Math 215B HW 4 Solutions

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1 Aspherical Space

When Y consists only of 0-cells, connectedness of Y guarantees that every map from $Y \to X$ is nullhomotopic. We now proceed by induction. Suppose all maps from the k-skeleton to X are nullhomotopic. Take one attachment map $f : (D^{k+1}, \partial D^{k+1}) \to Y \to X$. By the induction hypothesis, there exists a homotopy $\tilde{f}_t : \partial D^{k+1} \to Y \to X$ such that $\tilde{f}_0 = f|_{\partial D^k}$ and $\tilde{f}_1 = x_0$. For each t, by the homotopy extension property for $(D^{k+1}, \partial D^{k+1})$, $\tilde{f}_t$ extends to a homotopy $\tilde{F}_t$ on D^k with $\tilde{F}_1|_{\partial D^{k+1}} = x_0$. Hence the image of $\tilde{F}_1$ is now a S^{k+1} inside X. Since $\pi_{k+1}(X, x_0) = 0$, we can homotope $\tilde{F}_1$ to the constant map x_0. This completes the induction step.

If in addition, X has the homotopy type of a finite CW complex, then CW approximation tells us there is a finite CW complex $Z \sim X$ ($\sim$ here means homotopy equivalence). In particular, we can find $f : X \to Z, g : Z \to X$ such that $f \circ g \sim id_Z$ and $g \circ f \sim id_X$. Since g is nullhomotopic (by what we have proven), id_X is nullhomotopic and X is contractible.

2 Loop Space and Suspension

Recall the definitions:

$$\Sigma X = X \times S^1/(X \times \{1\} \cup x_0 \times S^1) \quad \Omega Y = \{f : (S^1, \{1\}) \to (Y, y_0)\} \tag{1}$$

(a) For any map $f : \Sigma X \to Y$, we can define the map $Gf : X \to \Omega Y$ via $Gf(x)(s) = f([x, s])$. The basepoints for $\Sigma X, \Omega Y, X, Y$ are $[x_0, s], \epsilon_0, x_0, y_0$ respectively. To check that this is a well-defined map of homotopy classes, we consider a basepoint preserving homotopy $f_t : \Sigma X \to Y$ with $f_t([x_0, s]) = y_0$. By definition, $Gf_t(x_0)(s) = f_t([x_0, s]) = y_0$ for all s. Therefore $Gf_t(x_0) = \epsilon_0$ is the basepoint of ΩY and G is a map between homotopy classes. The inverse can be constructed easily.

Now consider the special case where $X = S^{n-1}$. The map $G : [\Sigma S^{n-1}, Y] \to [S^{n-1}, \Omega Y]$ between homotopy classes preserve the group structures. Thus we have the chain of group isomorphisms

$$\pi_{n-1}(\Omega Y, \epsilon_0) = [S^{n-1}, \Omega Y] = [X, \Omega Y] \xrightarrow{G^{-1}} [\Sigma X, Y] = [S^n, Y] = \pi_n(Y, y_0)\,. \tag{2}$$

(b) Pick a basepoint $e_0 \in EG$ such that $p(e_0) = b_0 \in BG$. Since EG is contractible, we can find a homotopy $p \circ H_t : EG \to EG$ such that $p \circ H_{-1}(e) = b_0, p \circ H_1(e) = b$. Now consider the path space $P(BG) = \{\gamma : I \to BG | \gamma(0) = b_0\}$. The homotopy $p \circ H_t$ induces a map $\bar{F} : EG \to P(BG)$ via $\bar{F}(e)(t) = p \circ H_t(e)$. Now notice that the map $f : \Sigma G \to BG$ appearing in corollary 4.10 is just $f(g, t) = \bar{F}((b_0, g), t)$ where (b_0, g) represents an element in the fiber $p^{-1}(b_0)$. Via the correspondence in part (a), f induces a map $\bar{f} : G \to \Omega BG$ given by $\bar{f}(g)(t) = f(g, t)$. We have therefore completed the commutative diagram:

$$\begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{p} & BG \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \bar{F}} & & \downarrow{\scriptstyle id} \\ \Omega BG & \longrightarrow & P(BG) & \xrightarrow{p} & BG \end{array}$$

Since $EG, P(BG)$ are both contractible, $\pi_n(EG) = \pi_n(P(BG)) = 0$ for all n. Hence, when we extend both rows in the diagram via the long exact sequence of homotopy groups, the map $\bar{f}_* : \pi_n(G) \to \pi_n(\Omega BG)$ is sandwiched between four isomorphisms. By the five lemma, $\bar{f}_*$ must also be an isomorphism.

3 Joining Constructions

(a) We do this by induction. The base case $x_0 * x_1$ is clearly just a segment, the convex hull of two points. Suppose $x_0 \ldots * x_{k-1} = \Delta^{k-1}$, then by definition $\Delta^{k-1} * x_k$ is just the cone $C\Delta^{k-1}$ which is homeomorphic to Δ^k.

(b) Present the convex hull as $\Delta^k = \{(t_1, \ldots, t_k) | \sum_{i=0}^k t_i \leq 1, t_i \geq 0, \forall t\}$ (this is not the standard presentation because we drop the t_0's coordinate and change the sum condition to an inequality). Consider the map $\phi : \Delta^k \times G^{k+1} \to G^{*(k+1)}$ given by

$$\phi(\vec{t}, \vec{g}) = [(g_0, t_1, \ldots, g_{k-1}, t_k, g_k)] \tag{3}$$

where on the RHS, $[\ldots]$ is the quotient under the equivalence relation on $G^{*(k+1)}$. Denote by μ the action of G which is identity on Δ^k and diagonal on G^{k+1}. Then

$$\phi(\mu_{\tilde{g}}(\vec{t}, \vec{g})) = \phi(\vec{t}, \tilde{g}\vec{g}) = [(\tilde{g}g_0, t_1, \ldots, \tilde{g}g_{k-1}, t_k, \tilde{g}g_k)] = \tilde{g}\phi(\vec{t}, \vec{g})\,. \tag{4}$$

This shows G-equivariance. In $\tilde{\Delta}^k \times G^{k+1}$, $t_i \neq 0$ for all i and the equivalence relations imposed by the join construction on $t_i = \{0, 1\}$ do not matter. Thus ϕ is a homeomorphism onto its image.