Math 215B: Homework 4

Due Thursday, February 6, 2020

(1). Let X be an aspherical space, and Y a connected, finite CW-complex. Show that every map $f : Y \to X$ is null homotopic. Show that as a result, if X has the homotopy type of a finite CW-complex, X is contractible.

(2) Let X be a space with a basepoint $x_0 \in X$. Recall that the (reduced) suspension of X, ΣX, is the space

$$\Sigma X = X \times S^1 / \{X \times \{1\} \cup x_0 \times S^1\}$$

Here I am thinking of S^1 as the unit complex numbers. Let (Y, y_0) be another space with basepoint. Consider the (based) "loop space"

$$\Omega Y = Map((S^1, \{1\}), (Y, y_0)).$$

This is the space of maps from S^1 to Y that take $1 \in S^1$ to the basepoint $y_0 \in Y$, endowed with the compact - open topology.

(a). Prove that there is a bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Here the notation $[-,-]$ denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_o) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

(b). Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the proof of Corollary 4.10 in the text. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".

(3) (a). Let $\{x_0, \cdots, x_k\}$ be a collection of $k+1$ - distinct points in $\mathbb{R}^L$ for some large integer L. Prove that the k - fold join $x_0 * x_1 * \cdots * x_k$ is the convex hull of these points and hence is the k - dimensional simplex Δ^k with vertices $\{x_0, \cdots, x_k\}$.

(b). Prove that there is a natural G - equivariant map

$$\Delta^k \times G^{k+1} \to G^{*(k+1)}$$

which is a homeomorphism onto its image when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here G acts on $\Delta^k \times G^{k+1}$ trivially on the simplex Δ^k and diagonally on G^{k+1}. G acts diagonally on the iterated join $G^{*(k+1)}$ as well, as described in section 4.41 of the text.