j^{-1})$ is equivalent to the Jacobian of the transition function for the manifold M. Hence, orientability of M is equivalent to orientability on the bundle TM. (Notice that as a manifold, TM is always orientable. But we are talking about bundle orientation here, not manifold orientation.)

2 Moving Off of Itself

(a) Identify ν_N with the tubular neighborhood η. By theorem 8.6, we can find a section e of $\eta \to N^n$ arbitrarily close to the identity such that $e(N^n) \pitchfork N^n$. But since $2n < m$, $e(N^n) \pitchfork N^n$ implies $e(N^n) \cap N^n = \emptyset$.

(b) Suppose $\exists$ a section e such that $e(\mathbb{RP}^1)$ is moved off of the zero section $\mathbb{RP}^1$, then the mod-2 intersection number of $\mathbb{RP}^1$ inside $\mathbb{RP}^2$ must be zero by 8.12. However, we know that this is not the case since any two embeddings of $\mathbb{RP}^{\mathbb{1}}$ in $\mathbb{RP}^2$ intersect at a point and have nonzero intersection number.

3 Intersection Product

Since $N = [z_0, z_1, 0], K = [0, z_1, z_2]$, we know that $N \cap K = [0, z_1, 0]$ is just a point. Hence, $[N \cap K] = [N] \cdot [K] = \pm 1$. But N, K are related by an orientation-preserving diffeomorphism. Hence $[N] \cdot [K] = 1$. Consider now the following chain of maps:

$$H_2(\mathbb{CP}^2) \times H_2(\mathbb{CP}^2) \xrightarrow{D \times D} H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) \xrightarrow{\smile} H^4(\mathbb{CP}^2) \xrightarrow{D} H_0(\mathbb{CP}^2) \tag{1}$$

Recall that $H*(\mathbb{CP}^2) = \mathbb{Z}[x]/(x^3)$ with $x \in H^2(\mathbb{CP}^2)$. Since we end at a nontrivial element $1 \in H_0(\mathbb{CP}^2)$, the cup product must output the class x^2 in $H^4(\mathbb{CP}^2)$. Therefore, the cohomologies are $(\pm x, \pm x)$. By Poincare duality again, this means $[N], [K]$ each represents a generator of $H_2(\mathbb{CP}^2)$.

4 Shriek Map

Let $C_M([N]) = [N] \frown [M]$ be the Poincare duality map from cohomology to homology and let D_M be the inverse of C_M. Then $\Delta_! = C_M \circ \Delta^* \circ D_{M\times M}$ and $\Delta_! \circ C_{M\times M} = C_M \circ \Delta^*$. By Kunneth formula, $[M \times M] = [M] \times [M]$ and $C_{M\times M}(\alpha \times \beta) = (\alpha \times \beta) \frown [M \times M] = (\alpha \frown [M]) \times (\beta \frown [M])$. Thus letting $a = \alpha \frown [M], b = \beta \frown [M]$, we see $\Delta_! \circ C_{M\times M}(\alpha \times \beta) = \Delta_!(a \times b)$. Plugging this into the identity $\Delta_! \circ C_{M\times M} = C_M \circ \Delta^*$ we find that up to signs,

$$\Delta_!(a \times b) = C_M \circ \Delta^*(\alpha \times \beta) = (\alpha \smile \beta) \frown [M] = (D_M(a) \smile D_M(b)) \frown [M] = a \cdot b \qquad (2)$$