Math 215B HW 5 Solutions

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1 Stable Triviality

(a) Suppose there is an immersion $f : M \times \mathbb{R}^k \to \mathbb{R}^{n+k}$. Then since $\dim M \times \mathbb{R}^k = \dim \mathbb{R}^{n+k}$, df is a fiberwise isomorphism. Hence, we can define $\tilde{f} : \epsilon^{n+k} \simeq M \times \mathbb{R}^{n+k} \to TM \oplus \epsilon^k$ via the formula $\tilde{f}(m, v) = df_m^{-1} v$. This is clearly smooth with smooth inverse. Hence, we have found the desired bundle isomorphism.

(b) Map $S^n \times \mathbb{R}$ a thickened open neighborhood of S^n inside $\mathbb{R}^{n+1}$ (say a family of spheres with radius $(r - \epsilon, r + \epsilon)$. Since this map is clearly an immersion, we know from part (a) that TS^n is stably trivial.

(c) By the hairy ball theorem, there is no nonvanishing section of TS^n for even n. Since vector bundles are trivial iff they admit nonvanishing sections, we conclude that TS^2 is not trivial. However, $TS^2 \oplus \epsilon^1$ is trivial by the construction in part (a) along with the embedding in part (b).

2 Tubular Neighborhood

(a) Consider the normal bundle $\nu_{P \cap Q} = N(P \cap Q) = TM/T(P \cap Q)$. Since $T(P \cap Q) = TP \cap TQ$, we know that $N(P \cap Q) = TM/T(P \cap Q) = TM/(TP \cap TQ) = TP/(TP \cap TQ) + TQ/(TP \cap TQ)$ where the last step follows from transversality [1]. By dimension counting, $\dim TP/(TP \cap TQ) + \dim TQ/(TP \cap TQ) = p + q - 2(p + q - n) = 2n - p - q = n - (p + q - n) = \dim N(P \cap Q)$. Therefore, $N(P \cap Q) = TP/(TP \cap TQ) \oplus TQ/(TP \cap TQ)$. From this identity we get the desired fiberwise isomorphism at all points $x \in P \cap Q$

$$T_x Q/(T_x P \cap T_x Q) \simeq \frac{N_x(P \cap Q)}{T_x P/(T_x P \cap T_x Q)} \simeq \frac{T_x M}{T_x P} \simeq N_x P \tag{1}$$

which implies $\nu_P\big|_{P \cap Q} = \nu_{P \cap Q}$.

[1] It is a fact in linear algebra that if $A = B + C$, then $A/(B \cap C) = B/(B \cap C) + C/(B \cap C)$

(b) We work with the definition of tubular neighborhood in theorem 8.2. Let $P \xrightarrow{e} M$ and $P \xrightarrow{\zeta} \nu_P$ be inclusion maps. Then by the definition of tubular neighborhood, there exists $\tilde{e} : \nu_P \to M$ such that $\tilde{e}(\nu_P) = \eta_P$ and $\tilde{e} \circ \zeta = e$. Restricting all maps to $P \cap Q$, we get $P \cap Q \xrightarrow{e|_{P\cap Q}} Q \subset M$, $P \cap Q \xrightarrow{\zeta|_{P\cap Q}} \nu_P\big|_{P\cap Q}$, and $\tilde{e}(\nu_P\big|_{P\cap Q}) = \eta_P \cap Q$. But from part (a) we know that there is a diffeomorphism $\Phi : \nu_{P\cap Q} \to \nu_P\big|_{P\cap Q}$ and an inclusion $\iota : P \cap Q \to \nu_{P\cap Q}$ such that $\Phi \circ \iota = \zeta|_{P\cap Q}$. Define $e' = e|_{P\cap Q}, \zeta' = \iota$ and $\tilde{e}' = \tilde{e} \circ \Phi$. Since $\tilde{e}' \circ \zeta' = \tilde{e} \circ \zeta|_{P\cap Q} = e|_{P\cap Q}$, we see that $\tilde{e}'(\nu_{P\cap Q}) = \eta_P \cap Q$ is a tubular neighborhood of $P \cap Q$ in Q.