Math 215B: Homework 6

Due Thursday, February 27 , 2020

(1). Let M be a differentiable manifold. Show that the tangent bundle $p : TM \to M$ is orientable if and only if M is orientable.

(2). Let M^m be a C^∞ closed manifold, and let $N^n \subset M^m$ be a smooth embedded submanifold, where N^n is also assumed to be compact with no boundary. We say that N^n can be *"moved off of itself" in* M if a tubular neighborhood η of N with retraction map $\rho : \eta \to N$ admits a section $\sigma : N \to \eta$ that is disjoint from N. That is, $N \cap \sigma(N) = \emptyset \subset \eta \subset M$.

(a). Suppose the dimensions of the manifolds satisfy $2n < m$. Prove that N can be moved of itself in M.

(b). To see that the dimension requirement above is necessary in general, show that

$$\mathbb{RP}^1 \subset \mathbb{RP}^2$$

cannot be moved off of itself. *Hint:* Compute the self intersection number (mod 2) of $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

(3). Write $\mathbb{CP}^n$ in projective coordinates. $\mathbb{CP}^2 = \{[z_0, z_1, z_2] \in \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^\times\}$. That is $\mathbb{CP}^{n+1}$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action, via scalar multiplication, of the nonzero complex numbers $\mathbb{C}^\times$.

There are two natural copies of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$ given by $\{[z_0, z_1, 0]\}$ and by $\{[0, z_1, z_2]\}$. If we call one of these N and the other K,

Show that the intersection product $[N] \cdot [K] = 1 \in H_0(\mathbb{CP}^2)$. Conclude that each of these classes represent a generator of $H_2(\mathbb{CP}^2)$.

(4). Let M^n be a closed oriented n-dimensional manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Let $\Delta_! : H_q(M \times M) \to H_{q-n}(M)$ be the shriek map in homology. Show that for any homology classes α and β of M, then $\alpha \cdot \beta = \pm\Delta_!(\beta \times \alpha)$.