Math 215B HW 3 Solutions

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1 Clutching Functions

Let $\zeta \to B$ have trivializations $\{U_\alpha, \phi_\alpha\}$ and clutching maps $\phi_{\alpha,\beta}$.

(a) For the tensor product bundle, the clutching map acts the same way on each factor of the product. Therefore:

$$\phi^k_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R}) \times \ldots \times GL_n(\mathbb{R}) \quad \phi^k_{\alpha,\beta}(x) = (\phi_{\alpha,\beta}(x), \ldots, \phi_{\alpha,\beta}) \tag{1}$$

The three constraints $\phi_{\alpha\alpha} = I, \phi_{\alpha\beta} = \phi^{-1}_{\beta\alpha}, \phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$ are easy to check.

(b) As a set: $\Lambda^n\zeta = \sqcup_{b\in B}\Lambda^n\zeta_b$, where $\Lambda^n\zeta_b$ is just the n-th exterior power of the vector space ζ_b, which is isomorphic to $\Lambda^n(\mathbb{R}^n)$. Given the transition map $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ on ζ, the clutching function on the alternating bundle acts in the following way:

$$\Lambda^n\psi_{\alpha\beta}(x, w^1 \wedge w^2 \ldots \wedge w^n) = [\psi_{\alpha\beta}(x)w^1] \wedge [\psi_{\alpha\beta}(x)w^2] \ldots \wedge [\psi_{\alpha\beta}(x)w^n] \tag{2}$$

2 De Rham Map

Let $\int_\cdot d\omega = f(\cdot)$ and let $\delta(\int_\cdot \omega) = g(\cdot)$. Then we want to show $f = g$. Applying Stoke's theorem to a chain $\sigma : \Delta^k \to M$ (i.e. $\sigma \in C^k(M;\mathbb{R})$, we get

$$f(\sigma) = \int_\sigma d\omega = \int_{\partial\sigma} \omega = \delta(\int_\sigma \omega) = g(\sigma)\,. \tag{3}$$

3 Sections and Principal Bundles

Suppose that $p : E \to B$ has a section s. Then we can trivialize the principal bundle as

$$\phi : B \times G \to E \quad \phi(x, g) = s(x)g\,. \tag{4}$$

This map is surjective/injective because the G action is transitive/free. It is manifestly continuous. It is also equivariant because $\phi(x, g_1 g_2) = s(x) g_1 g_2 = \phi(x, g_1) g_2$. Since every morphism of G-bundles is an isomorphism, ϕ is a G-bundle isomorphism [1].

4 Line Bundles

(a) We get a binary operation for free because tensor products of line bundles remain line bundles. Associativity follows from the associativity of tensor multiplication. And commuting the order tensor multiplication clearly generates isomorphic bundles. Therefore, we have an abelian monoid. The unit element is simply the trivial bundle with clutching function $\phi_{\alpha,\beta}(x) = id$ for all $x \in U_\alpha \cap U_\beta$.

(b) To show that this is a group, we just need to find an inverse. But since $\phi_{\alpha,\beta}(x)$ maps to $\mathbb{R}_+$, we can always take $\psi_{\alpha,\beta}(x) = \phi_{\alpha,\beta}(x)^{-1}$. This is manifestly continuous and $\phi_{\alpha,\beta} \otimes \psi_{\alpha,\beta} \simeq id$.

[1] This essentially follows from the fact that G-bundles have local trivializations